Calculus II

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FRONT RANGE
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Contents

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We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its adolescence, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

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Chapter 0

Overview

0.1 A Three-Part Course

The topics of Calculus II fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Integration.** The first part of the course ties loose ends from Calculus I. The ending of Calculus I showed that antiderivatives can be used to evaluate integrals via the Fundamental Theorem of Calculus. However, by the end of Calculus I, only the very simplest antiderivatives can actually be computed. Part one expands the student's knowledge of **techniques of antidifferentiation**. These techniques are subsequently put to use computing **length, area, volume, and center of mass**.

- **Part II: Sequences and Series.** This is the topic that makes up the body of Calculus II. **Sequences and series** embody the beauty of mathematics; from simple beginnings (a sequence is just a list... a series is just adding up a list of numbers...) it quickly leads to incredible structure, surprises, complexity, and open problems. **Power series** redefine commonly used transcendental functions (functions that are not computed using algebra, e.g. cosine). If you've ever wondered **what your calculator does when you press the cosine button**, this is where you find out! (**Hint:** It does not have a circle of radius one spinning around with a team of elves that measure x coordinates.)

- **Part III: Coming Attractions.** By the end of Calculus II, the student is ready for a *lot* of other classes. The end of Calculus II thus ends with a sampler platter of topics that show the vast knowledge base built upon the foundation laid in Calculus II. Here the text takes a bite out of **Differential Equations**, serves some polar and parametric coordinates as a palate cleanser before **Calculus III**, and tastes some **Complex Analysis** to aid in digestion of Differential Equations. For dessert, it serves a scoop of **Probability** with both discrete and continuous colored sprinkles.

0.2 How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor

and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises".
2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises concurrently.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend

Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

0.3 Prerequisites in the Language of Mathematics

Types of Expressions

In Precalculus, there is a wide range of how much focus there is on the language of mathematics itself as opposed to calculation. To get everyone on the same page, here is a short list of some language, symbols, and ideas that we will use in this text.

Quantities, Functions, and Statements

Any valid collection of symbols in mathematics is called an *expression*. Every expression has a **type** which tells you what kind of expression that is. The most common types of expressions we will use in this course are the following:

- **Quantity.** Anything that represents a numerical value is a *quantity*. The following are some examples of expressions of type quantity:
  (i) 28
  (ii) $-\pi$
  (iii) The real root of the polynomial $x^3 - x + 10$.
- **Function.** A well-defined rule for mapping inputs to outputs is called a *function*. (See your precalculus text for a much more detailed and precise definition.) Here are some examples of expressions whose type would usually be interpreted as function:
  (i) cosine
  (ii) $f(x) = x^2$
  (iii) $f''(x)$
- **Statement.** Any expression that could be considered true or false is called a *statement*. You do not need to be certain which it is, just that it is possible to be true or false. Here are some examples of statements:
  (i) The number 28 is larger than the number 6.
  (ii) The number 28 is smaller than the number 6.
  (iii) The number 28 is smaller than my favorite number.

Note that all three of the above are perfectly good statements, even though the second and third may sound a bit odd! The first statement is true, the second statement is false, and the third statement is impossible to determine because you do not know my favorite number. But, it is a perfectly valid statement since it is either true or false.

If you have a background in computer programming, the above discussion of types should feel somewhat familiar; many programming languages require that one declares a data type when declaring a variable. The first type, quantity, is usually represented by something like `int` or `float` depending on what you want to use it for. The second type, function, usually corresponds to declaring a method or a subroutine. The third type, an expression which is true or false, is often called a `boolean`.

Also, be aware that our most common notation for functions, in which we write something like "$f(x) =$ formula" can easily be mistaken for a statement, since you could interpret the equals sign to be asking whether or not those two expressions are equal as opposed to creating an assignment of input to output. In programming languages, they often distinguish the different contexts by using a single equals to mean assignment and a double equals sign to mean a statement in which you are testing the equality of two expressions. It is extremely common in mathematics to use the same symbol for both meanings; we stick with this convention and will rely on context to interpret which is meant when.

Exercise 0.3.1. Types of Objects

Each of the following objects is either a quantity, a function, or a statement. Identify which is which!

- $\cos(x)$
- $\cos(\pi)$
- $\cos(\pi) = 0$

Be aware that we often identify a quantity with the corresponding constant function. That is, 3 is a quantity, but it often is useful to think of it as the constant function $f(x) = 3$.

Sets and Elements

There is another very important (and somewhat more complicated) expression type we will frequently use in this course: the type *set*. A *set* is just a collection of objects. Amazingly, this simple idea is often used as the foundation of all of modern mathematics! Here is some notation.

- If an object x is in a set A, we say x is an *element of* A and write $x \in A$.
- If an object x is not in a set A, we say x is *not an element of* A and write $x \notin A$.

Any particular object is either an element of a given set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in the same set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 0.3.2. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}.$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Invalid Expressions

Be aware that it is easy to write down expressions which do not have a valid type. In fact, most collections of symbols have no meaning in the language of mathematics, much as if you typed a random string of letters, it would be very unlikely to spell a valid word in the English language. We call these expressions *garbage* (and can be thought of as the equivalent of a compiler error in programming).

Example 0.3.3. Garbage

The expression

$$2 \in 3$$

might look like a statement. However, it is not. The relation $\in$ expects a set on the right, however we handed it a quantity. Thus, we did not assemble our types of objects into a valid expression. Thus the above expression is neither true nor false, but simply garbage. Nobody likes garbage.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used throughout mathematics.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,

  $$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

  Note that in many other sources, zero is not included in the natural numbers. Those who authored such sources are bad people, and you should tell them you are very disappointed in them when you see them.

- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,

  $$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.

- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.

- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : P(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $P(x)$ is true about x. Sometimes the "$\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as "the set of all x in B such that $P(x)$". One can think of this as running through all elements of B and throwing away any that do not meet the condition described by P.

Example 0.3.4. Interval Notation

Interval notation can be expressed in set-builder notation as follows.

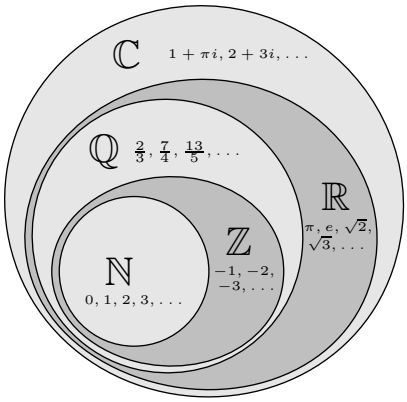
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 0.3.5. Rational, Real, and Complex in Set-Builder Notation

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$



Exercise 0.3.6. Sorting Recyclables and Taking Out the Trash ☕

Identify each of the following expressions as a **quantity**, **function**, **statement**, **set**, or **garbage**.

- The number of atoms in the universe.
- The process by which a US citizen is assigned a social security number.
- $+++$
- $\mathbb{N} < \mathbb{Z}$
- $\mathbb{N} \in \mathbb{Z}$
- $\pi \in \mathbb{Q}$
- $i \in \mathbb{C}$
- $3 \in \mathbb{C}$

Quantifiers

There are two symbols from logic that we will occasionally use.

- **Universal Quantifier:** The symbol $\forall$ is a shorthand for the phrase "for all", representing the A from All, but it tripped on a comma and landed on its head. For example, the expression

  $$\forall x \in \mathbb{R}, x^2 \geq 0$$

  is just a shorter way to say the sentence

  Every real number has a square that is greater than or equal to zero.

- **Existential Quantifier:** The symbol $\exists$ is a shorthand for the phrase "there exists", representing the E from Exists, but it too met a comma. One typically includes the phrase "such that" when reading an existential quantifier to make it sound more natural. For example, the expression

  $$\exists x \in \mathbb{R}, x^3 + x + 1 = 0.$$

  is just a shorter way to say the sentence

  There exists a real number x such that $x^3 + x + 1 = 0$.

A single quantifier is not usually that complicated to deal with. However, when a statement contains two or more quantifiers, it can quickly become difficult to extract exactly what it is saying! The following exercise demonstrates how slightly altering the order of quantifiers can drastically change the meaning of a sentence.

Exercise 0.3.7. Order Matters ☕☕

Let P be the set of all humans who have ever existed. Write each of the following statements out in words. Then, decide if it is true or false.

- $\forall x \in P, \exists y \in P$, x is the mother of y.
- $\exists x \in P, \forall y \in P$, x is the mother of y.
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, $x < y$.
- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}$, $x < y$.

0.4 Prerequisites from Algebra

Ever since Leonhard Euler's incredibly influential work in the 1700s, mathematics has largely become the study of functions. Today's algebra and trigonometry curricula reflect that! Here are the most important functions from those courses and a few very important things to know about them.

Polynomials

For our purposes in this course, a *polynomial* is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where n is a natural number and the numbers $a_0, a_1, \ldots, a_n$ are complex numbers (called *coefficients*) with $a_n \neq 0$. The coefficient a_n is called the *leading coefficient* and n is the *degree* of the polynomial. The number a_0 is called the *constant term* of the polynomial.

The above form of polynomials is often called *expanded form* or *standard form*. There is another form for polynomials called *factored form* in which a polynomial is written as the product of other smaller degree polynomials.

Polynomials are best understood through their *roots*. Any complex number r for which $p(r) = 0$ is called a *root* of the polynomial.

Exercise 0.4.1. Language of Polynomials

Identify each of the following statements as true or false.

- The function $p(x) = x^{1/2}$ is not a polynomial because $1/2$ is not a natural number.
- The polynomial $p(x) = x - 5$ has degree 1 and just a single root, namely 5.
- The polynomial $p(x) = 5$ has leading coefficient of 5 and degree zero.
- The polynomial $p(x) = 5$ has no roots.
- The polynomial $p(x) = x^2 - 9$ has exactly two roots, namely 3 and -3.
- The polynomial $p(x) = 0$ has degree zero and thus it has no roots.

Theorems about Polynomials

Here we state without proof several useful theorems regarding polynomials.

Theorem 0.4.2. Factor Theorem

The complex number r is a root of the polynomial $p(x)$ if and only if $(x - r)$ is a factor of $p(x)$. That is to say, $p(r) = 0$ if and only if $p(x) = (x - r)q(x)$ for some polynomial $q(x)$.

Note that the above theorem does not say that a polynomial $p(x)$ with root r has to be divisible by $(x - r)$ only once. Perhaps it is divisible by some higher power of $(x - r)$, like $(x - r)^2$. This leads to the idea of *multiplicity*: the multiplicity of a root r in a polynomial $p(x)$ is the highest power of $(x - r)$ that divides $p(x)$.

Theorem 0.4.3. Fundamental Theorem of Algebra

Every degree n polynomial has exactly n roots in the complex numbers when counted with multiplicity.

Example 0.4.4. Counting Roots with Multiplicity

Consider the polynomial

$$p(x) = (x - 1)^3(x + 2).$$

If we multiply everything out, we get

$$p(x) = -2 + 5x - 3x^2 - x^3 + x^4$$

which has degree 4. Thus, the Fundamental Theorem of Algebra promises four roots. When we count with multiplicity, we see that is the case: the list of roots is

$$1, 1, 1, -2.$$

Said another way, the polynomial has a root $r = 1$ with multiplicity 3 and a root $r = -2$ with multiplicity 1.

Exercise 0.4.5. Checking Algebra ☕

Multiply out the product $p(x) = (x - 1)^3(x + 2)$ and verify that the expanded form of $p(x)$ shown above is correct.

Because of the Fundamental Theorem of Algebra, when we say to factor a polynomial, we typically mean to factor it into degree 1 factors with complex roots (since it is always possible to do so). Sometimes, however, we simply factor over the real numbers, in which case one may end up with degree 2 factors that have no real roots (for example something like $x^2 + 1$ whose only roots are i and $-i$). It is not the slightest bit obvious why, but it turns out that any polynomial of degree 3 or more with real coefficients will factor into a product of smaller degree polynomials with real coefficients.

Notice that taking a polynomial from factored form to expanded form is not that difficult; one simply multiplies it out. However, going the other direction, from expanded form to factored form, is far more difficult. The next subsection is dedicated to this complicated task!

Polynomial Factoring Techniques

The most difficult part of working with polynomials is usually finding roots (or factoring, which is equivalent thanks to the Factor Theorem). The next few results give some methods towards that goal. Note that none of those are guaranteed to work in general; these results are merely a collection of special cases in which something works out nicely.

Quadratic Formula/Completing the Square

The famous *quadratic formula* gives an explicit formula for the roots of a degree 2 polynomial in terms of the coefficients. Specifically, the degree 2 polynomial

$$p(x) = ax^2 + bx + c$$

has roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and thus has factorization

$$p(x) = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

Notice that we need the leading coefficient a out in front in addition to the factors promised by the factor theorem; otherwise if you multiplied out the right-hand side, the leading term would be simply x^2 rather than ax^2.

Exercise 0.4.6. Trying out the Quadratic Formula

Find the roots of the polynomial $p(x) = 2x^2 + 2x + 2$ using the quadratic formula, and then use that to write it in factored form.

It is worth noting that there are cubic and quartic formulas (i.e., similar formulas for degree 3 and 4 polynomials, respectively) but they are far messier and thus typically not memorized, but rather used as theoretical tools or looked up when needed. There provably cannot exist a general formula for the roots when the degree is greater than or equal to five.

It is sensible to ask where the quadratic formula comes from! There is a process known as *completing the square* that can be used to prove it. Specifically, completing the square is just rewriting a quadratic in another form:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Exercise 0.4.7. Check It ☕

- Expand and simplify the right-hand side given above, namely

$$a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a},$$

  and verify that it does equal the left-hand side as claimed.

- Set $a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$ equal to zero and solve for x. Verify that it produces the quadratic formula.

One can apply the technique of completing the square by simply memorizing the above formula. However, in practice nobody does. It is usually implemented in a sequence of four steps:

1. Factor out the leading coefficient a from the x^2 and x terms.
2. Add and subtract the square of half of the remaining linear coefficient. That is, add and subtract the quantity $\left(\frac{b}{2a}\right)^2$.
3. Notice that $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2$ is now a perfect square trinomial, and factor it as $\left(x+\frac{b}{2a}\right)^2$.
4. Distribute the leading coefficient to the $-(b/2a)^2$ that is left over from step 2, and combine like terms.

Example 0.4.8. Revisiting Exercise 0.4.6

Suppose we once again consider the polynomial $p(x)=2x^2+2x+2$. Let us follow the four steps given above to complete the square on it.

1. We factor out 2, the leading coefficient:

$$p(x)=2\left(x^2+x\right)+2.$$

2. The remaining coefficient on x is just 1. We take half of that $(1/2)$ and then square that quantity to get $(1/2)^2=1/4$. This is the quantity we add and subtract, so it becomes

$$p(x)=2\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)+2.$$

3. Notice the first three terms inside the parentheses form a perfect square trinomial:

$$p(x)=2\left(\underbrace{x^2+x+\frac{1}{4}}_{\text{perfect square}}-\frac{1}{4}\right)+2.$$

   Factor that perfect square:

$$p(x)=2\left(\left(x+\frac{1}{2}\right)^2-\frac{1}{4}\right)+2.$$

4. Distribute the 2 and combine like terms:

$$\begin{aligned} p(x) &= 2\left(x+\frac{1}{2}\right)^2 - \frac{1}{2} + 2 \\ &= 2\left(x+\frac{1}{2}\right)^2 + \frac{3}{2}. \end{aligned}$$

At this point, we have successfully completed the square on the polynomial $p(x)$. If you like, you can then set that equal to zero and solve for the roots, which should match what we obtained via the quadratic formula. Trying this out, we have

$$\begin{aligned} 2\left(x+\frac{1}{2}\right)^2 + \frac{3}{2} = 0 &\iff 2\left(x+\frac{1}{2}\right)^2 = -\frac{3}{2} \\ &\iff \left(x+\frac{1}{2}\right)^2 = -\frac{3}{4} \\ &\iff x+\frac{1}{2} = \pm\sqrt{-\frac{3}{4}} \\ &\iff x+\frac{1}{2} = \pm\frac{\sqrt{3}}{2}i \\ &\iff x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{aligned}$$

Exercise 0.4.9. Try Some! ☕

Complete the square on the following polynomials.

- $p(x) = x^2 - 1$
- $p(x) = x^2 - x + 1$
- $p(x) = 3x^2 - 6x + 1$

Sums and Differences of Squares and Cubes

The following formulas come up so often they are worth simply memorizing.

- **Difference of Two Squares:**

$$A^2 - B^2 = (A-B)(A+B)$$

- **Sum of Two Squares:**

$$A^2 + B^2 = (A-Bi)(A+Bi)$$

- **Difference of Two Cubes:**

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

- **Sum of Two Cubes:**

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

Exercise 0.4.10. Testing the Formulas ☕☕☕

- For each of the above four formulas, multiply out the right-hand side. Verify that it does in fact equal the left-hand side.
- Use the above formulas to factor the polynomial $p(x) = 4x^4 - 9$.
- Factor the polynomial $x^6 - 1$ in two ways:
  - Start with the difference of two squares formula, rewriting the polynomial as $\left(x^3\right)^2 - 1^2$.
  - Start with the difference of two cubes formula, rewriting the polynomial as $\left(x^2\right)^3 - 1^2$.

Rational Root Theorem/Polynomial Long Division

The Rational Root Theorem gives us a list of educated guesses as to what a root of our polynomial might be.

Theorem 0.4.11. Rational Root Theorem

Let $p(x)$ be a polynomial with integer coefficients and let a and b be integers with b nonzero. If the rational number a/b is a root of $p(x)$, then a must divide the constant term of $p(x)$ and b must divide the leading coefficient.

The statement of the theorem might be a bit of a mouthful, but it is quite easy to apply.

Example 0.4.12. Applying the Rational Root Theorem

Consider the polynomial $p(x) = 5x^2 - 7x - 6$. The only integer divisors of the leading coefficient 5 are $1, 5$, and their negatives. The only integer divisors of the constant term -6 are $1, 2, 3, 6$, and their negatives. The Rational Root Theorem tells us that any rational root must have numerator that divides the constant term and denominator that divides the leading term. Thus, the only possible rational roots of $p(x)$ are

$$1, 2, 3, 6, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{6}{5},$$

and their negatives.
If we plug these numbers into the polynomial $p(x)$, we find that $p(-3/5) = 0$ and $p(2) = 0$. By the Factor Theorem, we have that $(x + 3/5)$ and $(x - 2)$ must be factors of $p(x)$. By the Fundamental Theorem of Algebra, we know that there are no other roots, since we already found two roots,

and the polynomial has degree two. Thus, the polynomial factors as

$$p(x) = 5(x + 3/5)(x - 2).$$

One note is that often it is cleaner to put the leading coefficient (or some part of it) into one of the factors to clean up the fractions. Here, if we put multiply the leading coefficient through the first factor, we get

$$p(x) = (5x + 3)(x - 2),$$

which is quite a bit cleaner.

One aspect of the above example that is perhaps a bit unsatisfying is that it would be extremely tedious to plug that whole list of possible roots into $p(x)$, desperately trying to find two roots. What is easier is after you find one root, just use *long division* to divide off the corresponding factor. We illustrate this below.

Example 0.4.13. Same Factorization with Long Division

Suppose we wanted to factor that same $p(x)$ from the previous example. We start by just plugging in easy numbers from the list, and find that $x = 2$ is a root. Not wanting to keep plugging in numbers, we recall from the Factor Theorem that $(x - 2)$ must be a factor of our polynomial. We now apply long division to find the quotient. In particular, we have the following:

$$\begin{array}{r|rrr}
 & & & 5x & +3 \\ \cline{2-4}
x-2 & 5x^2 & -7x & -6 \\
 & -5x^2 & +10x & \\ \cline{2-4}
 & & 3x & -6 \\
 & & -3x & +6 \\ \cline{3-4}
 & & & 0
\end{array}$$

which shows us that

$$5x^2 - 7x - 6 = (5x + 3)(x - 2).$$

Here is an example of using this same process on a larger degree polynomial.

Example 0.4.14. Factoring a Cubic

Suppose we wish to factor

$$p(x) = 4x^3 - 36x^2 + x - 9$$

using the Rational Root Theorem and long division. The list of possible roots is

$$1, 3, 9, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{1}{4}, \frac{3}{4}, \frac{9}{4},$$

and their negatives. We try the whole numbers first, and luckily find that $x = 9$ is a root. We proceed with long division:

$$
\begin{array}{r|l}
 & 4x^2 \qquad\quad +1 \\
\hline
x-9 & 4x^3 - 36x^2 + x - 9 \\
 & \underline{-4x^3 + 36x^2} \\
 & \qquad\qquad x - 9 \\
 & \qquad\qquad \underline{-x + 9} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

which tells us that

$$4x^3 - 36x^2 + x - 9 = (x-9)(4x^2+1).$$

If we are factoring over the real numbers, we are done. If we want to continue to factor using complex numbers, we can, using the Sum of Two Squares Formula. This produces

$$4x^3 - 36x^2 + x - 9 = (x-9)(2x+i)(2x-i).$$

Exercise 0.4.15. RRT/Long Division Factoring ☕☕

Apply the process from the previous two examples to factor the polynomial

$$p(x) = 2x^3 + x^2 - 4x - 3.$$

Specifically, generate a list of possible rational roots. Then, plug those numbers in until you find a root. Use the Factor Theorem to build a corresponding factor, and then use long division to find the quotient.

Exercise 0.4.16. Another Cubic ☕☕

Here we repeat the process of the previous examples, but to save a little tedium, we are given a root.

- Show that the number $x = 25/6$ is a root of the polynomial $p(x) = 6x^3 - 19x^2 - 13x - 50$.
- Note that the Factor Theorem itself tells us that our polynomial must be divisible by $(x - 25/6)$. Though this is true, it is often quite cumbersome to then go through the long division with all the fractions. A helpful strategy is to instead clear fractions (think multiplying both sides by six if the factor was set equal to zero) and instead use $6x - 25$. Perform the long division and find a quotient $q(x)$ such that $p(x) = (6x - 25)q(x)$.

Factor by Grouping

Factor by Grouping is a method in which we strategically take the greatest common factor out of different clumps of terms in the hope that we end up with yet another common factor to pull out.

Example 0.4.17. Factoring by Grouping

The polynomial $x^4 + x^3 + 2x^2 + x + 1$ can be factored as follows:

$$\begin{aligned} x^4 + x^3 + 2x^2 + x + 1 &= x^4 + x^3 + x^2 + x^2 + x + 1 \\ &= x^2\left(x^2 + x + 1\right) + 1\left(x^2 + x + 1\right) \\ &= \left(x^2 + x + 1\right)\left(x^2 + 1\right). \end{aligned}$$

We could leave it in that form if we were factoring over the real numbers, or we could continue by using complex roots to obtain

$$\begin{aligned} x^4 + x^3 + 2x^2 + x + 1 &= \left(x^2 + x + 1\right)\left(x^2 + 1\right) \\ &= \left(x - \left(\frac{1+\sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{1-\sqrt{3}i}{2}\right)\right)(x+i)(x-i). \end{aligned}$$

Exercise 0.4.18. Revisiting a Previous Example ☕

In Exercise 0.4.14, we factored the polynomial $4x^3 - 36x^2 + x - 9$. Try factoring that same polynomial again, but this time use factor by grouping. Verify the result comes out the same!

Exercise 0.4.19. Factor by Grouping Practice ☕☕

1. Factor the following polynomials by grouping:
   - $x^4 - x^3 + 2x^2 - x + 1$
   - $x^3 - x^2 + 2x - 2$
2. Consider the polynomial
$$x^4 + x^3 - x - 1$$
   - Factor by grouping the degree 3 and 4 terms together, while grouping the degree 1 and 0 terms together.
   - Factor by grouping the degree 4 and 0 terms together, while grouping the degree 3 and 1 terms together.

Pascal's Triangle

Pascal's Triangle can be thought of simply as a table of numbers. One starts with two diagonals of 1's, and then adds two numbers above to produce the number below.

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		3		3		1			
$n=4$			1		4		6		4		1		
$n=5$		1		5		10		10		5		1	
$n=6$	1		6		15		20		15		6		1

For example, in row $n = 4$, you can see the 6 and 4 add together to create the 10 below. The table goes on indefinitely, but we show just rows zero through six above.

Why is this in our section on polynomials? Well, it turns out that Pascal's Triangle gives you the coefficients upon expanding a power of a binomial. More specifically, the coefficients of the polynomial

$$(A+B)^n$$

can be found in row n of Pascal's Triangle. Notice that any term in the expansion of $(A+B)^n$ will have exactly n combined occurrences of A and B. Thus, the possible terms are

$$A^n, A^{n-1}B, A^{n-2}B^2, A^{n-3}B^3, \ldots, B^n,$$

which is to say one can start from A^n and then just decrease the power of A as you increase the power of B, ending at B^n. Then, the coefficients come from Pascal's Triangle.

This may sound like a complicated process, but in an example one will see it is quite quick and easy to implement!

Example 0.4.20. Expanding Using Pascal's Triangle

Suppose we wish to expand the polynomial

$$(x-2)^3.$$

We could just write out three copies of $(x-2)$ and multiply everything out, but that gets rather tedious. Instead, we apply Pascal's Triangle as described above.
First, we list the terms that will be produced:

$$x^3, -2x^2, (-2)^2x, (-2)^3.$$

Now, we attach the coefficients from row $n = 3$ of Pascal's Triangle (which are $1, 3, 3, 1$) to those terms, and add them up. This produces

$$(x-2)^3 = 1 \cdot x^3 + 3 \cdot (-2)x^2 + 3 \cdot (-2)^2x + 1 \cdot (-2)^3 = x^3 - 6x^2 + 12x - 8.$$

Ok, so why is this in a factoring section? Well, in reverse, if you happen to see a polynomial and can recognize the coefficients as coming from some row of Pascal's Triangle, then you can factor the entire polynomial in one step! For example, if we encountered the polynomial $x^3 - 6x^2 + 12x - 8$, we could say

$$x^3 - 6x^2 + 12x - 8 = 1 \cdot x^3 + 3 \cdot (-2)x^2 + 3 \cdot 4x + 1 \cdot (-8) = (x-2)^3,$$

which is far easier than RRT/long division or even Factor by Grouping.

Exercise 0.4.21. Practice with Pascal's Triangle ☕☕

Use Pascal's Triangle to factor the following polynomials:

- $-x^{10} + 5x^8 - 10x^6 + 10x^4 - 5x^2 + 1$
- $x^6 + 2x^3 + 1$

Rational Functions

A *rational function* is a function that can be written as a ratio (hence "rational") of two polynomials. That is, a rational function $r(x)$ is one expressible as

$$r(x) = \frac{p(x)}{q(x)}$$

for polynomials $p(x)$ and $q(x)$. Most of what one wants to know about rational functions can be determined by polynomial long division and the polynomial methods listed in the previous section.

Long Division

Long division with rational functions is a key step. If one has $r(x) = p(x)/q(x)$ and the degree of $p(x)$ is smaller than the degree of $q(x)$, then there is no need to perform division. For example, the function

$$r(x) = \frac{x+2}{x^2+2}$$

has smaller degree in the numerator than the denominator, so there is no need to use long division. But if it were the other way around,

$$r(x) = \frac{x^2+2}{x+2},$$

then we could perform long division. Specifically, it will allow us to write the function in form

$$r(x) = \texttt{quotient} + \frac{\texttt{remainder}}{\texttt{divisor}}.$$

One can think of this as being analogous to how improper fractions can be handled. For example, if one performs long division on 7 by 3, there is a quotient 2 with remainder 1. Thus, we have

$$\frac{7}{3} = 2 + \frac{1}{2}.$$

Let us now work through the example mentioned above.

Example 0.4.22. Long Division

We perform long division on

$$r(x) = \frac{x^2+2}{x+2}$$

as follows:

$$
\begin{array}{r|l}
 & \phantom{x^2 - 2x +{}} x - 2 \\
\hline
x + 2 & x^2 \qquad\quad + 2 \\
 & \underline{- x^2 - 2x} \\
 & \qquad - 2x + 2 \\
 & \qquad \underline{2x + 4} \\
 & \qquad\qquad 6
\end{array}
$$

From this, we conclude that

$$r(x) = x - 2 + \frac{6}{x+2}.$$

Notice that performing this long division tells you the *end behavior* of your rational function. The remainder term

$$\frac{6}{x+2}$$

will become arbitrarily small as x becomes a large positive or large negative number. Thus, the graph will converge to the quotient, which in this case is the line $y = x - 2$.

Exercise 0.4.23. Checking Work ☕

To really believe the calculation above, we should check it! Specifically, take the expression

$$x - 2 + \frac{6}{x+2}$$

and get a common denominator of $(x + 2)$ by turning the $x - 2$ into

$$\frac{x-2}{1} \cdot \frac{x+2}{x+2}.$$

Add the resulting numerators and recover the original function $r(x)$.

Roots of the Numerator and Denominator

Roots of the denominator of a rational function will cause division by zero, and thus produce either vertical asymptotes in the graph. Roots of the numerator of a rational function correspond to x-intercepts, since a fraction with zero in the numerator is zero. (Note that if the numerator and denominator share a zero, then it is more complicated and other things can happen. This situation will be explored later in the text.)

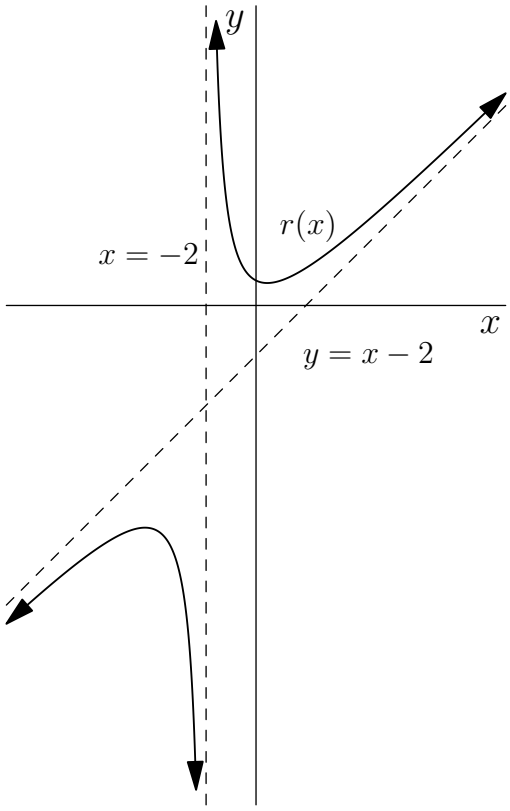
Example 0.4.24. Graphing a Rational Function

Let us combine all the information about the function

$$r(x) = \frac{x^2 + 2}{x + 2}$$

captured above in order to graph it. There are no real roots of the numerator, since the only roots are the complex numbers $\pm\sqrt{2}i$, which are valid roots of course, but they just don't show up on a graph. The denominator has $x = -2$ as its only root, so there is a vertical asymptote at $x = -2$.

It also never hurts to plot a random point or two. A nice one in this case is the y-intercept, $r(0) = 1$. Assembling this information along with the asymptote $y = x - 2$ found in the previous example produces the graph.



Exponentials and Logarithms

Though you were likely exposed to exponentials and logs in your college algebra/precalculus course, to really define exponentials and logarithms properly requires some construction from Calculus! Usually only an intuitive definition is given, something along the following lines:

The expression b^x is x copies of b multiplied together. The function $\log_b(x)$ is defined to be the inverse function of b^x.

The above definition is actually perfectly fine if x is a natural number. For example, one could say

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

and consequently

$$\log_2(8) = 3,$$

since inverse functions reverse the roles of inputs and outputs.

However, what if x is a fraction? Well, it turns out that isn't too bad to define, as one can use a radical. Specifically, if $x = n/m$ for some natural numbers n and m, we have

$$b^{n/m} = \sqrt[m]{b^n}.$$

However however, what if x is an irrational number? For example, what on earth does 2^π mean? To answer such questions, some form of calculus is required. So, in this section we do not dive deep at all, and instead just provide a list of commonly used identities. Note that all log and exponent identities come in pairs, as they are inverse functions: where one has an identity the other must have a corresponding opposite identity. In the table below, b always represents a positive real number.

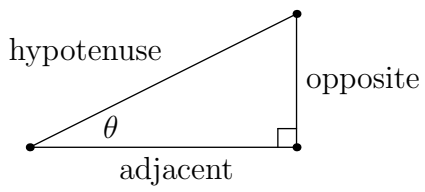
Name of Property	Property of Exponents	Property of Logarithms
Inverse Functions	$b^{\log_b(x)} = x$	$\log_b(b^x) = x$
Product to Sum	$b^x b^y = b^{x+y}$	$\log_b(xy) = \log_b(x) + \log_b(y)$
Difference to Quotient	$b^{x-y} = \frac{b^x}{b^y}$	$\log_b(x) - \log_b(y) = \log_b\left(\frac{x}{y}\right)$
Power to Product	$(b^x)^y = b^{xy}$	$\log_b(x^y) = y\log_b(x)$
Change of base	$b^x = e^{x\ln(b)}$	$\log_b(x) = \frac{\ln(x)}{\ln(b)}$

0.5 Prerequisites from Trigonometry

Trigonometry can be seen very geometrically as the study of triangles; it also can be seen as the study of the six trigonometric functions. Both perspectives will be used frequently throughout the calculus sequence!

Trigonometric Functions as Ratios of Sides

Consider the right triangle below, labelled with angle θ. It has two legs: one of which is *opposite* from the angle θ and one of which is *adjacent* to angle θ. There is only one side which can be called the *hypotenuse*: the side opposite the right angle.



The trigonometric functions are defined as ratios of ordered pairs of distinct sides of that triangle. Thus, there are 3 ways to select the numerator of the ratio and 2 remaining ways to select the denominator, so there are $3 \cdot 2 = 6$ trig functions. We name them and define them below.

- $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$
- $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$
- $\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$
- $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$
- $\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$

Exercise 0.5.1. Intertwined! ☕

It turns out all six trig functions can be written just in terms of sine and/or cosine. In particular, use the definitions given above to prove that the following four identities are true.

- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
- $\sec(\theta) = \frac{1}{\cos(\theta)}$
- $\csc(\theta) = \frac{1}{\sin(\theta)}$

Because the above exercise provides the other four trig functions for free once you have sine and cosine, the remainder of this section will focus largely on just those two functions rather than all six.

Sine and Cosine as Unit Circle Measurements

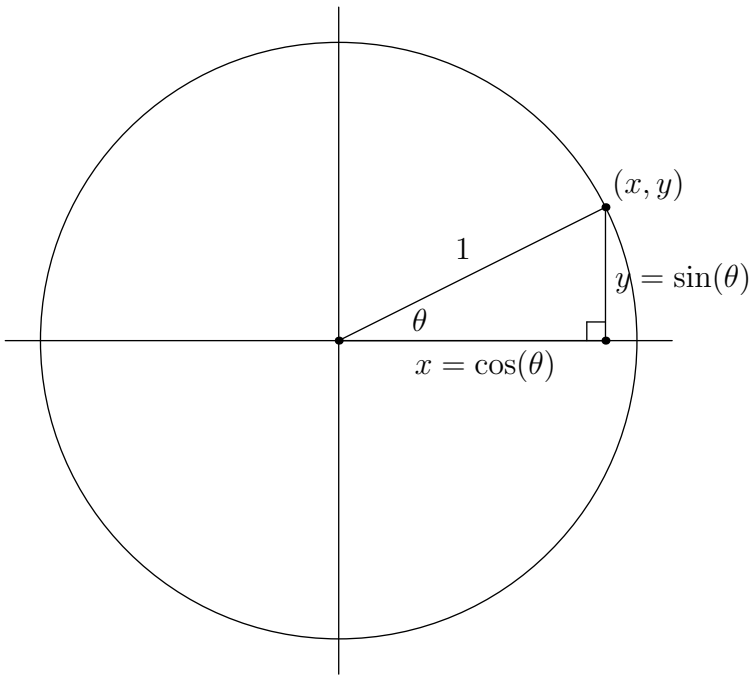
A very nice way to think about sine and cosine is as follows: for simplicity, pick the hypotenuse to equal 1. It is a ratio anyway, so you can always multiply the top and bottom by any nonzero amount. Then, place the adjacent side of the triangle along the positive x-axis with the vertex for angle θ at the origin. This means that if a point (x, y) is distance 1 from the origin, and the corresponding radius makes an angle θ with the positive x-axis, then we have

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x$$

and similarly

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y.$$

This is illustrated below.



On the unit circle, angles are typically measured in *radians* rather than degrees. Radian is simply the measure of arc length along the circumference of the unit circle. Since the circumference of a circle of radius r is

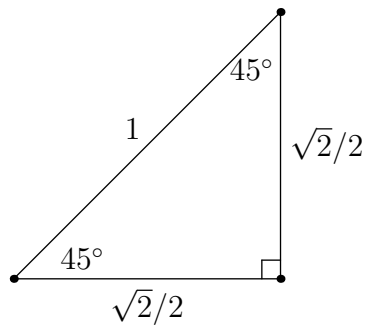
$$C = 2\pi r,$$

here we have a total circumference of 2π (since the radius is 1). Thus, one full lap of $360°$ is 2π radians. One can then scale that ratio up and down to get different equivalences of degrees and radians. Here are some common useful ones.

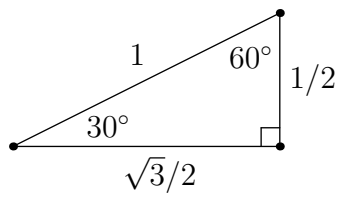
Degree Measure	Equivalent Radian Measure
$360°$	2π
$180°$	π
$90°$	$\pi/2$
$60°$	$\pi/3$
$45°$	$\pi/4$
$30°$	$\pi/6$

The acute angles listed above fit into two particularly special triangles of hypotenuse 1, whose measurements are shown below.

- **The** $45° - 45° - 90°$ **right triangle.** Since two angles are equal, the two legs must also be equal. One can then simply apply the Pythagorean Theorem for x in the equation $x^2 + x^2 = 1^2$ to obtain the measurement of $\sqrt{2}/2$.



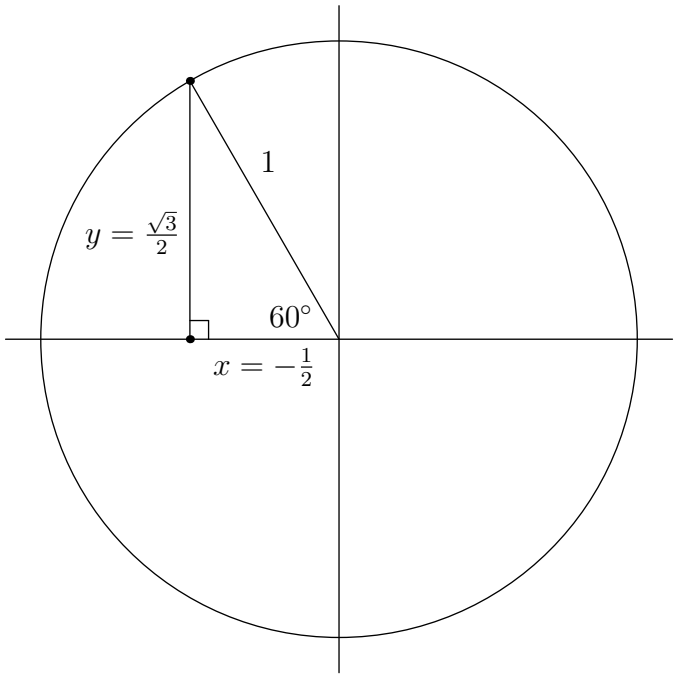
- **The** $30° - 60° - 90°$ **right triangle.** Notice that this triangle is simply half of an equilateral triangle of side length 1. That is how to remember the $1/2$, is it literally half of an equilateral. Then the Pythagorean Theorem provides the $\sqrt{3}/2$ by solving $x^2 + (1/2)^2 = 1^2$ for x.



While it is very common in reference sections like this to then provide a fully labeled unit circle which then students will fastidiously memorize, the author highly recommends against doing so. Instead, simply place the angle in question on the unit circle and find the x and y coordinates of the corresponding point using a special triangle whenever possible.

Example 0.5.2. Calculating Cosine and Sine of an Angle

Here we calculate cosine and sine of the angle $\theta = 2\pi/3$. We first draw the angle on the unit circle. To do so, we notice that since $\pi/3 = 60°$, then twice that must be $2\pi/3 = 120°$. This puts the angle at $60°$ from the negative x-axis. The hypotenuse is always 1, so we can label that side. We then see that we can fit a $30° - 60° - 90°$ triangle perfectly into that angle, which gives us the x and y coordinates.



Lastly, we recall that cosine is simply the x-coordinate and sine is the y-coordinate. We conclude that

$$\cos(2\pi/3) = -\frac{1}{2}$$

and

$$\sin(2\pi/3) = \frac{\sqrt{3}}{2}.$$

Here are a few other notes about unit circle computations:

- If an angle lands on an axis (i.e., is a multiple of ninety degrees) then no special triangle is needed as the x and y coordinates will just be 0, 1, or -1.
- Negative angles can be used; they simply wind clockwise rather than counterclockwise from the positive x-axis.

- Angles with magnitude larger than 2π can be used; this corresponds to taking more than one complete lap around the circle.

Trigonometric Identities

This is by no means a comprehensive list of trigonometric identities, but rather just a list of some that will come up frequently in this course.

- **Pythagorean Identities.**

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$
$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

- **Cosine Angle Sum.**

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

- **Sine Angle Sum.**

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

- **Cosine Angle Difference.**

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

- **Sine Angle Difference.**

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

- **Cosine Double Angle.**

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

- **Sine Double Angle.**

$$\sin(2A) = 2\sin(A)\cos(A)$$

- **Cosine Half Angle.**

$$\cos^2(A) = \frac{1 + \cos(2A)}{2}$$

- **Sine Half Angle.**

$$\sin^2(A) = \frac{1 - \cos(2A)}{2}$$

- **Cosine Even.**

$$\cos(-\theta) = \cos(\theta)$$

- **Sine Odd.**

$$\sin(-\theta) = -\sin(\theta)$$

- **Cofunction Identity.**

$$\cos(\pi/2 - \theta) = \sin(\theta)$$

Inverse Trigonometric Functions

The trigonometric functions are not one-to-one, so one must restrict their domains in order to build inverse functions. The table below gives one possible way of restricting the domains of the trigonometric functions and lists the corresponding domains and ranges of the inverse trigonometric functions.

Trig Function on Restricted Domain	Resulting Inverse Trig Function
$\sin : [-\pi/2, \pi/2] \to [-1, 1]$	$\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$
$\cos : [0, \pi] \to [-1, 1]$	$\arccos : [-1, 1] \to [0, \pi]$
$\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$	$\arctan : (-\infty, \infty) \to (-\pi/2, \pi/2)$
$\cot : (0, \pi) \to (-\infty, \infty)$	$\operatorname{arccot} : (-\infty, \infty) \to (0, \pi)$
$\sec : (0, \pi/2) \cup (\pi/2, \pi) \to (-\infty, -1] \cup [1, \infty)$	$\operatorname{arcsec} : (-\infty, -1] \cup [1, \infty) \to (0, \pi/2) \cup (\pi/2, \pi)$
$\csc : (-\pi/2, 0) \cup (0, \pi/2) \to (-\infty, -1] \cup [1, \infty)$	$\operatorname{arccsc} : (-\infty, -1] \cup [1, \infty) \to (-\pi/2, 0) \cup (0, \pi/2)$

To calculate with inverse trig functions once the domains and ranges are known, it is then just a matter of reversing the input and output from a trig function calculation.

Exercise 0.5.3. Inverse Trig Calculation

Calculate the values of the inverse trig functions listed below.

- $\arccos\left(-\frac{1}{2}\right)$
- $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

0.6 Prerequisites from Calculus

Though ideas of limits and continuity will also come up in Calculus II, the main skill from Calculus I that will be critical in this course is differentiation. We list some key formulas and properties of the derivative in this section. In the formulas below, x is a variable, $f(x)$ and $g(x)$ are differentiable functions, n is a real number, and b is a positive real number.

- **Limit Definition of the Derivative.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Power Rule.**

$$(x^n)' = nx^{n-1}$$

- **Linearity.**

$$(a \cdot f(x) + b \cdot g(x))' = a \cdot f'(x) + b \cdot g'(x)$$

- **Product Rule.**
$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- **Quotient Rule.**
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

- **Chain Rule.**
$$(f \circ g(x))' = f'(g(x)) \cdot g'(x)$$

- **Inverse Function Theorem.**
$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(f^{-1}(x)\right)}$$

- **Trig Functions.**
$$\begin{aligned}
(\sin(x))' &= \cos(x) \\
(\cos(x))' &= -\sin(x) \\
(\tan(x))' &= \sec^2(x) \\
(\cot(x))' &= -\csc^2(x) \\
(\sec(x))' &= \sec(x)\tan(x) \\
(\csc(x))' &= -\csc(x)\cot(x)
\end{aligned}$$

- **Inverse Trig Functions.**
$$\begin{aligned}
(\arcsin(x))' &= \frac{1}{\sqrt{1-x^2}} \\
(\arccos(x))' &= -\frac{1}{\sqrt{1-x^2}} \\
(\arctan(x))' &= \frac{1}{x^2+1} \\
(\operatorname{arccot}(x))' &= -\frac{1}{x^2+1} \\
(\operatorname{arcsec}(x))' &= \frac{1}{x\sqrt{x^2-1}} \\
(\operatorname{arccsc}(x))' &= -\frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

- **Logs and Exponentials.**
$$\begin{aligned}
(e^x)' &= e^x \\
(b^x)' &= b^x \cdot \ln(b) \\
(\ln(x))' &= \frac{1}{x} \\
(\log_b(x))' &= \frac{1}{x\ln(b)}
\end{aligned}$$

Part I

Integration

Chapter 1

Indefinite Integrals

The Fundamental Theorem of Calculus says that an integral (defined as the area under a curve) can be easily evaluated via antiderivative. However, it turns out to be very difficult and sometimes impossible to find an antiderivative! In this chapter, we give several commonly used methods for antidifferentiation.

Exercise 1.0.1. What is an Antiderivative Again?

- Complete the definition of antiderivative. That is, if $f(x)$ is a function, then we say $F(x)$ is an antiderivative of $f(x)$ if and only if...
- How do you use antiderivatives to evaluate definite integrals? Describe in a short sentence below.
- Once you found an antiderivative, what could you do to check that it is correct? (Besides just computing it again!)

1.1 The Method of u-Substitution

Undoing the Chain Rule

The technique of u-substitution (affectionately known as "u-sub" from here on) can be seen as the reverse of the chain rule for antiderivatives.

Exercise 1.1.1. What Was the Chain Rule Again?

- First, write down the chain rule.

$$(f(g(x)))' =$$

- Take the antiderivative of both sides of that equation.

$$\int \qquad \mathtt{d}x = f\left(g(x)\right) + C$$

In practice, we often make the substitution $u = g(x)$ to condense the notation. This will take a nastier integral with respect to x and replace it by a hopefully friendlier integral with respect to u. This process of transforming from x to u involves the following three steps:

1. **Choose u:** Pick u to be equal to some expression involving x. Frequently, it is helpful to pick u to be some "inner function" in a composition of functions that appears in the integrand. However, there is a *lot* of freedom regarding what substitution you make. Some choices of u will be helpful, and others will not be! It is important to be brave and just try some.
2. **Differentiate u:** Once you have a formula for u, differentiate with respect to x to get a formula for $\frac{\mathtt{d}u}{\mathtt{d}x}$. This will tell us what the conversion factor is between x units and u units.
3. **Solve for $\mathtt{d}x$:** Use your derivative to solve for $\mathtt{d}x$. Substitute that expression for the $\mathtt{d}x$ in the integral to replace it with $\mathtt{d}u$.

For the sake of having this process in a nice little formula box, here is the above paragraph rewritten concisely and precisely.

u-Substitution

$$\int f'\left(g(x)\right) g'(x)\,\mathtt{d}x = \int f'(u)\,\mathtt{d}u = f(u) + C = f\left(g(x)\right) + C$$

Example 1.1.2. An Example of Integration via u-sub

To evaluate $\int x\cos\left(x^2\right)\mathtt{d}x$, we identify $u = x^2$ as a plausible choice based on our recollection of chain rule. This gives the following change of variables:

Three Steps of u-Substitution

Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u = x^2$	$\frac{\mathtt{d}u}{\mathtt{d}x} = 2x$	$\mathtt{d}x = \frac{1}{2x}\,\mathtt{d}u$

We now replace x^2 by u and replace $\mathtt{d}x$ by $\frac{1}{2x}\,\mathtt{d}u$ in our integral.

$$\int x\cos\left(x^2\right)\mathtt{d}x = \int x\cdot\cos(u)\frac{1}{2x}\,\mathtt{d}u = \frac{1}{2}\int\cos(u)\,\mathtt{d}u = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin\left(x^2\right) + C$$

Exercise 1.1.3. Checking Our Work ☕

As a follow up to the previous example, differentiate the answer to verify that you end up with the original integrand!

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{1}{2}\sin\left(x^2\right)+C\right)=$$

Example 1.1.4. A Trickier u-sub

Suppose we wish to evaluate the following integral:

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\,\mathtt{d}x$$

One possible approach is to let u be the denominator. The denominator can be thought of as the "inner function" inside a reciprocal function and thus often makes a good choice for u.

Three Steps of u-Substitution		
Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u=\sqrt[3]{x}+1$	$\frac{\mathtt{d}u}{\mathtt{d}x}=\frac{1}{3}x^{-2/3}$	$\mathtt{d}x=3x^{2/3}\,\mathtt{d}u$

We now perform the substitutions on the denominator and the $\mathtt{d}x$.

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\,\mathtt{d}x=\int \frac{\sqrt[3]{x}}{u}3x^{2/3}\,\mathtt{d}u=3\int \frac{x}{u}\,\mathtt{d}u$$

At the moment, it seems like things are going very poorly! We hoped that x in the numerator would nicely cancel out, like it did back in the more civilized age of Exercise 1.1.2. To fix this, we solve for x in the equation $u=\sqrt[3]{x}+1$ to obtain $x=(u-1)^3$. We now substitute that expression for x in the integral.

$$\begin{aligned}
3\int \frac{x}{u}\,\mathtt{d}u &= 3\int \frac{(u-1)^3}{u}\,\mathtt{d}u\\
&= 3\int \frac{u^3-3u^2+3u-1}{u}\,\mathtt{d}u\\
&= 3\int u^2-3u+3-\frac{1}{u}\,\mathtt{d}u\\
&= u^3-\frac{9}{2}u^2+9u-3\ln|u|+C\\
&= \left(\sqrt[3]{x}+1\right)^3-\frac{9}{2}\left(\sqrt[3]{x}+1\right)^2+9\left(\sqrt[3]{x}+1\right)-3\ln\left|\sqrt[3]{x}+1\right|+C\\
&= x-\frac{3}{2}\sqrt[3]{x}^2+3\sqrt[3]{x}-3\ln\left|\sqrt[3]{x}+1\right|+C
\end{aligned}$$

Exercise 1.1.5. Missing Constants ☕

In the above example, all of the constant terms disappeared on the final step! Was that ok?

Exercise 1.1.6. Practice with u-sub ☕☕

- Evaluate $\int \frac{6x+3}{x^2+x+8}\,\mathtt{d}x$.

- Evaluate $\int \frac{(\ln(x))^2}{x}\,\mathtt{d}x$.

- Evaluate $\int xe^{-x^2}\,\mathtt{d}x$.

- Consider the integral
$$\int e^{\left(x^2\right)}\,\mathtt{d}x$$
Explain in words why the substitution $u = x^2$ will not work in this case. Where do you get

stuck?

Exercise 1.1.7. Create an Integral! ☕☕☕

Come up with your own integral that can be evaluated by u-sub. Find a partner and trade! See if you can evaluate each other's integrals with u-sub, or explain to your partner why theirs cannot be evaluated using u-sub.

Antiderivatives of the Six Trig Functions

In Calculus I, we found the derivatives of all six trig functions. List those below.

Exercise 1.1.8. Recalling the Derivatives of the Six Trig Functions ☕

Write the derivative of each of the following trig functions:

- $\frac{\mathtt{d}}{\mathtt{d}x}(\sin(x)) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}(\cos(x)) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}(\tan(x)) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}(\cot(x)) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}(\sec(x)) =$
- $\frac{\mathtt{d}}{\mathtt{d}x}(\csc(x)) =$

From these, we easily obtain the antiderivatives of sine and cosine.

Exercise 1.1.9. Integrals of Sine and Cosine ☕

Use the derivatives above to compute the following antiderivatives.

- $\int \sin(x)\,\mathtt{d}x =$
- $\int \cos(x)\,\mathtt{d}x =$

For tangent and cotangent, we need u-sub.

Example 1.1.10. Antiderivative of Tangent

We compute the antiderivative of tangent by rewriting as $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and then using the substitution $u = \cos(x)$. Differentiating both sides produces $\mathtt{d}x = \frac{\mathtt{d}u}{-\sin(x)}$. We now apply these substitutions:

$$\begin{aligned}
\int \tan(x)\,\mathtt{d}x &= \int \frac{\sin(x)}{\cos(x)}\,\mathtt{d}x \\
&= \int \frac{\sin(x)}{u}\frac{\mathtt{d}u}{-\sin(x)} \\
&= -\int \frac{1}{u}\,\mathtt{d}u \\
&= -\ln|u| + C \\
&= -\ln|\cos(x)| + C
\end{aligned}$$

The method used to antidifferentiate tangent can be adapted to also antidifferntiate cotangent.

Exercise 1.1.11. Integral of Cotangent ☕☕

Find the antiderivative of cotangent.

$$\int \cot(x)\,\mathtt{d}x =$$

The antiderivative of secant is much trickier! The process is not intuitive and requires a rabbit out of a hat.

Example 1.1.12. Integral of Secant

Since multiplication by 1 does not change the integrand, we are free to multiply by 1 whenever it is helpful. Here, it turns out to be helpful to multiply by $\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$. This is the rabbit.

$$\begin{aligned}
\int \sec(x)\,\mathtt{d}x &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,\mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,\mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{u}\,\frac{1}{\sec(x)\tan(x)+\sec^2(x)}\,\mathtt{d}u \\
&= \int \frac{1}{u}\,\mathtt{d}u \\
&= \ln|u| + C \\
&= \ln|\sec(x)+\tan(x)| + C
\end{aligned}$$

The above method can be adapted to antidifferentiate cosecant.

Exercise 1.1.13. Integral of Cosecant ☕☕

Find the antiderivative of cosecant.

$$\int \csc(x)\,\mathtt{d}x =$$

1.2 Integration by Parts

Integration by parts (IBP) is the Product Rule spun around backwards to become a rule for antiderivatives rather than derivatives.

Exercise 1.2.1. Reversing the Product Rule ☕☕

Fill in the blanks in the following construction of integration by parts:

- Recall the Product Rule for derivatives.

$$(f(x)g(x))' =$$

- Take an antiderivative of both sides.

$$= \int (f'(x)g(x))\,\mathtt{d}x + \int (f(x)g'(x))\,\mathtt{d}x$$

- Rewrite the equation by subtracting the term $\int (f'(x)g(x))\,\mathtt{d}x$ from both sides.

$$=$$

- To condense the notation, it is customary to make the substitutions $u = f(x)$ and $v = g(x)$. Thus, we say $\frac{\mathtt{d}u}{\mathtt{d}x} = f'(x)$ and similarly $\frac{\mathtt{d}v}{\mathtt{d}x} = g'(x)$. Multiply the $\mathtt{d}x$ to the right-hand side in both of those equations, we obtain

$$\mathtt{d}u =$$

and

$$\mathtt{d}v =$$

- Use these substitutions to replace all instances of x, f, and g by u and v and conclude the IBP formula.

Just for sake of having it in its own box, here it is again!

Integration by Parts Formula

$$\int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u$$

We typically use this to integrate a product of functions in the case that u-substitution does not work. You can identify one factor of your integrand as u, the remaining factor as $\mathtt{d}v$, and plug into the IBP formula. There are three main types:

1. A product with one factor that becomes much simpler upon differentiation
2. A not-quite-a product that we turn into a product
3. An integrand that reappears after applying IBP

We illustrate each of these methods with an example.

A Product with One Factor That Becomes Much Simpler Upon Differentiation

We let u be whichever factor becomes simpler when it is differentiated. The other factor by default must then be set equal to $\mathtt{d}v$.

Example 1.2.2. Integrating a Product

Suppose we wish to find an antiderivative for the function $x \cdot \cos(x)$. We can either choose $u = x$ or $u = \cos(x)$. Since $u = x$ has lovely little constant function 1 as its derivative, whereas $u = \cos(x)$ would produce just another trig function as its derivative, we conclude $u = x$ is the better choice.

Choice of u and $\mathtt{d}v$	
$u = x$	$v = \sin(x)$
$\mathtt{d}u = \mathtt{d}x$	$\mathtt{d}v = \cos(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int x \cdot \cos(x)\,\mathtt{d}x = \int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u = x \cdot \sin(x) - \int \sin(x)\,\mathtt{d}x = x \cdot \sin(x) + \cos(x) + C$$

Exercise 1.2.3. Checking Our Work ☕

Take the derivative of our result, $x\sin(x) + \cos(x) + C$, to verify that it is in fact the correct antiderivative!

Exercise 1.2.4. An Integral via both u-sub and IBP ☕☕☕

Consider the integral

$$\int x\sqrt{x+1}\,\mathtt{d}x$$

- Evaluate the integral using the u-sub $u = x + 1$.

- Evaluate the integral using IBP, choosing $u = x$ and $\mathtt{d}v = \sqrt{x+1}\,\mathtt{d}x$.

- Your answers will appear very different! Is one incorrect? Or are they compatible?

A Not-Quite-a Product That We Turn into a Product

Often, an integrand that does not appear to be a product can be rewritten as product in a helpful way. This often includes **rewriting the integrand as the integrand times one**. We let u be the entire integrand, leaving $\mathtt{d}v$ to just be the invisible 1 times $\mathtt{d}x$.

Example 1.2.5. Multiplying by 1 in an IBP

Suppose we wish to find an antiderivative for the function $\arccos(x)$. We identify $u = \arccos(x)$ which leaves $\mathtt{d}v = 1 \cdot \mathtt{d}x$. Thus we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = \arccos(x)$	$v = x$
$\mathtt{d}u = -\frac{1}{\sqrt{1-x^2}}\,\mathtt{d}x$	$\mathtt{d}v = 1 \cdot \mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int \arccos(x) \cdot 1 \cdot \mathtt{d}x = \int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u = x \cdot \arccos(x) - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right)\mathtt{d}x$$

$$= x\arccos(x) + \int \frac{x}{\sqrt{1-x^2}}\mathtt{dx} = x\arccos(x) - \sqrt{1-x^2} + C$$

Exercise 1.2.6. Filling in the Details ☕☕

Notice that the very last step of the above example was in fact a u-substitution! Show the details of how that antiderivative was carried out.

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathtt{d}x =$$

Exercise 1.2.7. The Antiderivative of the Natural Logarithm ☕☕

Apply the same technique to find an antiderivative for the function $\ln(x)$.

An Integrand that Reappears After Applying IBP

Sometimes, we can get the original expression to come back after applying integration by parts one or more times. Once this occurs, you can **give some name to the integral (we will use I)** and solve for it as you would solve any equation in algebra!

Example 1.2.8. An Integrand that Reappears After IBP

Suppose we wish to find an antiderivative for the function $e^{2x}\cos(x)$. Call I the desired antiderivative. That is:

$$I = \int e^{2x}\cos(x)\,\mathtt{d}x$$

We now wish to apply IBP, so we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = e^{2x}$	$v = \sin(x)$
$\mathtt{d}u = 2e^{2x}\,\mathtt{d}x$	$\mathtt{d}v = \cos(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\begin{aligned} I &= \int u\,\mathtt{d}v \\ &= uv - \int v\,\mathtt{d}u \\ &= e^{2x}\sin(x) - \int \sin(x)\cdot 2e^{2x}\,\mathtt{d}x \\ &= e^{2x}\sin(x) - 2\int e^{2x}\sin(x)\,\mathtt{d}x \end{aligned}$$

We notice now that the new integral is again a product of functions (and does not appear to be doable via u-sub) so we apply IBP once again with the following declarations (using new u and v):

Choice of u and $\mathtt{d}v$	
$u = e^{2x}$	$v = -\cos(x)$
$\mathtt{d}u = 2e^{2x}\,\mathtt{d}x$	$\mathtt{d}v = \sin(x)\,\mathtt{d}x$

We now proceed with the previous expression, using the new IBP setup and notice that the original integral I reappears:

$$\begin{aligned}
I &= e^{2x}\sin(x) - 2\int u\,\mathtt{d}v \\
&= e^{2x}\sin(x) - 2\left(uv - \int v\,\mathtt{d}u\right) && (\bowtie) \\
&= e^{2x}\sin(x) - 2\left(e^{2x}(-\cos(x)) - \int(-\cos(x))2e^{2x}\,\mathtt{d}x\right) && (\bowtie) \\
&= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,\mathtt{d}x \\
&= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4I
\end{aligned}$$

At first glimpse this seems troubling; we have reduced the problem we are trying to solve to solving the exact same problem that we are trying to solve! Yet upon further inspection, it becomes clear that this is in fact an equation involving I, and thus we can solve for it! Proceeding:

$$\begin{aligned}
I &= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4I \\
5I &= e^{2x}\sin(x) + 2e^{2x}\cos(x) \\
I &= \frac{e^{2x}\sin(x) + 2e^{2x}\cos(x)}{5}
\end{aligned}$$

and we are done, concluding that

$$\int e^{2x}\cos(x)\,\mathtt{d}x = \frac{e^{2x}\sin(x) + 2e^{2x}\cos(x)}{5} + C$$

Exercise 1.2.9. Carefulness ☕

In the example above, there are two lines labeled with bowties ($\bowtie$). Explain briefly in a sentence or two why those giant parentheses are present. What would go wrong if those parentheses were

not there?

Example 1.2.10. Another Reappearing IBP Integral

Suppose we wish to find an antiderivative for the function $\tan(x)\sec^2(x)$. We identify $\mathtt{d}v = \sec^2(x)\,\mathtt{d}x$ as having a nice clean antiderivative, which leaves $u = \tan(x)$ by default. Thus we make the following declarations:

Choice of u and $\mathtt{d}v$	
$u = \tan(x)$	$v = \tan(x)$
$\mathtt{d}u = \sec^2(x)\,\mathtt{d}x$	$\mathtt{d}v = \sec^2(x)\,\mathtt{d}x$

We are now ready to calculate the antiderivative via IBP:

$$\int \tan(x)\sec^2(x)\,\mathtt{d}x = \tan(x)\tan(x) - \int \tan(x)\sec^2(x)\,\mathtt{d}x$$

We notice that the original integral has reappeared! We give it the name I and solve. The equation becomes $I = \tan^2(x) - I$, which implies that $2I = \tan^2(x)$. Dividing by two produces the following result:

$$\int \tan(x)\sec^2(x)\,\mathtt{d}x = \frac{1}{2}\tan^2(x) + C$$

Exercise 1.2.11. Alternate Solutions ☕☕

Find the antiderivative of $\tan(x)\sec^2(x)$ yet again but by two different methods! In particular, try...

- ...a u-sub with $u = \tan(x)$.

- ...an IBP with $u = \sec(x)$ and $\mathtt{d}v = \tan(x)\sec(x)\,\mathtt{d}x$.

Confirm that your answers match the result of Exercise 1.2.10.

Exercise 1.2.12. A Tricky but Important One: Secant Cubed ☕☕☕

Find an antiderivative for the function $\sec^3(x)$. (**Hint:** Split the cube as $\sec^3(x) = \sec^2(x)\sec(x)$. Also, the Pythagorean Identity $\tan^2(x) = \sec^2(x) - 1$ will be useful.)

Mixed Practice with Substitution/IBP

Sometimes it is not obvious which technique to use in solving a particular problem. One must often use more than one technique of integration in combination.

Exercise 1.2.13. Practice on u-sub and/or IBP ☕☕☕

- Find an antiderivative for the function $\cos(\sqrt{x})$.

- Evaluate $\int e^{\sqrt{2x}}\,\mathtt{d}x$.

- Evaluate $\int \arcsin(5x)\,\mathtt{d}x$.

- Evaluate $\int e^{2x} \sin(2x)\, \mathtt{d}x$.

Exercise 1.2.14. Who is u vs Who is $\mathtt{d}v$? ☕☕

Suppose we wish to find an antiderivative for the function $x^{2.5} \ln(x)$. There are two natural choices for u. We can let $u = x^{2.5}$ and $\mathtt{d}v = \ln(x)\, \mathtt{d}x$, or we can let can let $u = \ln(x)$ and $\mathtt{d}v = x^{2.5}\, \mathtt{d}x$.

- Apply just the first step of IBP with $u = x^{2.5}$ and $\mathtt{d}v = \ln(x)\, \mathtt{d}x$.

$$\int x^{2.5} \ln(x)\, \mathtt{d}x =$$

- Apply just the first step of IBP with $u = \ln(x)$ and $\mathtt{d}v = x^{2.5}\, \mathtt{d}x$.

$$\int x^{2.5} \ln(x)\, \mathtt{d}x =$$

- Write a short explanation regarding which choice of u will be easier to use to evaluate the integral and why.

- Carry out the integral using whichever choice you decided was easier.

$$\int x^{2.5} \ln(x)\, \mathtt{d}x =$$

- Differentiate your answer to check that your antiderivative is correct.

1.3 Integrating Products of Powers of Sine and Cosine

In this section, we give an algorithm to find an antiderivative of the form

$$\int \sin^n(x)\cos^m(x)\,\mathtt{d}x$$

for $n, m \in \mathbb{N}$.

Exercise 1.3.1. Knowledge is Power ☕

There are two exponents in the integrand above.

- What symbol above is the exponent of sine?
- What symbol above is the exponent of cosine?

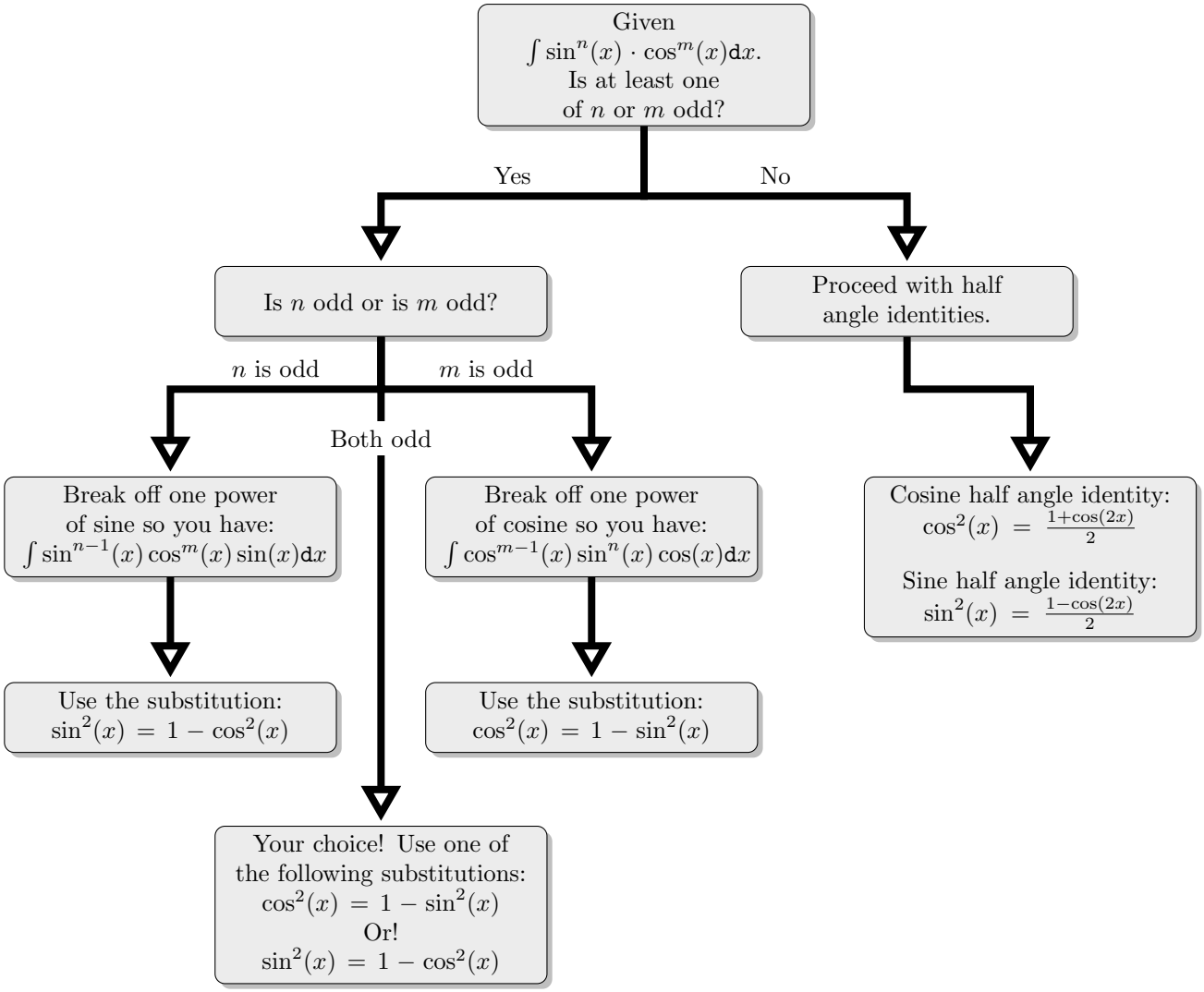
Note that some sine-cosine integrals can be done by techniques you have already learned. For example, n or m is equal to 1, ordinary u-substitution will work just fine!

Exercise 1.3.2. u-sub with Sines and Cosines ☕☕

Evaluate the following integral using the substitution $u = \sin(x)$:

$$\int \sin^2(x)\cos(x)\,\mathtt{d}x$$

There are two types of integrals containing powers of sine and cosine. The first type is the case where we have at least one odd exponent; the second type is where both exponents are even. We show an overview of how to handle each case in the following awesome flow chart:



At Least One Odd Power

Recall the Pythagorean identity for sine and cosine (written in two useful forms here):

Pythagorean Theorem Slightly Rewritten	
$\cos^2(x) = 1 - \sin^2(x)$	$\sin^2(x) = 1 - \cos^2(x)$

If at least one exponent is odd, we pull one of those functions out for the "$\mathtt{d}u$" and perform u-sub. We then use the Pythagorean trig identity to rewrite sine and cosine in terms of each other as needed.

Example 1.3.3. Odd Power Case

Here we compute the integral

$$\int \sin^7(x)\cos^2(x)\,\mathtt{d}x.$$

In this case, we proceed using the substitution $u = \cos(x)$, so $\mathtt{d}x = \frac{1}{-\sin(x)}\,\mathtt{d}u$.

$$\begin{aligned}
\int \sin^7(x)\cos^2(x)\,\mathtt{d}x &= \int \sin^6(x)\cos^2(x)\sin(x)\,\mathtt{d}x \\
&= \int \left(\sin^2(x)\right)^3 \cos^2(x)\sin(x)\frac{1}{-\sin(x)}\,\mathtt{d}u \\
&= \int \left(1-\cos^2(x)\right)^3 \cos^2(x)(-1)\,\mathtt{d}u \\
&= -\int \left(1-u^2\right)^3 u^2\,\mathtt{d}u \\
&= -\int \left(1-3u^2+3u^4-u^6\right) u^2\,\mathtt{d}u \\
&= -\int \left(u^2-3u^4+3u^6-u^8\right)\mathtt{d}u \\
&= -\left(\frac{1}{3}u^3-\frac{3}{5}u^5+\frac{3}{7}u^7-\frac{1}{9}u^9\right)+C \\
&= -\frac{1}{3}\cos^3(x)+\frac{3}{5}\cos^5(x)-\frac{3}{7}\cos^7(x)+\frac{1}{9}\cos^9(x)+C
\end{aligned}$$

Exercise 1.3.4. Why Odd Mattered ☕

In Example 1.3.3, the exponent of sine (in this case, the number 7) being odd really mattered. If that 7 were replaced by an even number instead, why would this approach have failed? Answer in a few short sentences below.

Exercise 1.3.5. Try a Few with Odd Exponents ☕☕

- Find an antiderivative for the function $\sin^5(x)\cos^2(x)$.

- Evaluate $\int \cos^9(x)\,\mathtt{d}x$. (**Hint:** Pascal's Triangle will be extremely helpful!)

Exercise 1.3.6. Two Different Options ☕☕

- Consider $\int \cos(x)\sin^3(x)\,\mathtt{d}x$.

- Compute this integral using $u = \cos(x)$.

- Compute this integral using $u = \sin(x)$.

- Your two answers will appear very different! Show that they are in fact compatible.

- Consider $\int \cos^3(x) \sin^{11}(x) \, \mathtt{d}x$.
  - Can you compute this integral using $u = \cos(x)$? Explain.

  - Can you compute this integral using $u = \sin(x)$? Explain.

– Which of the two above substitutions will be easier to use? Carry out the integration, using the easier of the two.

Both Even Powers

Recall the Half-Angle Identities!

Half-Angle Identities	
$\cos^2(x) = \frac{1+\cos(2x)}{2}$	$\sin^2(x) = \frac{1-\cos(2x)}{2}$

If the powers of sine and cosine are both even, we use the half-angle identities for both sine and cosine. This can get quite messy, but it works!

Exercise 1.3.7. Just Cosines without Sine ☕

Consider the following integral:

$$\int \cos^6(x)\,\mathtt{d}x$$

Here the exponent on cosine is the even number 6. What is the exponent of sine in that integrand? Is that an even number?

Example 1.3.8. Carrying Out Antidifferentiation with the Half-Angle Identities

We now show how the half-angle identities help antidifferentiate the sixth power of cosine.

$$\begin{aligned}
\int \cos^6(x)\,\mathtt{d}x &= \int \left(\cos^2(x)\right)^3 \mathtt{d}x \\
&= \int \left(\frac{1+\cos(2x)}{2}\right)^3 \mathtt{d}x \\
&= \frac{1}{8}\int 1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\,\mathtt{d}x \\
&= \frac{1}{8}\left(\int 1\,\mathtt{d}x + \int 3\cos(2x)\,\mathtt{d}x + \int 3\cos^2(2x)\,\mathtt{d}x + \int \cos^3(2x)\,\mathtt{d}x\right)
\end{aligned}$$

Notice that we now have four integrals. The first is easy, the second is a u-substitution, and the third is another even power of cosine (where we again use the half-angle identity). Finally, the fourth is an odd power of cosine, so we can use the technique from the previous section.

Exercise 1.3.9. Finishing the Example

Carry out each of these processes to compute the four integrals:

- $\int 1\,\mathtt{d}x$
- $\int 3\cos(2x)\,\mathtt{d}x$
- $\int 3\cos^2(2x)\,\mathtt{d}x$
- $\int \cos^3(2x)\,\mathtt{d}x$

Add your antiderivatives together and combine like terms to produce your final answer for the integral! Oh and remember that one-eighth.

$$\int \cos^6(x)\,\mathtt{d}x =$$

Exercise 1.3.10. Checking the Previous Example ☕☕☕

Differentiate your answer and verify you get the original integrand back.

Exercise 1.3.11. Practice with the Even Case ☕☕

- Find an antiderivative for the function $\sin^2(3x)$.

- Find an antiderivative for the function $\sin^4(x)$.

- Find an antiderivative for the function $\sin^2(x)\cos^2(x)$.

1.4 Trigonometric Substitution

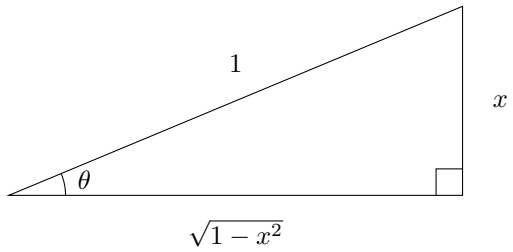
There are many other integrals that do not yield to the techniques that we have already covered, but at least some integrals can be turned into trigonometric integrals through clever substitution. Consider the following integral.

Example 1.4.1. Why Bother?

Suppose we wish to evaluate

$$\int \frac{1}{1-x^2}\,\mathtt{d}x.$$

We can't get away with a clean u-substitution, and IBP doesn't help at all either. So we need to try a different kind of substitution. In particular, if we build a right triangle with a hypotenuse of length 1 and an opposite side of length x, the remaining side would have length $\sqrt{1-x^2}$ by the Pythagorean Theorem.



This suggests two useful substitutions:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{1} \text{ and } \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1}.$$

This gives us that

$$x = \sin(\theta) \text{ and } 1 - x^2 = \cos^2(\theta).$$

We can differentiate both sides of $x = \sin(\theta)$ to get that $\mathtt{d}x = \cos(\theta)\,\mathtt{d}\theta$. Then we can substitute:

$$\begin{aligned}\int \frac{1}{1-x^2}\,\mathtt{d}x &= \int \frac{1}{\cos^2(\theta)}\cos(\theta)\,\mathtt{d}\theta \\ &= \int \frac{1}{\cos(\theta)}\,\mathtt{d}\theta \\ &= \int \sec(\theta)\,\mathtt{d}\theta \\ &= \ln|\sec(\theta)+\tan(\theta)| + C.\end{aligned}$$

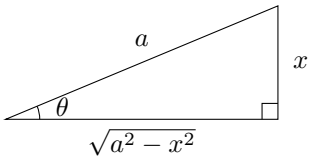
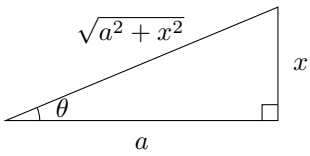
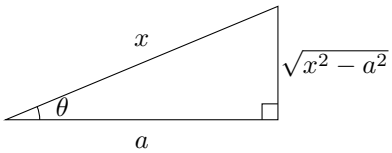
Of course, we need to undo our substitution. When we refer back to our triangle from before, we note that $\sec(\theta) = \frac{1}{\sqrt{1-x^2}}$ and $\tan\theta = \frac{x}{1-x^2}$. So we get the result:

$$\int \frac{1}{1-x^2}\,\mathtt{d}x = \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C.$$

Though in theory, you could use any trigonometric function, the three commonly used trigonometric

substitutions are sine, tangent, and secant. These substitutions arise from building triangles for three different binomial structures, $a^2 - x^2$, $a^2 + x^2$, and $x^2 - a^2$ respectively. These three substitutions are shown in the table below.

Trigonometric Substitutions

If you see...	...use the triangle...	... where
$a^2 - x^2$		$x = a \sin\theta$
$a^2 + x^2$		$x = a \tan\theta$
$x^2 - a^2$		$x = a \sec\theta$

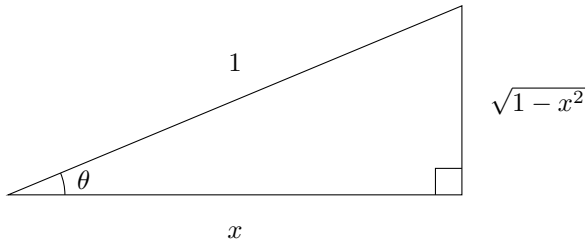
Exercise 1.4.2. Why Only Three Cases?

In the table above, we have cases for how to clean up expressions of the form $a^2 - x^2$, $a^2 + x^2$, and $x^2 - a^2$. Why is there not a fourth case for $x^2 + a^2$?

Exercise 1.4.3. What About the Cofunctions?

When we built the example at the beginning of this section, we used a triangle that specifically yielded the substitution $x = \sin(\theta)$.

- Instead, evaluate
$$\int \frac{1}{1 - x^2}\,\mathtt{d}x$$
using the triangle below that gives the substitution $x = \cos(\theta)$.



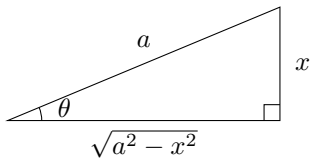
- Which substitution do you prefer? Why?
- Can you think of a reason to try to avoid the cofunction substitutions, $x = \cos(\theta)$, $x = \cot(\theta)$ and $x = \csc(\theta)$?

Sine Substitution

When we see an expression of the form $a^2 - x^2$ in the integrand, we think of the identity $1 - \sin^2(\theta) = \cos^2(\theta)$. This motivates the following substitution:

Sine Substitution

$$x = a \cdot \sin(\theta)$$



The next example will require use of the Double-Angle Identities for sine and cosine. We recall these before we dive in!

Exercise 1.4.4. Recalling the Double-Angle Formulas ☕

- The double-angle formula for sine is $\sin(2\theta) =$
- The double-angle formula for cosine is $\cos(2\theta) =$
- What do you get if you apply the sine double-angle identity to $\sin(4\theta)$? Specifically, think of $\sin(4\theta)$ as $\sin(2 \cdot 2\theta)$.

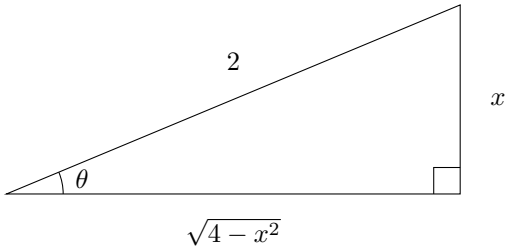
We now put our sine substitution to use to evaluate an antiderivative!

Example 1.4.5. Using a Sine Substitution

Suppose we wish to evaluate

$$\int \left(4 - x^2\right)^{3/2} \mathtt{d}x.$$

Let's see if we can substitute! In particular, notice the binomial $4 - x^2$. This difference of squares suggests that we can build the following triangle. Note that if we allow the hypotenuse of the triangle to have a length of 2 and the opposite leg to have a length of x, then the remaining leg of the triangle must be $\sqrt{4 - x^2}$ by the Pythagorean Theorem!



There are two critical substitutions suggested here:

$$\sin\theta = \frac{x}{2} \text{ and } \cos\theta = \frac{\sqrt{4 - x^2}}{2}.$$

We can multiply both sides by 2 to get

$$2\sin\theta = x \text{ and } 2\cos\theta = (4 - x^2)^{\frac{1}{2}}.$$

We then differentiate both sides of the former to find the conversion between the differentials and then multiply both sides by $\mathtt{d}\theta$:

$$\mathtt{d}x = 2 \cdot \cos(\theta)\, \mathtt{d}\theta$$

We now use the above equations to substitute for $(4 - x^2)^{\frac{1}{2}}$ and $\mathtt{d}x$ in the integral:

$$\begin{aligned}
\int \left(4 - x^2\right)^{3/2} \mathtt{d}x &= \int (2\cos(\theta))^3 \cdot 2\cos(\theta)\mathtt{d}\theta \\
&= 2\int 8\cos^3(\theta) \cdot \cos(\theta)\mathtt{d}\theta \\
&= 16\int \cos^4(\theta)\mathtt{d}\theta
\end{aligned}$$

Recall the previous section where we learned how to antidifferentiate even powers of sine and cosine! Using those techniques, we can find that

$$\int \cos^4(\theta)\ \mathtt{d}\theta = 6\theta + 4\sin(2\theta) + \frac{1}{2}\sin(4\theta) + C.$$

It still remains to unwind the trigonometric substitution back in terms of x rather than θ. Our original substitution argument is θ, whereas currently we have 2θ and 4θ as arguments. In order

to resolve this, we use the sine and cosine double angle formulas and the Pythagorean identity. Proceeding:

$$\begin{aligned}
\int \left(4-x^2\right)^{3/2} \mathtt{d}x &= 6\theta + 4\cdot\sin(2\theta) + \frac{1}{2}\sin(4\theta) + C \\
&= 6\theta + 4\cdot 2\cdot\sin(\theta)\cos(\theta) + \sin(2\theta)\cos(2\theta) + C \\
&= 6\theta + 4\cdot 2\cdot\sin(\theta)\cos(\theta) + 2\cdot\sin(\theta)\cos(\theta)\left(\cos^2(\theta)-\sin^2(\theta)\right) + C \\
&= 6\cdot\arcsin\left(\frac{x}{2}\right) + 4\cdot 2\frac{x}{2}\frac{\sqrt{4-x^2}}{2} + 2\frac{x}{2}\frac{\sqrt{4-x^2}}{2}\left(\frac{4-x^2}{4}-\frac{x^2}{4}\right) + C \\
&= 6\cdot\arcsin\left(\frac{x}{2}\right) + 4\cdot 2\frac{x}{2}\frac{\sqrt{4-x^2}}{2} + \frac{x\sqrt{4-x^2}}{2}\left(1-\frac{x^2}{2}\right) + C \\
&= 6\arcsin\left(\frac{x}{2}\right) + 2x\sqrt{4-x^2} + \frac{x\sqrt{4-x^2}}{2} - \frac{x^3\sqrt{4-x^2}}{4} + C
\end{aligned}$$

Exercise 1.4.6. Checking Our Work ☕☕☕

Use techniques from the previous section to verify that

$$\int \cos^4(\theta)\ \mathtt{d}\theta = 6\theta + 4\sin(2\theta) + \frac{1}{2}\sin(4\theta) + C.$$

Exercise 1.4.7. Try One on your own! ☕☕☕

Evaluate the following antiderivative:

$$\int x^3 \left(16 - x^2\right)^{5/2} \, \mathtt{d}x$$

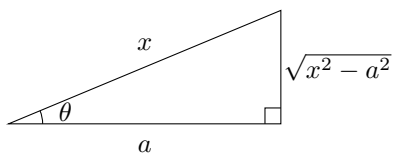
(**Hint:** Recall our methods for integrating powers of sines and cosines!)

Secant Substitution

When we see an expression of the form $x^2 - a^2$ in the integrand, we think of the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$, so we use the following substitution:

Secant Substitution

$$x = a \cdot \sec(\theta)$$

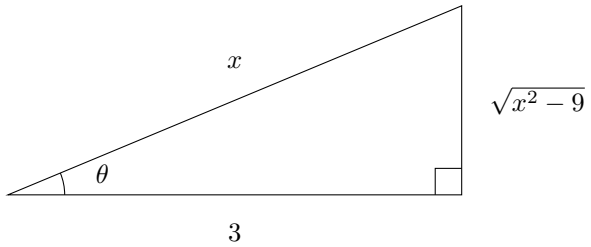


Example 1.4.8. A Secant Substitution

Suppose we wish to evaluate the following integral:

$$\int \frac{1}{x^4 - 9x^2} \, \mathtt{d}x$$

Since $x^4 - 9x^2 = x^2\left(x^2 - 9\right)$, we let $a = 3$ and use the following triangle:



$$\begin{aligned} x &= 3\sec(\theta) \\ \mathtt{d}x &= 3\sec(\theta)\tan(\theta)\,\mathtt{d}\theta \\ \sqrt{x^2-9} &= 3\tan(\theta) \\ x^2-9 &= 9\tan^2(\theta) \end{aligned}$$

We now apply these substitutions to rewrite the integral in terms of θ.

$$\begin{aligned} \int \frac{1}{x^4-9x^2}\,\mathtt{d}x &= \int \frac{1}{x^2\left(x^2-9\right)}\,\mathtt{d}x \\ &= \int \frac{3\sec(\theta)\tan(\theta)}{9\sec^2(\theta)\left(9\tan^2(\theta)\right)}\,\mathtt{d}\theta \\ &= \int \frac{3\sec(\theta)\tan(\theta)}{81\sec^2(\theta)\tan^2(\theta)}\,\mathtt{d}\theta \\ &= \frac{1}{27}\int \frac{1}{\sec(\theta)\tan(\theta)}\,\mathtt{d}\theta \\ &= \frac{1}{27}\int \frac{\cos^2(\theta)}{\sin(\theta)}\,\mathtt{d}\theta \\ &= \frac{1}{27}\int \frac{1-\sin^2(\theta)}{\sin(\theta)}\,\mathtt{d}\theta \\ &= \frac{1}{27}\int \frac{1}{\sin(\theta)}\,\mathtt{d}\theta - \frac{1}{27}\int \frac{\sin^2(\theta)}{\sin(\theta)}\,\mathtt{d}\theta \\ &= \frac{1}{27}\int \csc(\theta)\,\mathtt{d}\theta - \frac{1}{27}\int \sin(\theta)\,\mathtt{d}\theta \\ &= -\frac{1}{27}\ln\left|\csc(\theta)+\cot(\theta)\right| + \frac{1}{27}\cos(\theta) + C \end{aligned}$$

Here we have successfully taken the antiderivative, and now need to just get back to x from θ. We can use the triangle from the substitutions earlier. This enables us to compute the other trig functions using this triangle.

Exercise 1.4.9. Getting from θ back to x

Complete the above example by using the triangle to find the values of the other trig functions.

$$\cos(\theta) =$$
$$\cot(\theta) =$$
$$\csc(\theta) =$$

Then plug these expressions back into our antiderivative to get a final answer in terms of x rather than θ. Make these substitutions and then simplify to verify the final answer shown below. Show your work below.

$$\begin{aligned}\int \frac{1}{x^4 - 9x^2}\,\mathtt{d}x &= -\frac{1}{27}\ln\left|\csc(\theta) + \cot(\theta)\right| + \frac{1}{27}\cos(\theta) + C \\ &= \\ &= \\ &= \frac{1}{9x} - \frac{1}{27}\ln\left|\frac{x+3}{\sqrt{x^2-9}}\right| + C\end{aligned}$$

Exercise 1.4.10. Yes You Can! Take the Cant Out of Secant! ☕☕☕

Evaluate the following antiderivative:

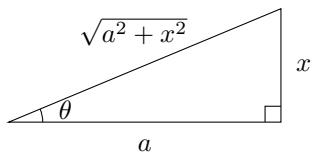
$$\int \sqrt{x^2 - 4}\,\mathtt{d}x.$$

Tangent Substitution

When we see an expression of the form $a^2 + x^2$ or $x^2 + a^2$ (which are the same) in the integrand, we think of the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$, so we use the following substitution:

Tangent Substitution

$$x = a \cdot \tan(\theta)$$



Exercise 1.4.11. Revisiting an Old Friend

- Recall the derivative of arctangent:

$$\frac{\mathtt{d}}{\mathtt{d}x}(\arctan(x)) =$$

- We should be able to reverse the above by taking the antiderivative of the right-hand side. Perform this antiderivative using the substitution $x = \tan(\theta)$:

$$\int \frac{1}{1 + x^2}\,\mathtt{d}x =$$

Preprocessing with Algebra or u-sub

Often we need to do a little algebra and/or u-sub to get the integrand into a form where we can then perform trig sub.

Exercise 1.4.12. A Bit of Algebra to Help Us ☕☕☕

- Explain why the two following expressions are equal:

$$(4x^2+1)^2 = \left((2x)^2+1^2\right)^2$$

- Use the equality above to develop a tangent substitution to evaluate the following antiderivative:

$$\int \frac{1}{(4x^2+1)^2}\,\mathtt{d}x$$

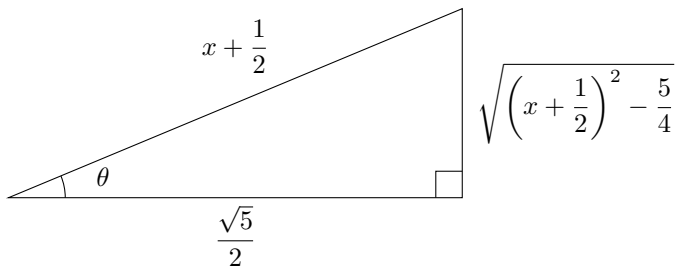
A trick from algebra that is often used with trigonometric substitution is completing the square. You might need to complete the square to get it into a form where a trig sub will work.

Example 1.4.13. Completing the Square

Suppose we wish to find an antiderivative for the function $\left(x^2+x-1\right)^{-2}$. We begin by completing the square on the quadratic polynomial:

$$\begin{aligned} x^2+x-1 &= x^2+x+\frac{1}{4}-\frac{1}{4}-1 \\ &= \left(x+\frac{1}{2}\right)^2-\frac{5}{4} \end{aligned}$$

We now use the substitution that this quadratic motivates. Namely, we pick $a=\frac{\sqrt{5}}{2}$ since we want its square to be five-fourths. Where x used to go in the problems above, we now have an $x+\frac{1}{2}$. Thus our substitution is found in the following triangle.



Exercise 1.4.14. Completing the Example ☕☕☕

Use the substitutions suggested in the example above to find the antiderivative.

$$\int \frac{1}{(x^2+x-1)^2}\,\mathtt{d}x =$$

Exercise 1.4.15. Try One On Your Own ☕☕☕

Evaluate the following antiderivative:

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,\mathtt{d}x$$

1.5 Partial Fraction Decomposition

In this section, we will combine the techniques of all previous sections and learn how to antidifferentiate rational functions!

Exercise 1.5.1. What is a Rational Function Again? ☕

What is the definition of a *rational function*?

A partial fraction decomposition (PFD) is a way to decompose a rational function (a polynomial divided by a polynomial) as a sum of simpler rational functions. This is purely an algebraic trick that fundamentally does not involve calculus. It is useful in many contexts! Here we apply it to (of course) finding antiderivatives. Typically, a given rational function is too challenging to antidifferentiate as is. Once we break it up into smaller pieces via PFD, it becomes manageable.

The fundamental idea is simple. If we have a fraction that has more than one factor in the denominator, we can rewrite it as a sum of fractions whose denominators have the original denominator as their least common multiple.

Exercise 1.5.2. Trying This with Integers Before We Go to Polynomials ☕

Consider the fraction $\frac{1}{6}$. We notice that the denominator, six, is equal to two times three. Thus, we attempt to write one-sixth as a sum of fractions whose denominators are two and three.
Find integers A and B such that:

$$\frac{1}{6} = \frac{A}{2} + \frac{B}{3}$$

Check your answer by adding the fractions on the right hand side back together to verify you get one-sixth.

Warming Up with a Small Example

Partial fraction decomposition is the same idea, except we are working with polynomials rather than just integers.

Example 1.5.3. Our First Decomposition!

To decompose the fraction $\frac{1}{x^2-1}$, we first factor the denominator into $x^2 - 1 = (x-1)(x+1)$. Thus, we look for an expression of the form

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

for some numbers A and B. To find such A and B, we multiply both sides by $x^2 - 1$ to produce

the polynomial equation

$$1 = A(x+1) + B(x-1)$$

Since we want the expressions to be equal for all values of x, we pick convenient values of x to plug in to solve for A and B.

- Set $x = 1$:

$$1 = A \cdot (2) + B \cdot (0) \implies A = \frac{1}{2}$$

- Set $x = -1$:

$$1 = A \cdot (0) + B \cdot (-2) \implies B = -\frac{1}{2}$$

At last, we have obtained the partial fraction decomposition!

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

Exercise 1.5.4. Checking Our Work ☕

Take the right-hand side of the above equation and add the two fractions together by finding a common denominator. Verify that their sum is the original rational function $\frac{1}{x^2-1}$.

Example 1.5.5. Finding the Same PFD by Expanding and Equating Coefficients

We repeat the above example but demonstrate an alternate way to find our coefficients. Recall the equation

$$1 = A(x+1) + B(x-1)$$

In the previous example, we proceeded by plugging in numerical values for x. Instead, we could fully multiply out the polynomials and combine like terms. This produces

$$1 = (A+B)x + (A-B)$$

We can pad the left-hand side with a degree one term with coefficient zero to put both sides in the form "number times x plus number".

$$0x + 1 = (A+B)x + (A-B)$$

Now we can construct a system of two equations in two unknowns by equating one coefficient at a time. Specifically, we build it as:

Degree zero coefficient of LHS = Degree zero coefficient of RHS	$\Longrightarrow$	$1 = A - B$
Degree one coefficient of LHS = Degree one coefficient of RHS	$\Longrightarrow$	$0 = A + B$

The resulting linear system in two equations and two unknowns can then be solved via any applicable method (substitution, elimination, matrices, etc).

Exercise 1.5.6. Solve the System ☕

Solve the linear system of two equations and two unknowns in the example above. Verify you obtain the same values for A and B that we found in Example 1.5.3.

Exercise 1.5.7. Using a PFD to Find an Antiderivative ☕☕

- Find an antiderivative of $\frac{1}{x^2-1}$ by antidifferentiating

$$\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

- Verify the answer is the same as what you would get if you had taken the antiderivative of $\frac{1}{x^2-1}$ using the trigonometric substitution $x = \sec(\theta)$. Compare this answer to the negative of the answer obtained in Example REF, noting that all three must in fact be equivalent by the transitive property of equality and since

$$\frac{1}{1-x^2} = -\frac{1}{x^2-1}.$$

It turns out there are three strange things that can happen when finding a PFD, namely:

1. The degree of the numerator is greater than or equal to the degree of the denominator.
2. The denominator has one or more irreducible quadratic factors (where irreducible quadratic means a degree two polynomial that has no real roots).
3. The denominator has one or more repeated factors.

Each has a particular workaround. Below, we describe these methods and show a corresponding hideous example that demonstrates all of these steps.

Exercise 1.5.8. Reminding Ourselves of Some Language

- What exactly does *irreducible quadratic* mean?

- Give an example of a quadratic polynomial that is irreducible.

- Give an example of a quadratic polynomial that is not irreducible.

- Is the polynomial x^2 an irreducible quadratic? Explain why or why not.

The General Method of PFD

The process for performing a partial fraction decomposition of $\frac{p(x)}{q(x)}$ is as follows:

1. **Polynomial Long Division:** If the degree of $p(x)$ is not strictly smaller than the degree of $q(x)$, start by performing polynomial long division to split the fraction into a quotient and remainder. In the remainder term, the numerator will now have degree less than the denominator.
2. **Factor Denominator:** Factor the denominator into a product of powers of linear and irreducible quadratic polynomials.
3. **Set Up Terms in the Summation:**
   (a) **Linear Factors:** If the denominator is divisible by $(x-r)^n$ for some real number r and positive natural number n, we build terms that look like
   $$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_n}{(x-r)^n}$$
   where the A_i represent unknown real constants. That is, you use all consecutive powers of a linear factor as denominators and have arbitrary constants as numerators.
   (b) **Irreducible Quadratic Factors:** Let b and c be real numbers and suppose x^2+bx+c is an irreducible quadratic. If the denominator is divisible by $(x^2+bx+c)^n$ for some positive natural number n, we build terms that look like
   $$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \frac{A_3x+B_3}{(x^2+bx+c)^3} + \cdots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$$
   where the A_i and B_i represent unknown real constants. That is, you use all consecutive powers of a linear factor as denominators and have arbitrary constants as numerators.
4. **Clear Denominators:** Multiply each side of your equation by the denominator $q(x)$ to clear all fractions.
5. **Solve for Unknowns:** Solve for the unknown constants by plugging in convenient values of x (since we want the expression to be true for all values of x). The roots of $q(x)$ are always good choices for x values, but other friendly numbers like zero or one are also often helpful.
6. **Plug Values Back into the Previously Unknown Numerators:** Plug your constants back in to conclude the equality of your original rational expression with its PFD.

Example 1.5.9. An Epic PFD

We now find the partial fraction decomposition of the rational function

$$r(x) = \frac{x^7 + 8x^6 + 25x^5 + 52x^4 + 79x^3 + 13x^2 - 61x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x}$$

This rational function has quotient $x - 1$ and remainder $6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81$ upon long division. So, for our first step in the decomposition we have

$$r(x) = x - 1 + \frac{6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x}$$

We now ignore the quotient and work on breaking up the fractional piece. The denominator is divisible by x, so we factor that out. Next, we use the Rational Root Theorem to form a list of possible roots and divide off the corresponding factors as we find them. Working out all the algebra, we conclude the denominator factors as

$$x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x = x(x + 3)^3(x^2 + 1)$$

In this particular setting, x, $x + 3$, $(x + 3)^2$, and $(x + 3)^3$ are the relevant powers of linear factors. The factor $x^2 + 1$ is the only irreducible quadratic. (**Note:** $(x+3)^2$ is not an irreducible quadratic term; it is a common mistake to consider it so. It is a power of a linear term and should be treated as such.) We now set up our sum.

$$\frac{6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81}{x(x + 3)^3(x^2 + 1)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2} + \frac{D}{(x + 3)^3} + \frac{Ex + F}{x^2 + 1}$$

Since fractions are a pain, we get rid of them! Multiplying both sides by $x(x + 3)^3(x^2 + 1)$, our equation becomes

$$6x^5 + 44x^4 + 88x^3 + 13x^2 - 34x + 81$$
$$= A(x + 3)^3(x^2 + 1) + Bx(x + 3)^2(x^2 + 1) + Cx(x + 3)(x^2 + 1) + Dx(x^2 + 1) + (Ex + F)x(x + 3)^3$$

We now solve for our unknown coefficients. It is highly convenient to set $x = 0$. This produces the equation $81 = A(3)^3$ which implies $A = 3$. Similarly, we set $x = -3$. This produces the equation

$$6(-3)^5 + 44(-3)^4 + 88(-3)^3 + 13(-3)^2 - 34(-3) + 81 = D(-3)((-3)^2 + 1)$$

which simplifies to $30 = D(-30)$ which implies $D = -1$. We have now run out of the most convenient values to choose for x, namely the roots of the denominator. At this point, we unfortunately need to do something messy! We can either plug in less than optimal values of x, for example $x = 1$, then $x = -1$, then $x = 2$, etc, and solve the resulting simultaneous system of equations that results. Or, we can multiply out the polynomials and equate coefficients one degree at a time (the method of Example 1.5.5). Carrying out either of these methods will produce

$$B = 2, C = 1, E = 1, F = -5$$

At last, we plug the values for the constants A, B, C, D, E, and F back into the original decomposition (with quotient). Our final PFD is

$$\frac{x^7 + 8x^6 + 25x^5 + 52x^4 + 79x^3 + 13x^2 - 61x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x} = x - 1 + \frac{3}{x} + \frac{2}{x + 3} + \frac{1}{(x + 3)^2} - \frac{1}{(x + 3)^3} + \frac{x - 5}{x^2 + 1}$$

Exercise 1.5.10. Identifying the Steps of PFD ☕

In the ridiculous example above, label each of the six steps of partial fraction decomposition. Where exactly does each step occur?

Exercise 1.5.11. Which Type of Numerator Goes Where? ☕

In the above example, notice that the factor $(x+3)^2$ corresponded to a term of the form

$$\frac{C}{(x+3)^2}$$

and not a term of the form

$$\frac{Cx+D}{(x+3)^2}.$$

Why was this the case?

Well, that's the process of partial fraction decomposition! Why are we doing it in a calculus course? Because a generic rational function is really hard to integrate, but the partial fraction decomposition is made up of simpler terms that are much easier to integrate. Let's find the antiderivative of that beast above!

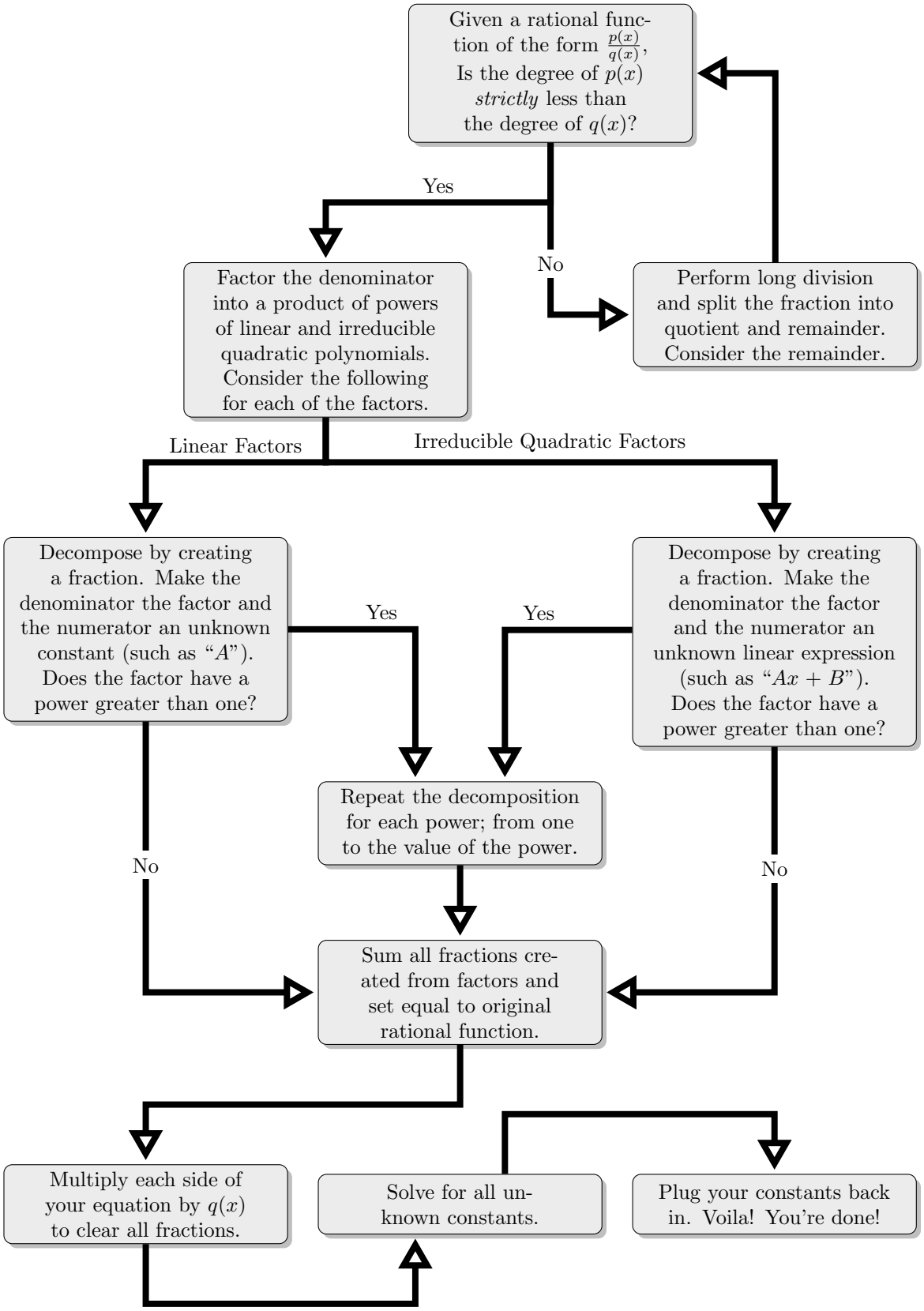
Example 1.5.12. Return of the Son of *Using a PFD to Find an Antiderivative*

We apply our PFD to compute the following antiderivative:

$$\begin{aligned}
&\int \left(\frac{x^7 + 8x^6 + 25x^5 + 52x^4 + 79x^3 + 13x^2 - 61x + 81}{x^6 + 9x^5 + 28x^4 + 36x^3 + 27x^2 + 27x} \right) \mathtt{d}x \\
&= \int \left(x - 1 + \frac{3}{x} + \frac{2}{x+3} + \frac{1}{(x+3)^2} - \frac{1}{(x+3)^3} + \frac{x-5}{x^2+1} \right) \mathtt{d}x \\
&= \frac{x^2}{2} - x + 3\ln(x) + 2\ln(x+3) + -\frac{1}{x+3} + \frac{1}{2(x+3)^2} + \int \frac{x}{x^2+1}\,\mathtt{d}x + \int \frac{-5}{x^2+1}\,\mathtt{d}x \\
&= \frac{x^2}{2} - x + 3\ln(x) + 2\ln(x+3) + -\frac{1}{x+3} + \frac{1}{2(x+3)^2} + \frac{1}{2}\ln(x^2+1) - 5\arctan(x)
\end{aligned}$$

Oh, and um, plus C.

Sweet PFD Flow Chart



Exercise 1.5.13. Now you cry! I mean, try! ☕☕☕

Find the following antiderivatives. Keep in mind that not every step of PFD will necessarily occur in every problem!

- $\displaystyle\int \frac{1}{x^2 - 9x + 20}\,\mathtt{d}x$

- $\displaystyle\int \frac{1}{x^4 - 9}\,\mathtt{d}x$

- $\displaystyle\int \frac{x^4}{x^2+1}\,\mathtt{d}x$

- $\displaystyle\int \frac{2}{x^5+2x^3+x}\,\mathtt{d}x$

- $\displaystyle\int \frac{x-2}{x^3+x^2+3x-5}\,\mathtt{d}x$

Exercise 1.5.14. Revisiting an Old Friend ☕☕☕

Recall Example 1.4.8, where we found the antiderivative of

$$\frac{1}{x^4-9x^2}$$

via trig sub. Find this antiderivative again but via PFD! Verify your answer is compatible with

what trig sub produced.

1.6 Chapter Summary

In this chapter, we tackled a very difficult question, namely

Given a function $f(x)$, how does one find an antiderivative?

Though there are many functions out there that do not have a closed form antiderivative, we explored **five** techniques that can get you there in a great many cases! Here are brief descriptions of the five:

1. **U-substitution:** Try to clean up an integral by making a substitution of the form $u = g(x)$. Often $g(x)$ is chosen to be the inner function in some function composition appearing in the integrand.

2. **Integration by Parts:** This is the product rule for antiderivatives. We identify two factors in the integrand and call one u while the other is called $\mathtt{d}v$. We then apply the IBP formula:

$$\int u\,\mathtt{d}v = uv - \int v\,\mathtt{d}u.$$

   In general, one tries to pick u to be something that is cleaner when differentiated and $\mathtt{d}v$ to be something we can antidifferentiate.

3. **Products of sines and cosines:** Any expression of the form

$$\int \sin^n(x)\cos^m(x)\,\mathtt{d}x$$

   for $n, m \in \mathbb{N}$ can be integrated by using the appropriate trig identities based on the parity of n and m.

4. **Trigonometric Substitution:** If you see quadratic polynomials in your integrand, you can likely clean things up with a trigonometric substitution. In particular,

If you see...	...make the substitution...	...because...
$a^2 - x^2$	$x = a\sin(\theta)$	$a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta)$
$a^2 + x^2$	$x = a\tan(\theta)$	$a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta)$
$x^2 - a^2$	$x = a\sec(\theta)$	$a^2\sec^2(\theta) - a^2 = a^2\tan^2(\theta)$

5. **Partial Fraction Decomposition:** This is the general method by which we can integrate any expression of the form

$$\int \frac{p(x)}{q(x)}\,\mathtt{d}x$$

   where $p(x)$ and $q(x)$ are polynomials.

Don't forget that you can check your work on any antiderivative by differentiating your answer. The result should be the original integrand!

1.7 Mixed Practice

Exercise 1.7.1. ☕

Find the antiderivative of $\frac{1}{1+x}$ using the substitution $u = 1 + x$.

Exercise 1.7.2. ☕☕

Find the antiderivative

$$\int \frac{\sqrt{x}}{\sqrt{x}+1}\,\mathtt{d}x$$

using the substitution $u = \sqrt{x} + 1$.

Exercise 1.7.3. ☕☕

Compute the exact value of the following definite integral:

$$\int_{x=1}^{x=\sqrt{3}} \frac{1}{\sqrt{x^2+1}}\,\mathtt{d}x.$$

Exercise 1.7.4. ☕☕

Calculate the antiderivative:

$$\int \frac{1}{x^4 - x^2}\,\mathtt{d}x$$

via partial fraction decomposition.

Exercise 1.7.5. Mixed bag of u-subs ☕☕

Find the following antiderivatives.

- $\int 2x + 1\,\mathtt{d}x$
- $\int \frac{1}{2x+1}\,\mathtt{d}x$
- $\int e^x \sec(e^x)\tan(e^x)\,\mathtt{d}x$
- $\int \frac{1}{e^x}\,\mathtt{d}x$

- $\int \frac{\cos(\ln(2x))}{x}\,\mathtt{d}x$
- $\int (2^x)^2\,\mathtt{d}x$
- $\int \frac{1}{2+2x+x^2}\,\mathtt{d}x$
- $\int \frac{1}{1+2x+x^2}\,\mathtt{d}x$

Exercise 1.7.6. ☕☕

Consider $\int \cos^{13} x \sin^5 x\,\mathtt{d}x$.

- Can you compute this integral using $u = \cos x$? Explain.
- Can you compute this integral using $u = \sin(x)$? Explain.
- Which of the two above substitutions will be easier to use? Carry out the integration, using the easier of the two.

Exercise 1.7.7. ☕☕☕

Evaluate the integral

$$\int \csc^3(x)\,\mathtt{d}x$$

via IBP.

Exercise 1.7.8. ☕☕☕

Consider the following antiderivative:

$$\int \frac{1}{x^2-16}\,\mathtt{d}x$$

- Compute the above antiderivative via a partial fraction decomposition.
- Compute the above antiderivative via trigonometric substitution.
- Your answers may appear very different! Verify that they are in fact equivalent.

Exercise 1.7.9. ☕☕☕☕

- Perform a Partial Fraction Decomposition on the following rational function:

$$\frac{x^3}{x^3 - 3x^2 + 4}$$

- Use your work from the previous part to evaluate the following antiderivative:

$$\int \frac{x^3}{x^3 - 3x^2 + 4}\,\mathtt{d}x$$

Exercise 1.7.10. ☕☕☕

Evaluate the following antiderivative using Integration by Parts:

$$\int \sec^5 x\,\mathtt{d}x.$$

Hint: The two integrals from Subsections 1.1 and 1.2 listed below may be helpful!

$$\int \sec x\,\mathtt{d}x = \ln|\sec x + \tan x| + C$$
$$\int \sec^3 x\,\mathtt{d}x = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

Chapter 2

Definite Integrals

Now that we are much better at the process of antidifferentiation, we apply integrals to the classic problems of geometry. We find lengths, areas, volumes, and centers of mass. Before we begin, we state L'Hospital's Rule, which will assist in computing areas of unbounded regions.

2.1 L'Hospital's Rule and Improper Integrals

L'Hospital's Rule (LHR) allows us to evaluate indeterminate limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It says that in either of these cases, we can simply differentiate the numerator and the denominator and try again.

Theorem 2.1.1. L'Hospital's Rule

Let c be a real number, ∞, or $-\infty$. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

For the moment, we will just accept LHR and use it. In Section 4.7, we will prove LHR using power series. Notice that here we do not differentiate with a Quotient Rule. We instead simply differentiate the top and differentiate the bottom.

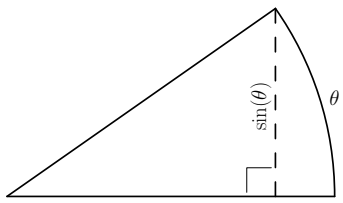
Example 2.1.2. Sine of a Small Angle

Consider the following limit:

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$$

It is indeterminate of the form $\frac{0}{0}$. Thus it is valid to apply LHR.

$$\begin{aligned}\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} &= \lim_{\theta\to 0} \frac{(\sin(\theta))'}{(\theta)'} \\ &= \lim_{\theta\to 0} \frac{\cos(\theta)}{1} \\ &= 1\end{aligned}$$



Small angle θ versus $\sin(\theta)$.

Exercise 2.1.3. Interpreting the Above Example ☕

Since the ratio of $\sin(\theta)$ to θ approaches 1 as θ gets small, it would be appropriate to say the following (fill in the blanks):

For small values of θ, _____$\approx$_____.

Note that this property comes up frequently in physics! For example, when modeling the motion of a mass hanging from a spring, Hooke's Law tells us that force is proportional to displacement. We use the same model to describe motion of a pendulum, even though in that case force is not technically proportional to displacement, but rather the *sine* of displacement. Why can we throw away the sine? It is because for small displacements, the sine of the displacement is roughly equal to the displacement!

Exercise 2.1.4. Practice with LHR ☕☕

Evaluate the following limits using L'Hospital's Rule. In each case, justify why it is ok to use it!

- $\lim\limits_{x\to\infty} \frac{x^2}{2^x}$

- $\lim\limits_{x\to 2} \frac{x-2}{\sin(\pi x)}$

- $\lim\limits_{x\to\infty} \frac{\arctan(x)-\pi/2}{\sin(1/x)}$

Other Indeterminate Forms

There are many other indeterminate forms besides just $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Others that come up include:

- $0 \cdot \infty$
- 0^0
- 1^∞
- $\infty - \infty$

Often these other forms can be rearranged algebraically to become $\frac{0}{0}$ or $\frac{\infty}{\infty}$. After this rearrangement, they can then be evaluated with LHR (or perhaps the algebra itself resolves the indeterminate form and LHR will not be needed). Common helpful strategies include:

- Multiplying the top and bottom of the limit by the same expression (especially the conjugate of an expression involving a radical).
- Taking e to the ln of the limit.
- Rewriting a product as a fraction via $a \cdot b = \frac{b}{\frac{1}{a}}$.

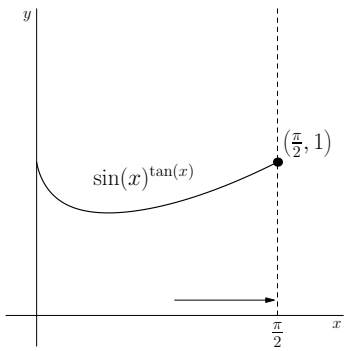
Example 2.1.5. Rewriting a Different Indeterminate Form

Consider the function $\sin(x)^{\tan(x)}$. As x approaches $\frac{\pi}{2}$ from the left, the function takes on the indeterminate form 1^∞. Thus, we try the second strategy described above, where we take e to the ln of the limit. Proceeding:

$$\begin{aligned}\lim_{x\to\pi/2^-} \sin(x)^{\tan(x)} &= \lim_{x\to\pi/2^-} e^{\ln\left(\sin(x)^{\tan(x)}\right)} \\ &= \lim_{x\to\pi/2^-} e^{\tan(x)\ln(\sin(x))}\end{aligned}$$

Notice the exponent is now the indeterminate form $0\cdot\infty$. Since the exponential function is continuous, we can move the limit inside and use LHR!

$$\begin{aligned}\lim_{x\to\pi/2^-} \sin(x)^{\tan(x)} &= \lim_{x\to\pi/2^-} e^{\tan(x)\ln(\sin(x))} \\ &= e^{\lim_{x\to\pi/2^-}(\tan(x)\ln(\sin(x)))} \\ &= e^{\lim_{x\to\pi/2^-}\left(\frac{\ln(\sin(x))}{\cot(x)}\right)} \\ &= e^{\lim_{x\to\pi/2^-}\left(\frac{(\ln(\sin(x)))'}{(\cot(x))'}\right)} \\ &= e^{\lim_{x\to\pi/2^-}\left(\frac{\frac{\cos(x)}{\sin(x)}}{-\csc^2(x)}\right)} \\ &= e^{\lim_{x\to\pi/2^-}(-\cos(x)\sin(x))} \\ &= e^{-0\cdot 1} \\ &= 1\end{aligned}$$



Exercise 2.1.6. Identifying LHR ☕

In the example above, circle the exact step where LHR was applied. Why was it ok to use LHR on that step? Write a short sentence to explain.

Exercise 2.1.7. Rewriting ☕☕☕

Utilize these strategies to rewrite the limits below as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then evaluate. Note that some of these may need LHR after rewriting and some may not!

- $\lim_{x\to\infty} \quad x \cdot \sin(1/x)$

- $\lim_{x\to\infty} \quad x - \sqrt{x^2 + 4x + 3}$

- $\lim_{x\to\infty} \quad \left(1 + \frac{1}{x}\right)^x$ (**Note:** This limit is often taken as the definition of the constant you get

here!)

Exercise 2.1.8. Polyexposaurus ☕☕☕

- Any positive real number raised to the zero is...
- Zero raised to any positive real number is...
- So, what is $\lim_{x \to 0^+} x^x$?

Be careful when using LHR to only apply it in the two indeterminate forms specified above. Applying LHR to an expression that is not either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ will most likely produce incorrect results.

Exercise 2.1.9. L'Urgent Care ☕☕

Consider the following limit.

$$\lim_{x\to\pi} \frac{\sin(x)}{x}$$

- Why would it be wrong to apply LHR to the above limit?
- What do you get if you blindly apply LHR?
- What should the limit actually be?

Growth Orders

LHR is often used for comparing growth orders of functions. To compare the growth orders of functions, we take the limit of their ratio as x approaches infinity and then see if the ratio approaches zero, a nonzero constant, or infinity to see which is growing faster. More formally:

Definition 2.1.10. Growth Order

Let $f(x)$ and $g(x)$ be functions on the real numbers.

- If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $g(x)$ has *larger growth order* than $f(x)$.
- If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is a nonzero constant, then $f(x)$ and $g(x)$ have the *same growth order*.
- If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then $f(x)$ has *larger growth order* than $g(x)$.

Exercise 2.1.11. Comparing the Growth Orders of Two Lines ☕☕

Consider the following two linear functions:

$$f(x) = 6x + 1$$
$$g(x) = 2x - 1$$

Fill out the table below to study some of their values and corresponding ratios. Use decimal approximations for values that aren't integers.

x	1	10	100	1,000	10,000
$f(x)$					
$g(x)$					
$f(x)/g(x)$					

- From the table, does it appear that the ratio $f(x)/g(x)$ is approaching zero, infinity, or a nonzero constant?

- Use LHR to compute the limit
$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$
How does it relate to the values in the data table?

- See the above definition of Growth Order. In this case, would you say f and g have the same growth order, or does one function have larger growth order than the other?

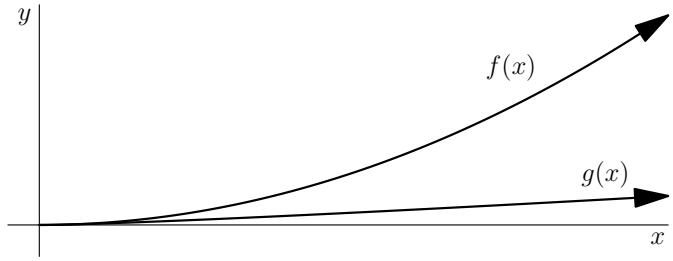
The above calculation justifies why it is ok to talk about something having *linear growth order* or *growing linearly*. Any two lines have the same growth order (ignoring vertical and horizontal), so it is perfectly well-defined to talk about linear growth even if the slope or intercepts of the lines being discussed are unknown.

This concept comes up frequently in computer science when you try to measure the runtime of algorithms. If $f(x)$ is the number of operations performed by an algorithm that is handed an input of size x, then a logarithmic growth order of $f(x)$ is generally more desirable than a linear growth order, which is more desirable than quadratic growth order, and so on.

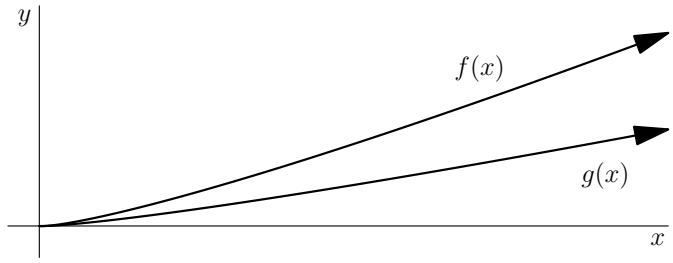
Exercise 2.1.12. A Visual Representation of Growth Order

In each of the graphs below, there is a graph of $f(x)$ and a graph of $g(x)$. Based on the graphs, do you expect that f and g have the same growth order, or is one larger?

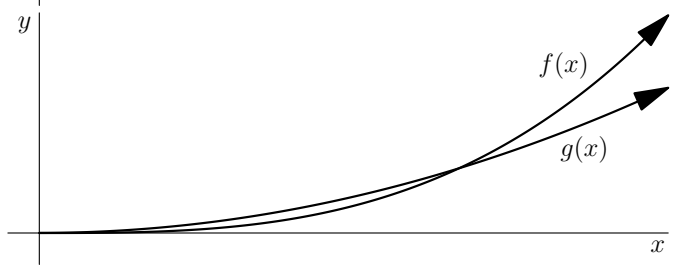
1.



2.



3.

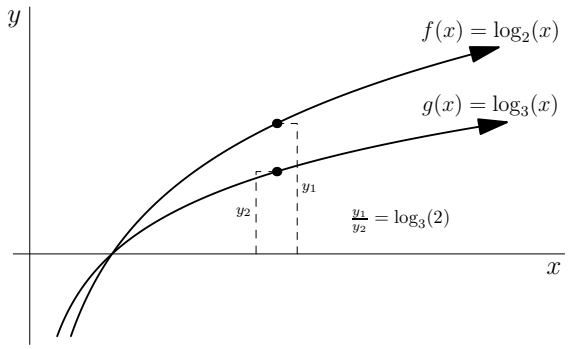


Example 2.1.13. Logarithmic Growth Order

Here we compare the growth orders of the following logarithmic functions:

$$f(x) = \log_2(x)$$
$$g(x) = \log_3(x)$$

Definition 2.1.10. shows that we should take the limit of their ratio and then see if we get zero, infinity, or a nonzero constant.



$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{\log_2(x)}{\log_3(x)} \\ &= \lim_{x\to\infty} \log_3(2) \\ &= \log_3(2)\end{aligned}$$

Since their ratio came out to a nonzero constant, we conclude that those two functions in fact have the same growth order (even though the y-coordinate of $f(x)$ is always bigger). By a similar calculation, we could observe that in fact any two logarithms have the same growth order regardless of what the base is!

Exercise 2.1.14. Race of the Turtles ☕☕

Rank the following functions by growth order from slowest to fastest by comparing their growth orders two at a time:

$$f(x) = x\ln(x)$$
$$g(x) = x^{1.1}$$
$$h(x) = x\left(\ln(x)\right)^2$$

Exercise 2.1.15. Exponential vs Polynomial ☕☕☕

Explain why an exponential function will always have larger growth order than an polynomial function.

Improper Integrals

In Calculus I, all of our definite integrals corresponded to the area of a bounded region. A definite integral over an unbounded region is called *improper*.

Vertically Unbounded Regions

If the integrand has a vertical asymptote between the limits of integration, we must proceed by approximating the unbounded region with a bounded region and then taking a limit.

Exercise 2.1.16. Analyzing a Vertical Asymptote ☕☕

Consider the following integral:

$$\int_0^1 \ln(x)\,\mathrm{d}x$$

- Explain why the above integral would be called improper.

- Find the antiderivative of the function $\ln(x)$.

- Fill out the following table. For each definite integral, include a rough sketch of the region whose signed area it corresponds to.

c	$\int_c^1 \ln(x)\,\mathrm{d}x$	Graph of Region
0.1		
0.01		
0.001		
0.0001		

- As c approaches 0 from the right, what does the area seem to be approaching?

The above calculation motivates the definition of an improper integral.

Definition of Improper Integral I

If $f(x)$ has a vertical asymptote at $x = a$

but is continuous on the interval $(a, b]$,

then the improper integral is defined as

$$\int_a^b f(x)\,\mathtt{d}x = \lim_{c\to a^+} \int_c^b f(x)\,\mathtt{d}x$$

If this limit converges to a number, then we say the improper integral *converges*. Otherwise, we say the improper integral *diverges*. We apply the definition to finish off the above exercise.

Example 2.1.17. Area Between the Axes and Natural Log

To calculate the area between $y = 0$, $x = 0$, and $y = \ln(x)$, we use the definition of improper integral. We proceed with the following calculation:

$$\begin{aligned}\int_0^1 \ln(x)\,\mathtt{d}x &= \lim_{c\to 0^+} \int_c^1 \ln(x)\,\mathtt{d}x \\ &= \lim_{c\to 0^+} x\ln(x) - x\Big]_{x=c}^{x=1} \\ &= \lim_{c\to 0^+} (1\ln(1) - 1) - (c\ln(c) - c)\end{aligned}$$

In the above limit, all terms are harmless except for $\lim_{c\to 0} c\ln(c)$, which is indeterminate of the form $0 \cdot \infty$. We can rewrite as

$$\lim_{c\to 0^+} c\ln(c) = \lim_{c\to 0^+} \frac{\ln(c)}{\left(\frac{1}{c}\right)}$$

to get it in the form $\frac{\infty}{\infty}$ (up to a minus sign which is harmless), where we can apply LHR.

Exercise 2.1.18. We'll Actually Finish This Problem Here, Promise ☕☕

Finish evaluating the limit above and verify the area matches your estimations from Exercise 2.1.16.

Let us now construct the analogous definition for a vertical asymptote occurring at the right-hand endpoint rather than the left-hand endpoint.

Exercise 2.1.19. Completing the Definition ☕☕☕

Suppose a function $f(x)$ is continuous on $[a, b)$ but had a vertical asymptote at $x = b$. Use the diagram below to help complete the definition of such an improper integral. Fill in the boxes

below.

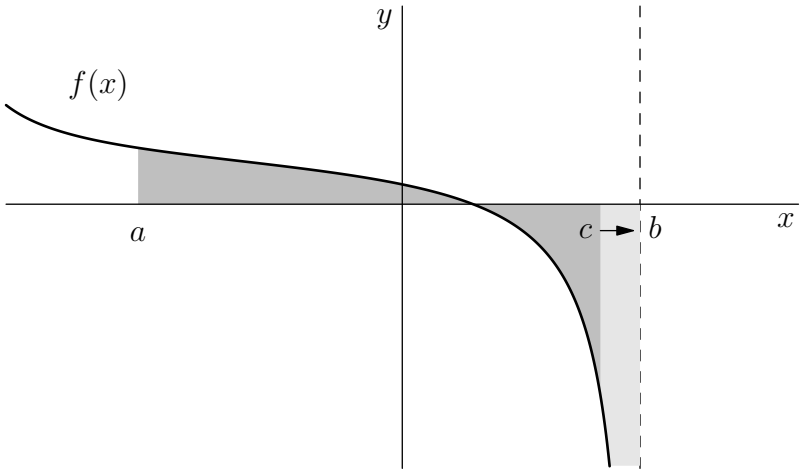
Definition of Improper Integral II

If $f(x)$ has a vertical asymptote at $x = b$

but is continuous on the interval $[a, b)$,

then the improper integral is defined as

$$\int_a^b f(x)\,\mathtt{d}x = \lim_{c\to\square} \int_{\square}^{\square} f(x)\,\mathtt{d}x$$

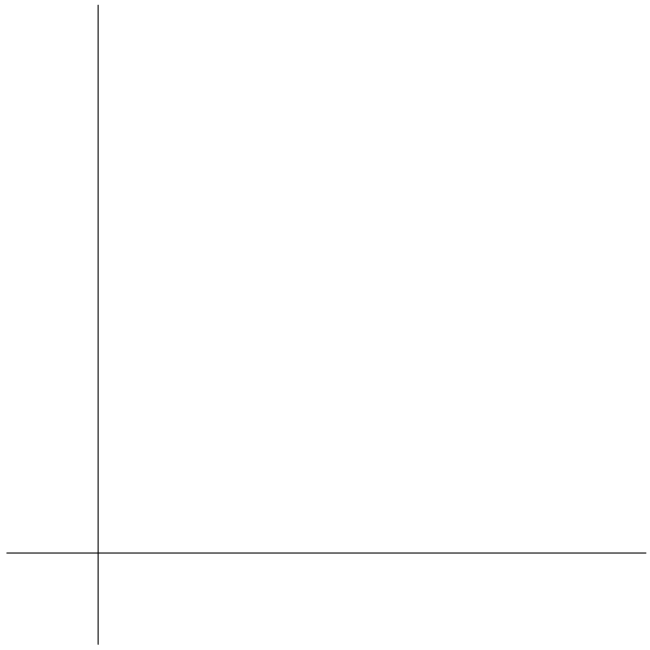


Once again, if this limit converges to a number, then we say the improper integral *converges*. Otherwise, we say the improper integral *diverges*.

Exercise 2.1.20. Illustrate the Computation ☕

Here we demonstrate an example of computing an improper integral with the vertical asymptote at the right-hand endpoint. Illustrate the computation on the axes below. Show the graph of the integrand and the locations of the bounds a, b, and c.

$$\begin{aligned}
\int_{x=0}^{x=\pi/2} \tan(x)\,\mathtt{d}x &= \lim_{c\to\pi/2^-} \int_{x=0}^{x=c} \tan(x)\,\mathtt{d}x \\
&= \lim_{c\to\pi/2^-} -\ln\left(\cos(x)\right)\Big]_{x=0}^{x=c} \\
&= \lim_{c\to\pi/2^-} -\ln\left(\cos\left(c\right)\right) + \ln\left(\cos(0)\right) \\
&= \infty
\end{aligned}$$



If the vertical asymptote is in the interior (rather than at an endpoint) of the interval over which you are integrating, it may be necessary to split it into several integrals.

Example 2.1.21. A Particular Unbounded Region

Suppose we wish to calculate the area bounded by the x-axis, the line $x = 1$, the line $x = -1$, and the graph of $f(x) = \frac{1}{\sqrt{|x|}}$. Notice the function has a vertical asymptote at $x = 0$, which is in the interior of the interval over which we wish to integrate. Thus, we split the integral into two integrals. The first has a vertical asymptote at the right-hand endpoint and the second has a vertical asymptote at the left-hand endpoint. We handle each accordingly.

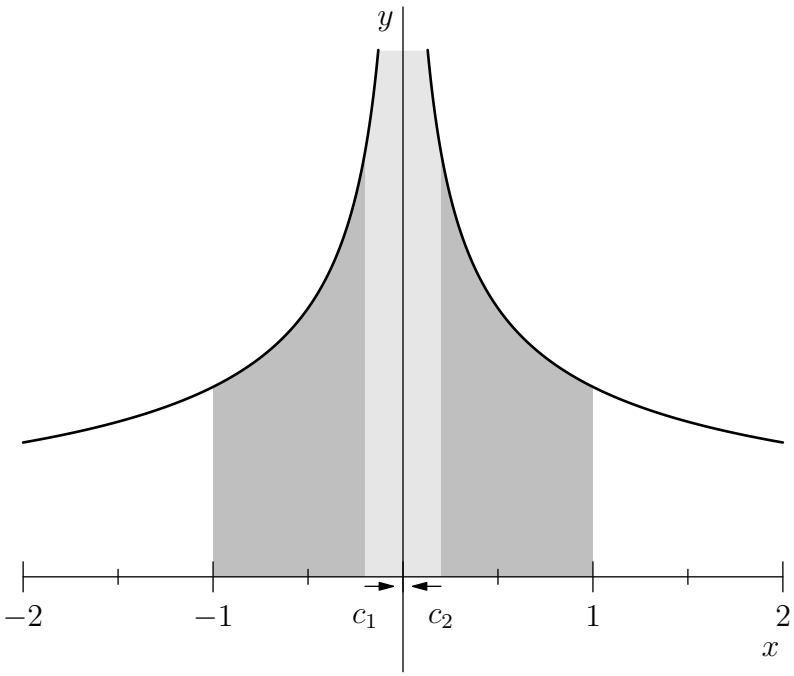
$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\,\mathtt{d}x &= \int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,\mathtt{d}x + \int_{0}^{1} \frac{1}{\sqrt{|x|}}\,\mathtt{d}x \\ &= \lim_{c_1\to 0^-} \int_{-1}^{c_1} \frac{1}{\sqrt{|x|}}\,\mathtt{d}x + \lim_{c_2\to 0^+} \int_{c_2}^{1} \frac{1}{\sqrt{|x|}}\,\mathtt{d}x\end{aligned}$$

We now evaluate each of those integrals separately and add their totals.

$$\begin{aligned}\lim_{c_1\to 0^-} \int_{-1}^{c_1} \frac{1}{\sqrt{|x|}}\,\mathtt{d}x &= \lim_{c_1\to 0^-} -2\sqrt{|x|}\Big|_{x=-1}^{x=c_1} \\ &= \lim_{c_1\to 0^-} -2\sqrt{-c_1} + 2\sqrt{1} \\ &= 0 + 2 \\ &= 2\end{aligned}$$

The other region is just a reflection across the y-axis and thus must also have area 2. We conclude

$$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\,\mathtt{d}x = 4$$



Exercise 2.1.22. Some Subtle Sign Business ☕

In the above computation, why is there a negative sign on the antiderivative, producing $-2\sqrt{|x|}$ instead of just $2\sqrt{|x|}$? (**Hint:** Graph the function $2\sqrt{|x|}$!)

Ok, now you give it a shot!

Exercise 2.1.23. Improper Integral Practice ☕☕

Evaluate the following integrals and draw graphs similar to the figure above. Show how you are evaluating the improper integral as a limit of integrals of bounded regions.

- $\int_2^4 \frac{1}{\sqrt{x-2}}\,\mathtt{d}x$

- $\int_2^4 \frac{1}{x^2-4}\,\mathtt{d}x$

- $\int_0^{\pi/2} \sec(x)\,\mathtt{d}x$

- $\int_{-\pi/2}^{\pi/2} \csc^2(x)\,\mathtt{d}x$

Horizontally Unbounded Regions

If the integral is over an interval that includes plus or minus infinity as one of the endpoints, we must proceed by approximating via a bounded interval and then taking the limit as the endpoint goes to plus or minus infinity.

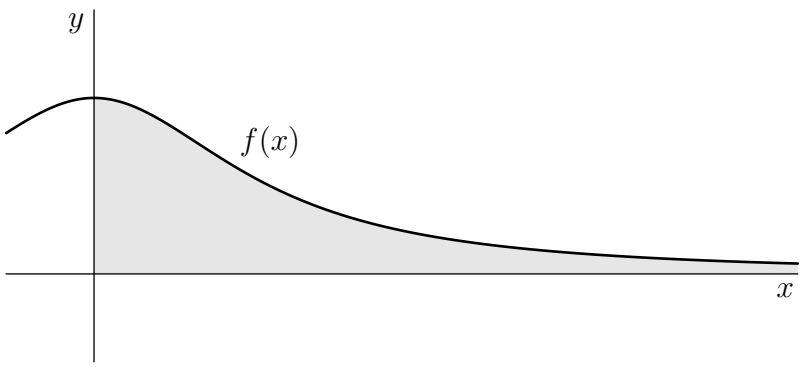
Definition of Improper Integral III

Let $f(x)$ be continuous on the interval $[a, \infty)$ for some real number a.

Then we define $\int_a^\infty f(x)\,\mathtt{d}x = \lim_{c\to\infty} \int_a^c f(x)\,\mathtt{d}x$.

An integral to negative infinity is defined analogously via the corresponding limit.

Exercise 2.1.24. Finding c ☕

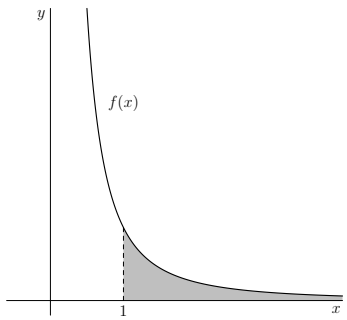


Here is a graph that represents the above definition for $a = 0$. Interpret the definition by labeling

c on the graph and explaining the role it plays.

Example 2.1.25. Area Under $f(x) = \frac{1}{x^2}$

Suppose we wish to compute the area under the curve $f(x) = \frac{1}{x^2}$ over the interval $[1, \infty)$. We apply the definition of the improper integral as a limit of bounded integrals.



$$\begin{aligned}
\int_{x=1}^{x=\infty} \frac{1}{x^2}\,\mathtt{d}x &= \lim_{c\to\infty} \int_{x=1}^{x=c} \frac{1}{x^2}\,\mathtt{d}x \\
&= \lim_{c\to\infty} -\frac{1}{x}\Big]_{x=1}^{x=c} \\
&= \lim_{c\to\infty} -\frac{1}{c} + \frac{1}{1} \\
&= 1
\end{aligned}$$

Thus, the area under the curve is 1.

Exercise 2.1.26. Area Under $1/x^p$ ☕☕☕

- Calculate the improper integral $\int_{x=1}^{x=\infty} \frac{1}{x^3}\,\mathtt{d}x$.

- Calculate the improper integral $\int_{x=1}^{x=\infty} \frac{1}{x^1}\,\mathtt{d}x$.

- Calculate the improper integral $\int_{x=1}^{x=\infty} \frac{1}{x^{1/2}}\,\mathtt{d}x$.

- For what real numbers p will $\int_{x=1}^{x=\infty} \frac{1}{x^p}\,\mathtt{d}x$ converge? For what p will it diverge?

If the integral is across the entire real number line, one must split into two separate integrals, similar to how we handled a vertical asymptote in the interior of our interval.

Definition of Improper Integral III

Let $f(x)$ be continuous on the entire real number line.

For any real number a, we define $\int_{-\infty}^{\infty} f(x)\,\mathtt{d}x = \int_{-\infty}^{a} f(x)\,\mathtt{d}x + \int_{a}^{\infty} f(x)\,\mathtt{d}x$.

For the following problems, you may spot yourself the following fact we will prove in Calc 3:

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathtt{d}x = \sqrt{\pi}$$

Exercise 2.1.27. More Practice with Improper Integrals ☕☕

Now, try the following integrals and for each draw a graph like the above figure that represents your integral as a limit of integrals of bounded regions. Also, for the following problems, you may spot yourself the following fact we will prove in Calc 3:

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathtt{d}x = \sqrt{\pi}$$

- $\int_0^{\infty} xe^{-x^2}\,\mathtt{d}x$

- $\int_{-\infty}^{\infty} xe^{-x^2}\,\mathtt{d}x$

- $\int_{-\infty}^{\infty} x^2e^{-x^2}\,\mathtt{d}x$

- $\int_{2}^{\infty} \frac{1}{x\ln(x)}\,\mathtt{d}x$

- $\int_{2}^{\infty} \frac{1}{x(\ln(x))^2}\,\mathtt{d}x$

- Consider the integral $\int_0^{\infty} \sin(x)\,\mathtt{d}x$. Explain why it would be incorrect to say that all the positive and negative area cancel each other out to be zero. In particular, cite the definition

of the improper integral as a limit in your explanation.

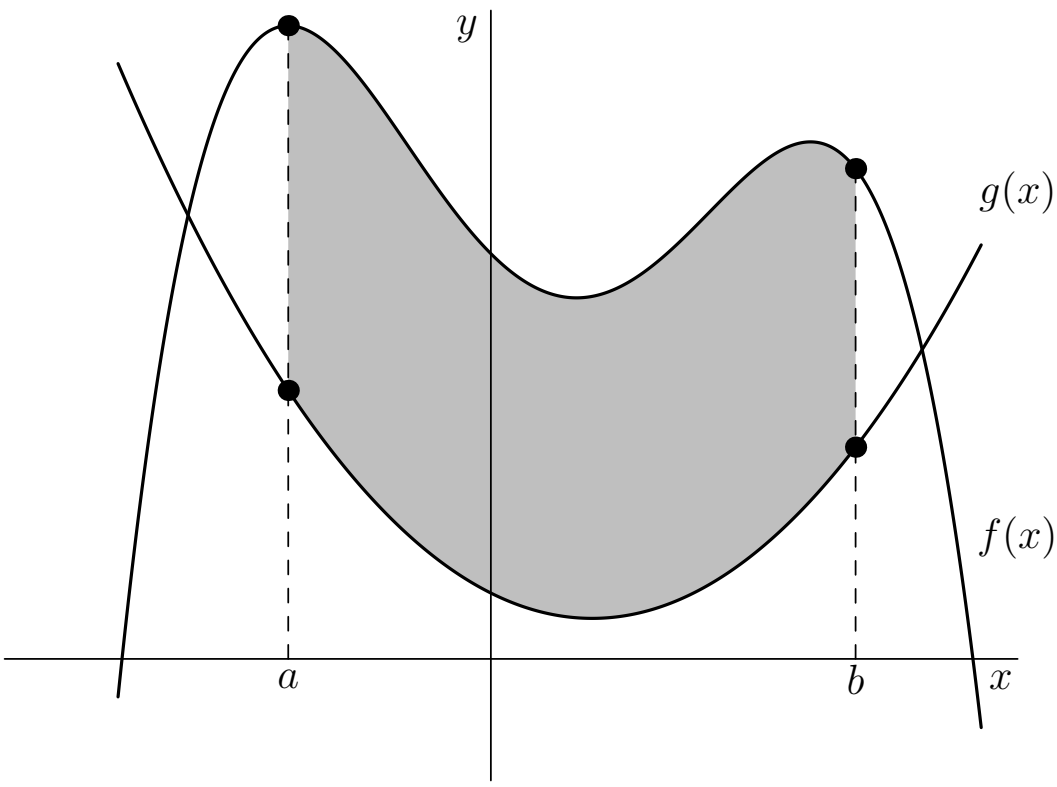
2.2 Area Between Curves and Volume by Cylindrical Shells

Recall that a definite integral calculates the signed area under a curve. Thus, we can find the signed area between two curves by taking their difference and integrating.

Formula 2.2.1. Area Between the Graphs of Two Functions

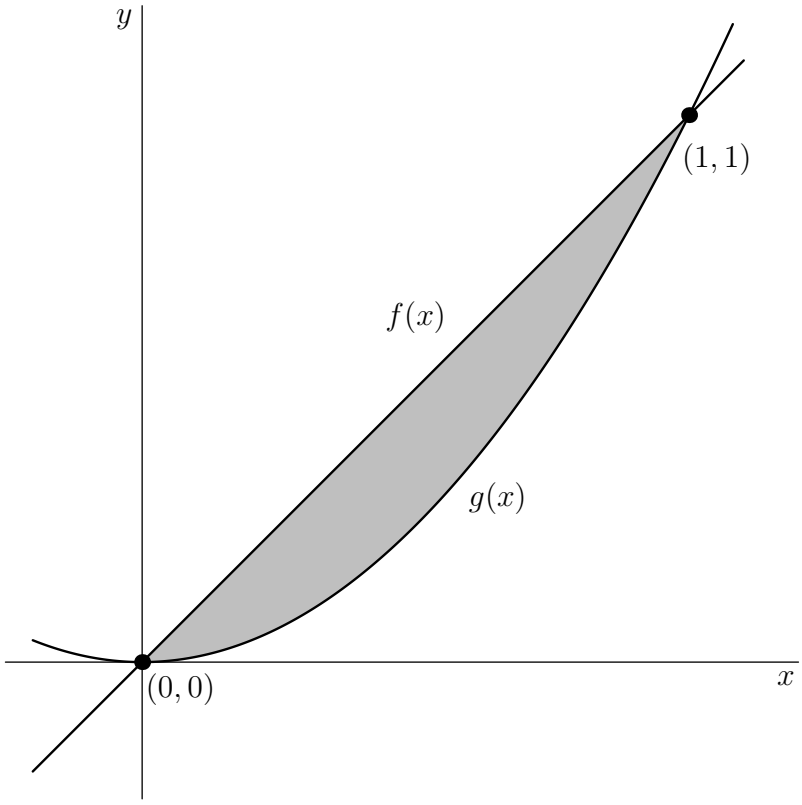
Let $g(x) \leq f(x)$ for all x in an interval $[a, b]$. Then the area of the region bounded by the graphs $x = a$, $x = b$, $y = f(x)$, and $y = g(x)$ is

$$A = \int_a^b (f(x) - g(x))\, \mathtt{d}x.$$



Example 2.2.2. Quadrature of a Parabola

Suppose we wish to find the area between curves $f(x) = x$ and $g(x) = x^2$. To accomplish this, we set the two formulas equal to each other to solve for the points of intersection. The line and parabola meet at $(0, 0)$ and $(1, 1)$.



Thus, the area between curves is

$$\begin{aligned}\int_{x=0}^{x=1} \left(x - x^2\right) \mathtt{d}x &= \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{x=1} \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{1}{6}.\end{aligned}$$

Exercise 2.2.3. Finding Intersection Points ☕

In the above exercise, the claim was made that the two curves intersect at $(0, 0)$ and $(1, 1)$. Verify this claim by setting the two functions equal to each other and doing the algebra to solve for the intersection points!

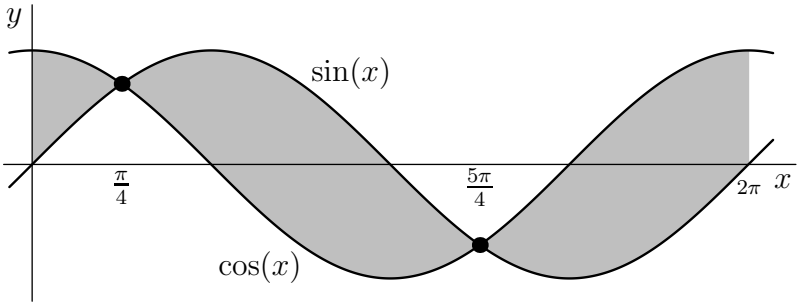
Note that if the curves intersect multiple times, you might have to split the integral onto the corresponding intervals.

Example 2.2.4. A Region with More Crossings

Find the area between the graphs of sine and cosine between $x = 0$ and $x = 2\pi$.
Again, to accomplish this, we set the two formulas equal to each other to solve for the points of intersection.

$$\sin(x) = \cos(x) \implies \tan(x) = 1 \implies x = \pi/4 \text{ or } x = 5\pi/4$$

Thus the points of intersection are at $\left(\pi/4, \sqrt{2}/2\right)$ and $\left(5\pi/4, -\sqrt{2}/2\right)$.



We now compute the area as follows, being careful to always keep the integrand as "top function minus bottom function" on each interval:

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos(x) - \sin(x))\,\mathtt{d}x + \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,\mathtt{d}x + \int_{5\pi/4}^{2\pi} (\cos(x) - \sin(x))\,\mathtt{d}x \\ &= (\sin(x) + \cos(x))\,\Big|_0^{\pi/4} + (-\cos(x) - \sin(x))\,\Big|_{\pi/4}^{5\pi/4} + (\sin(x) + \cos(x))\,\Big|_{5\pi/4}^{2\pi}. \end{aligned}$$

Exercise 2.2.5. Complete the Example ☕☕

Finish the computation and verify the area is $4\sqrt{2}$.

Exercise 2.2.6. A Common Mistake ☕

Briefly write in words, why would simply evaluating

$$\int_{x=0}^{x=2\pi} \cos(x) - \sin(x)\,\mathtt{d}x$$

in the example above not give the area of the shaded region?

Area of a Circle

Let's now prove an old friend, the formula for the area of a circle!

Exercise 2.2.7. Area of a Circle ☕☕

- Recall the equation for a circle of radius r is $x^2 + y^2 = r^2$. Draw a diagram that shows that this equation is a consequence of the Pythagorean Theorem.

- Solve for y and note that the square root requires a "plus or minus". To get the top curve $f(x)$, choose the positive square root. To get the bottom curve $g(x)$, choose the negative square root. Write your formulas for $f(x)$ and $g(x)$ below.
  - Top Half: $f(x) =$
  - Bottom Half: $g(x) =$

- Use an integral to find the area between f and g to obtain the formula for the area of a

circle of radius r.

Some Other Regions for Practice

Find the area between the following curves. Graph the curves and shade the region!

Exercise 2.2.8. Other Regions

- $f(x) = x^3 - x^2 - x + 1$ and $g(x) = x^3 + x^2 - x - 1$

- $y = \sqrt{1 - x^2}$ and $y = 1/2$

- $y = \tan(x)$ restricted to the domain $(-\pi/2, \pi/2)$ and $y = \frac{4}{\pi}x$

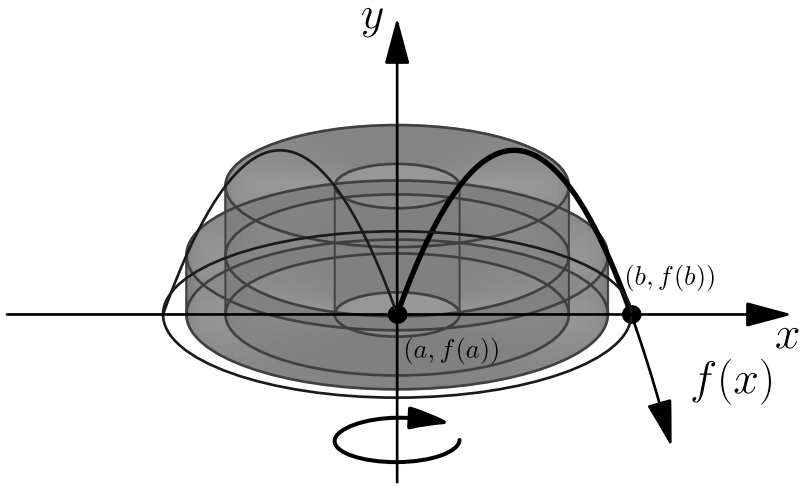
Cylindrical Shells

When using an integral to find the area between two curves, you are essentially breaking up the area into rectangles of a certain width, then taking the limit of that approximation as the number of rectangles grows (and their corresponding width shrinks). Remember, this is exactly the definition of the integral that you learned in Calculus I! This is a technique that we can expand to find the volumes of 3-dimensional objects that have circular symmetry. These types of objects are commonly found in real world settings, from rocket nozzles to doorknobs and are commonly manufactured by milling on a lathe. This technique is called volume by cylindrical shells.

Formula 2.2.9. Volume by Cylindrical Shells

If the region under of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is revolved around the y-axis, the volume is given by

$$V = \int_{x=a}^{x=b} 2\pi x f(x)\, \mathtt{d}x.$$

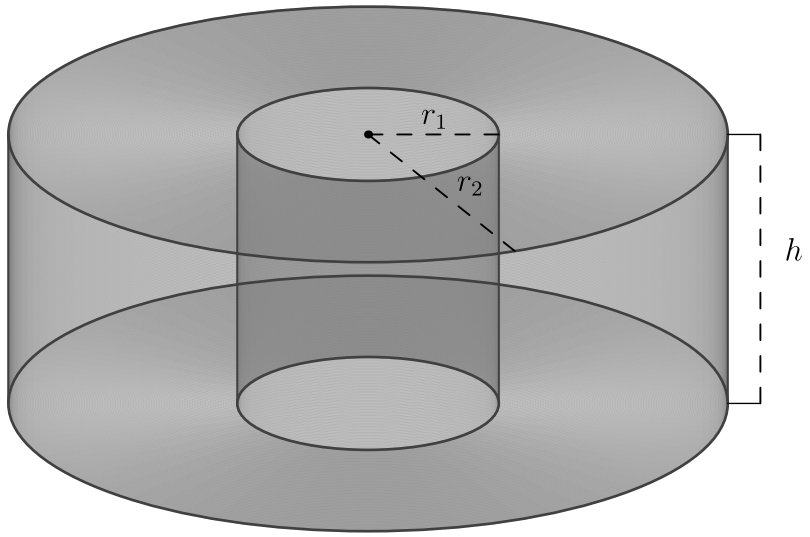


Note that $f(x)$ in the formula above is shown as the graph of a function, but it also could represent the height of a shell at location x, that does not have to strictly be situated in a manner where it starts at the x-axis and goes up from there. For example, if the region is bounded on top by a function $h(x)$ and below by a function $g(x)$, then $f(x) = h(x) - g(x)$ would be perfectly valid in the formula above, representing the height of the shells.

This formula comes from approximating the volume of the region by using nested cylinders with smaller cylinders deleted from their middle (hence *cylindrical shells*). In particular, we are cutting the region into shells that approximate the volume, and then taking the limit as the thickness of these shells goes to zero, and correspondingly, the number of shells goes to infinity.

Definition 2.2.10. Cylindrical Shells

A *cylindrical shell* is a cylinder with a second cylinder of equal height but smaller radius deleted out of the middle of it.



Exercise 2.2.11. The Volume of a Single Cylindrical Shell ☕

See the diagram above of a cylindrical shell with height h, outer radius of r_2, and inner radius r_1. Show the volume of that shell is given by $V = h\pi\left(r_2^2 - r_1^2\right)$.

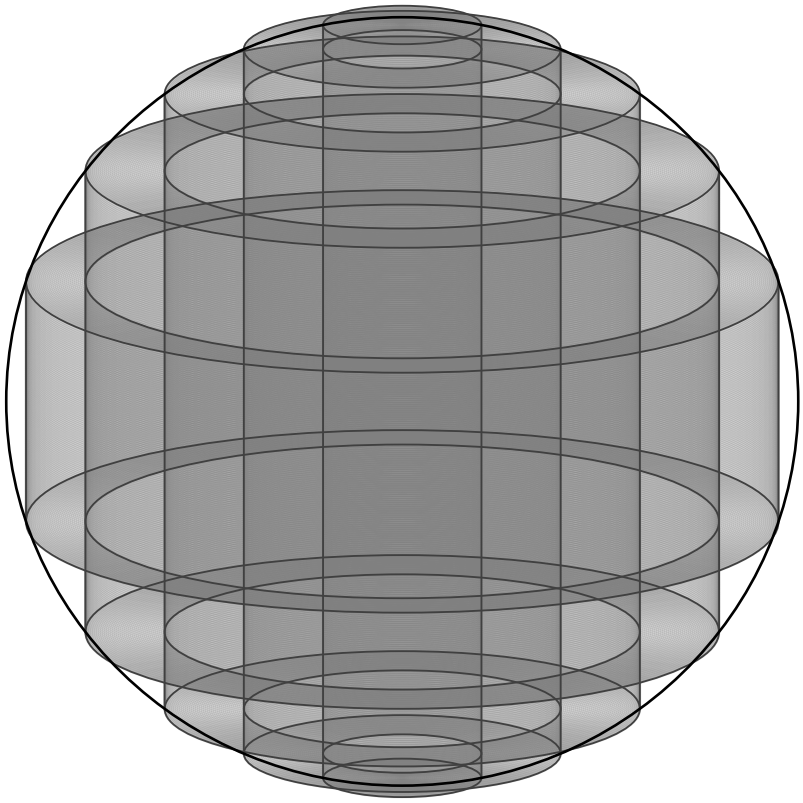
We test this new method out on a familiar object, the sphere! Notice that here we don't have just a single function $f(x)$ that bounds the region which stops at the x-axis. However, we can treat it as if that were the case and just double our answers, because the region is symmetric about the x-axis anyway.

Exercise 2.2.12. Volume of a Sphere ☕☕

Suppose we have a sphere of radius 1.

- To start, approximate the volume of a sphere in a very crude manner. Obtain an upper bound for the volume by enclosing the sphere in just a single cylinder of height 2 and radius 1.

- To get a better estimate, we now approximate the volume using six shells. In the diagram below, assume the center of the sphere is the origin. Then label the points with x-coordinates $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{2}{6}, x_3 = \frac{3}{6}, x_4 = \frac{4}{6}, x_5 = \frac{5}{6}$, and $x_6 = 1$. Compute the volume of each cylindrical shell using the formula from Exercise 2.2.11. Add those six volumes to estimate the volume of the sphere. Label the points and corresponding shell volumes in the diagram below.



- How does the single-cylinder estimate compare to the six-shell estimate? What would happen if we continually cut the sphere into smaller and smaller shells and let the number of shells

go to infinity?

We can now build the cylindrical shells volume formula much in the same manner you did for Riemann sums when you first defined the integral.

Suppose we wish to find the volume of the region under the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ revolved around the y-axis. We begin by splitting into n cylindrical shells. Specifically, let $x_0, x_1, x_2, \ldots, x_n$ be equally spaced points along the x-axis from a to b. That is, $x_0 = a$, $x_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta x = x_{i+1} - x_i = \frac{b-a}{n}$.

With this setup, if we want the volume of the cylindrical shells between points x_{i+1} and x_i, we would use our volume of a cylindrical shell formula to obtain

$$f(x_{i+1})\pi\left(x_{i+1}^2 - x_i^2\right)$$

as the volume. We then add up the volumes of all shells and take the limit as the number of shells goes to infinity:

$$\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_{i+1})\pi\left(x_{i+1}^2 - x_i^2\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_{i+1})\pi\left(x_{i+1} - x_i\right)\left(x_{i+1} + x_i\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_{i+1})\pi\left(x_{i+1} - \Delta x + x_{i+1}\right)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_{i+1})\pi\left(2x_{i+1} - \Delta x\right)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_{i+1})\pi\left(2x_{i+1}\right)\Delta x - \lim_{n\to\infty} \Delta x \sum_{i=0}^{n-1} f(x_{i+1})\pi\Delta x \\
&= \int_{x=a}^{x=b} 2\pi x f(x)\,\mathtt{d}x - \lim_{n\to\infty} \frac{b-a}{n} \int_{x=a}^{x=b} f(x)\pi\,\mathtt{d}x \\
&= \int_{x=a}^{x=b} 2\pi x f(x)\,\mathtt{d}x - 0.
\end{aligned}$$

Thus, the exact volume is given by

$$V = \int_{x=a}^{x=b} 2\pi x f(x)\,\mathtt{d}x.$$

We now use this to finish our analysis of the sphere via shells that we began in the above exercise.

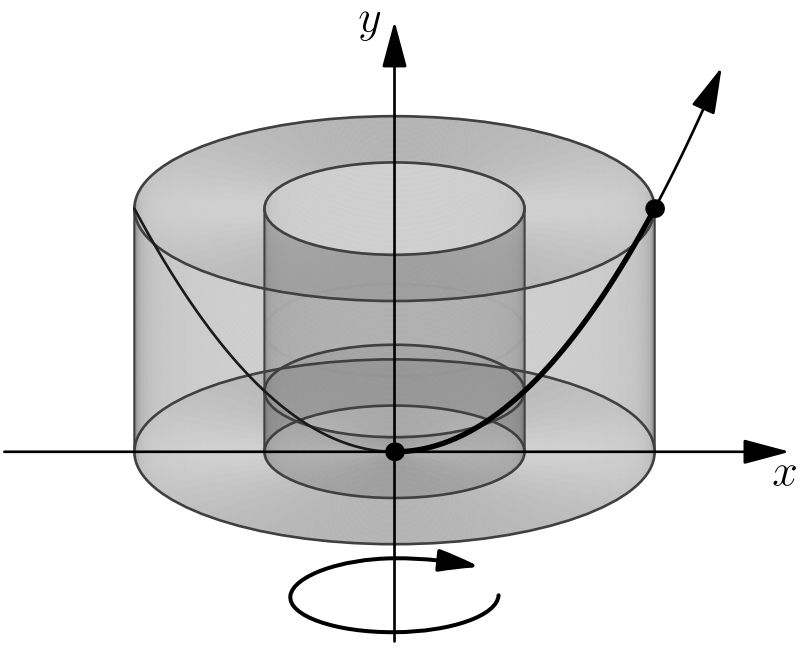
Exercise 2.2.13. Exact Volume of a Sphere via Shells ☕☕

- Since the points on the unit circle satisfy the equation $x^2 + y^2 = 1$, we can solve for y to obtain a function $g(x)$ that represents the QI y-coordinate of the point on the circle at location x.

- Rather than using just $g(x)$ in our cylindrical shells integral, we'll use $f(x) = 2g(x)$. Why is this the case?

- Use our shells formula $V = \int_{x=a}^{x=b} 2\pi x f(x)\,\mathrm{d}x$ with bounds $x = 0$ and $x = 1$ and height function $f(x) = 2\sqrt{1 - x^2}$ to find the exact volume of the unit sphere.

We now analyze another 3D shape using shells. Suppose we take the region under $f(x) = x^2$ between $x = 0$ and $x = 1$ and revolve it around the y-axis. We could estimate the volume using two cylindrical shells as follows:



- One cylinder of height one-fourth and radius one-half, centered at the y-axis. We can consider this to be a shell where the inner deleted cylinder had radius zero.
- One cylinder of height one and radius one, centered at the y-axis, but with a cylinder of height one and radius one-half deleted out of the middle of it.

Exercise 2.2.14. Volumes Approximated by Shells ☕☕

- Compute the approximate volume of that region by adding the volumes of the two cylindrical shells described above.

- Draw the same region but this time split it into four cylindrical shells with x-coordinates at zero, one-quarter, one-half, three-quarters, and one. Draw a diagram showing the shells and compute the approximate volume. How does this compare to the previous approximation?

- Use the cylindrical shells formula to compute the exact volume of the region under the parabola, revolved about the y-axis. Specifically, evaluate the integral

$$V = \int_{x=0}^{x=1} 2\pi x f(x)\,\mathtt{d}x = \int_{x=0}^{x=1} 2\pi x \cdot x^2\,\mathtt{d}x.$$

How does the exact volume compare to the approximations?

Exercise 2.2.15. Volume of a Cone ☕☕

Use integration by cylindrical shells to compute the volume of a cone with circular base of radius r and height h. Verify you get the same result! (**Hint:** To set up this region, place the center of the circular base at the origin and then obtain your $f(x)$ from slope-intercept form of the line connecting the points $(0, h)$ and $(r, 0)$.)

Note that we can extend our formula for volume by shells using our area between curves and find the volume of a solid formed by rotating the area between two functions $f(x)$, $g(x)$ where $f(x) > g(x)$ on the interval (a, b) about the y-axis as

$$V = \int_{x=a}^{x=b} 2\pi x \left(f(x) - g(x)\right) \mathtt{d}x.$$

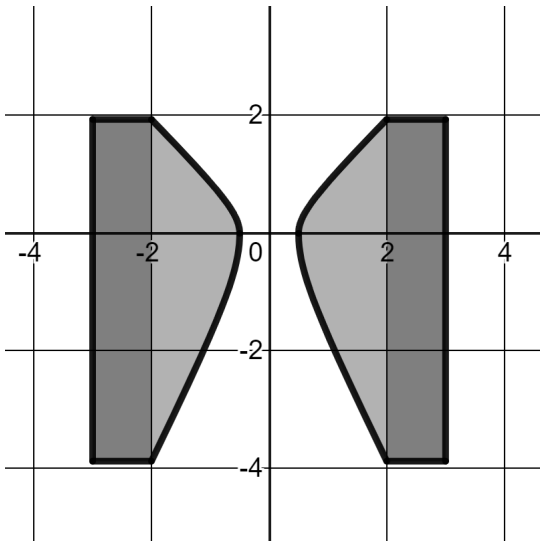
In the sphere example, one could think of that extra factor of 2 that sneaks in via this formula instead of by symmetry. That is, the top half of a unit circle is given by $f(x) = \sqrt{1-x^2}$ and the bottom half is given by $g(x) = -\sqrt{1-x^2}$, so their difference is

$$f(x) - g(x) = \left(\sqrt{1-x^2}\right) - \left(-\sqrt{1-x^2}\right) = \sqrt{1-x^2} + \sqrt{1-x^2} = 2\sqrt{1-x^2}.$$

Exercise 2.2.16. Volume of a Rotational Solid

Most rocket motors use a type of nozzle called a **de Laval** nozzle (after the engineer Gustaf de Laval, who adapted Giovanni Venturi's design for use in an impulse turbine). The de Laval nozzle is also called a CD or converging diverging nozzle. The nozzle design takes flowing gas and converges it to a restricted flow, then diverges it back out more gradually. The rocket nozzle has

circular symmetry. Suppose we're going to mill a de Laval nozzle out of a 3 inch radius cylinder. A graph of the cross section of the nozzle is below.



It will be easiest to find the volume of the nozzle in two parts. Note that the outermost section is just a cylinder! The equation for the upper portion of the inner section is $f(x) = \frac{1}{2}\sqrt{4x^2 - 1}$ and the equation for the lower portion of the inner section is $g(x) = -\sqrt{4x^2 - 1}$. A calculator will be useful to find decimal approximations for these solutions.

- What should the bounds for integration be to find the volume of the inner section of the nozzle?

- What is the volume of the interior portion of the nozzle?

- What is the volume of the outer portion of the nozzle?

- What is the overall volume of the nozzle?

2.3 Volume by Cross Sections and Discs

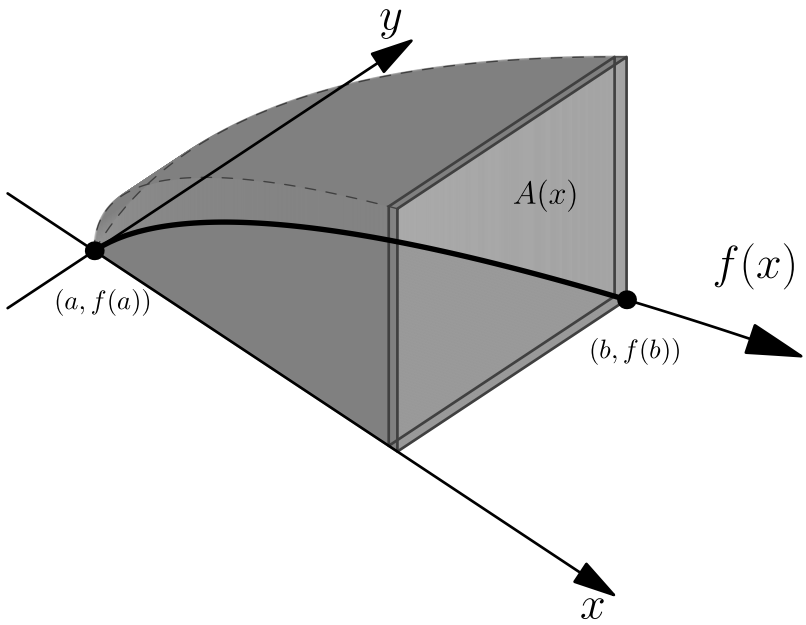
Here we introduce a second method of computing volume: volume by cross sections.

Volume by Cross-sectional Area

Recall that we calculate the area of a planar region by integrating the height at each x-coordinate; here we compute volume of a 3D solid by integrating the area at each x-coordinate. More formally, we say that the volume of a 3D figure that starts at $x = a$ and ends at $x = b$ with the function $A(x)$ representing the area of the cross-section at coordinate x is given by the integral of the cross-sectional areas.

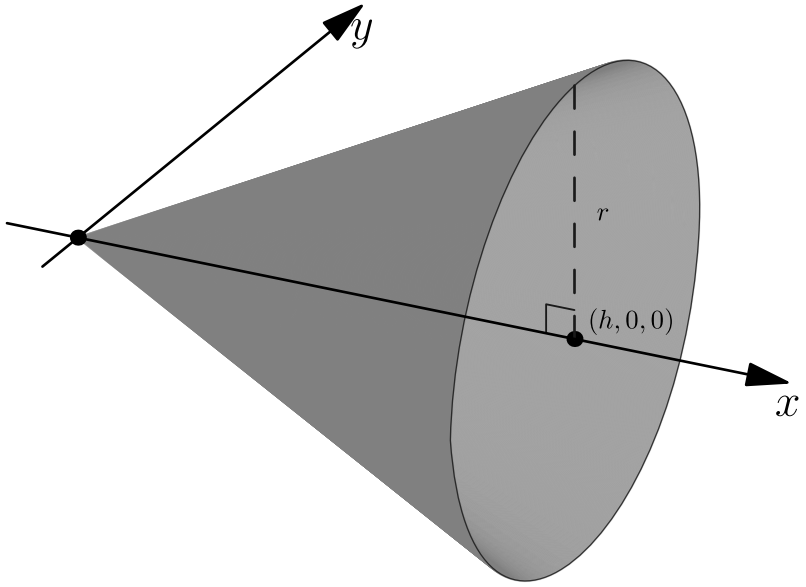
Formula 2.3.1. Volume by Cross Sections

The volume of the solid from $x = a$ to $x = b$ with cross-sectional area $A(x)$ is $V = \int_{x=a}^{x=b} A(x)\,\mathtt{d}x$.

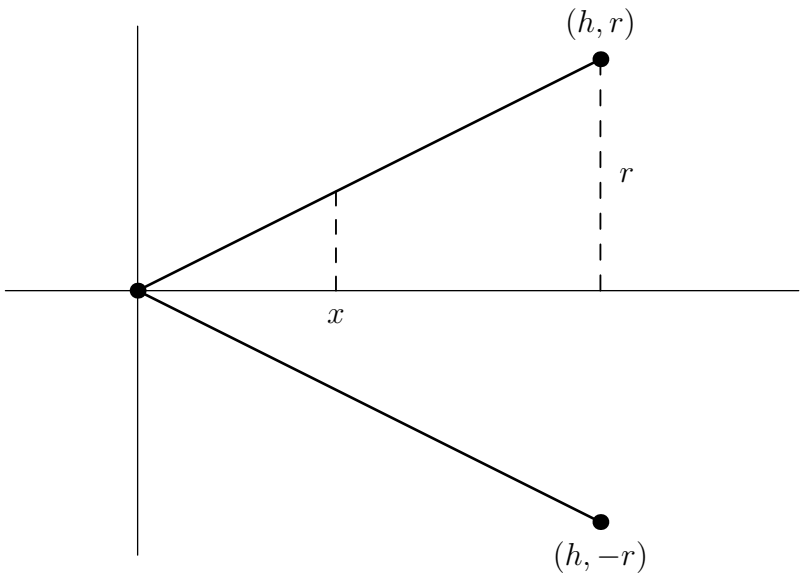


The Cone, Pyramid, and Tetrahedron

Let's try this out on a cone! Suppose we have a right circular cone of height h and radius r. Place the cone so that the vertex lies at the origin and the center of the base lies at the point $(h, 0, 0)$.



Any cross section parallel to the base is clearly a circle. Thus to compute the area of each circle we just need to find the radius of an arbitrary cross section at location x. To help us, we imagine a 2D "side view" of the middle of the cone.



Notice the top boundary of this shape is the graph of the linear function $f(x) = \frac{r}{h}x$. This height is exactly the radius of the circular cross section of the cone at location x.

Exercise 2.3.2. Check the Boundary ☕

Briefly explain why the given formula $f(x) = \frac{r}{h}x$ is the correct formula for the top boundary!

Example 2.3.3. Volume of a Cone by Cross Sections

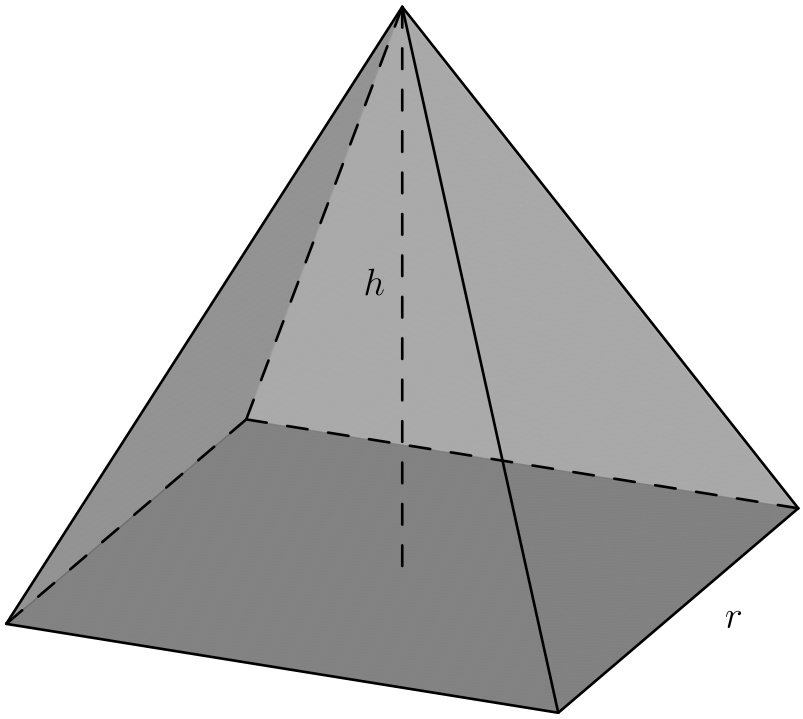
We can now set up and evaluate our volume integral:

$$\begin{aligned} V &= \int_{x=0}^{x=h} A(x)\,\mathtt{d}x \\ &= \int_{x=0}^{x=h} \pi\left(\frac{r}{h}x\right)^2 \mathtt{d}x \\ &= \int_{x=0}^{x=h} \pi\frac{r^2}{h^2}x^2\,\mathtt{d}x \\ &= \pi\frac{r^2}{3h^2}x^3\Big|_{x=0}^{x=h} \\ &= \pi\frac{r^2}{3h^2}h^3 - 0 \\ &= \frac{1}{3}\pi r^2 h. \end{aligned}$$

Note that this is actually a very clean formula; it says the area of a cone is one-third times the area of the base times the height.

Exercise 2.3.4. The Pyramid ☕☕

- Consider a square base pyramid of side length r and height h. What would you conjecture for the volume of this solid based on our cone computation above?



- Use integration of cross sectional area to verify your conjecture and formally compute the volume of the pyramid. **Hint!** The setup of the integral will be very similar to the cone, except our cross sections are squares instead of circles.

Exercise 2.3.5. A Tetrahedron ☕☕☕

- In three dimensions, plot a tetrahedron that has vertices $(0,0,0), (a,0,0), (0,b,0)$, and $(0,0,c)$. Based on how the volumes came out for the cone and pyramid, what would you suspect for the volume of this figure?

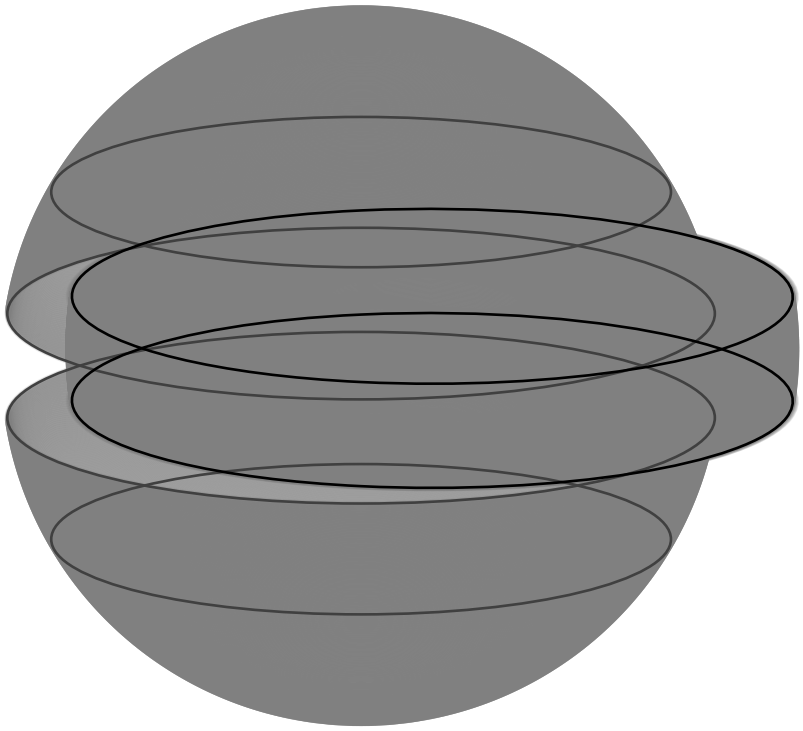
- Use integration of cross sections to find the volume of that tetrahedron.

Exercise 2.3.6. Other Bases ☕☕☕☕

What happens if you start with other shapes as the base of your figure? If you form a solid by connecting the boundary of the base to a point with line segments, do you always get just one-third times the area of the base times the height as the volume? Or, can you find some bases for which this formula does not hold?

The Sphere

A sphere of radius r is a solid of revolution constructed by rotating a circle of radius r centered at the origin about either the x or y axis. Here, we use cross-sectional area, which many sources call the "disc method" since a cross-section of a sphere is a circular disc.



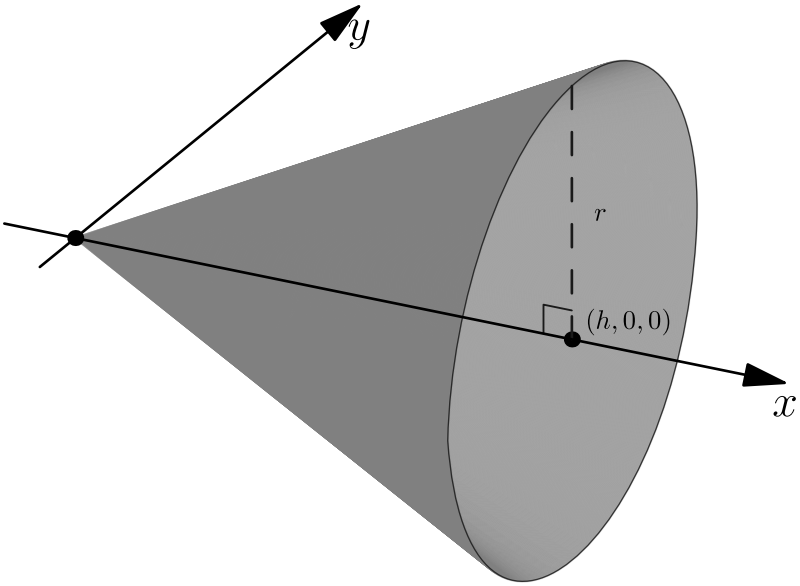
Exercise 2.3.7. The Volume of a Sphere ☕☕

- Draw a 2D "side view" of the sphere much like we did for the cone. What is the formula for the top boundary curve?

- Compute the volume of a sphere via an integral of cross-sectional area.

The Methods of Discs and Washers

This method of finding areas by cross-sections is often adapted into what is often called volume by discs or volume by washers. This method has a similar use case to the volume by cylindrical shells we saw earlier, but with some key differences. While volume by cylindrical shells naturally gives the volume of an object with circular symmetry made up of functions rotated about the y-axis, volume by discs or washers naturally gives the volume of an object with circular symmetry made up of functions rotated about the x-axis.



In fact, many of the examples in this section can be considered using the method of discs. The method of discs is just cross-sectional areas where the cross sections are circles with radius equal to the value of our function at a given point.

Formula 2.3.8. Method of Discs

The volume of an object created by revolving $f(x)$ about the x-axis over the interval $[a, b]$ is

$$V = \pi \int_a^b (f(x))^2 \, \mathtt{d}x.$$

The method of washers is similar, but allows for both an outer and an inner function.

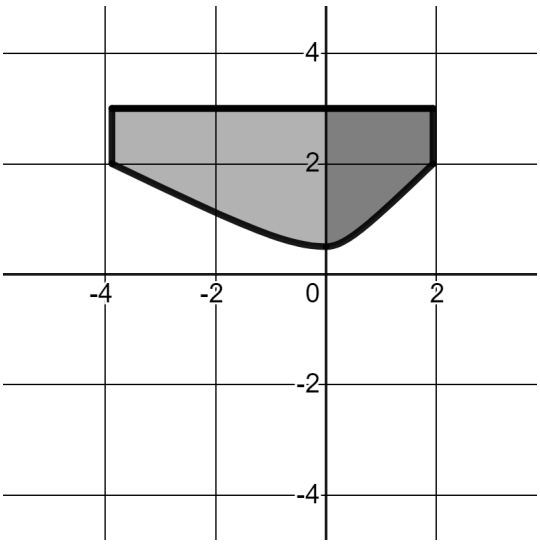
Formula 2.3.9. Method of Washers

The volume of an object created by revolving the region between $f(x)$ and $g(x)$ where $f(x) > g(x)$ about the x-axis over the interval $[a, b]$ is

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2 \, \mathtt{d}x.$$

Exercise 2.3.10. The Rocket Nozzle Again

We can set up the same rocket nozzle that we made in exercise 2.2.16, though this time we'll rotate about the x-axis.



We'll again have to break up the volume into two parts, this time a left part and a right part.

- On the left section, the upper function is $f(x) = 3$ and the lower function is $g(x) = \frac{1}{2}\sqrt{1+x^2}$ and the interval is from -3.873 to 0. Find the volume of the left section.

- On the right section, the upper function is $f(x) = 3$ and the lower function is $g(x) =$

$\frac{1}{2}\sqrt{1+4x^2}$ and the interval is from 0 to 1.936. Find the volume of the right section.

- What is the total volume of the nozzle? How does this compare to the volume found in exercise 2.2.16?

It is worth noting that this process works equally well if we slice along the y-axis instead of the x-axis.

Exercise 2.3.11. Volume of a Parabolic Bowl (Para-bowl-a?)

Consider the region bounded by $y = x^2, y = 0, x = 0$, and $x = 1$. Revolve this 2D region about the y-axis to create a bowl. Note that this is exactly the shape described in exercise 2.2.14. This would be an absolute mess to examine via cross sections in the x direction, since they do not have an easily describable shape. We can use the method of washers here. However, since the region is rotated about the y-axis, we'll need our upper and lower functions to be $f(y)$ and $g(y)$ instead of $f(x)$ and $g(x)$ respectively.

- Rewrite $y = x^2$ as x as a function of y.
- What function of y should represent the outer portion of our bowl?
- What should the bounds for integration be?

- Set up and solve the integral to find the area of the parabolic bowl.

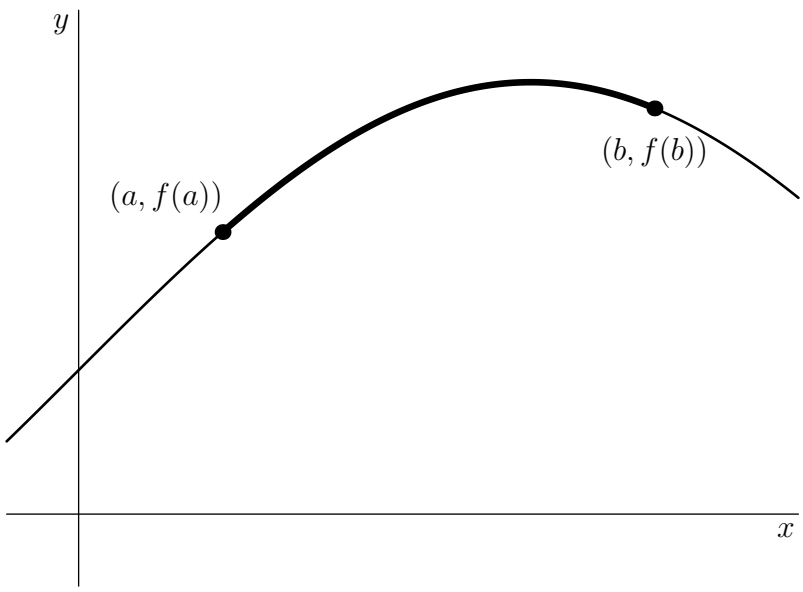
- Consider the cylinder centered at the y-axis with height 1 and radius 1. What percent of the volume of that cylinder is occupied by the parabolic bowl?

- How do your results compare to what you computed in Exercise 2.2.14?

2.4 Arc Length and Surface Area

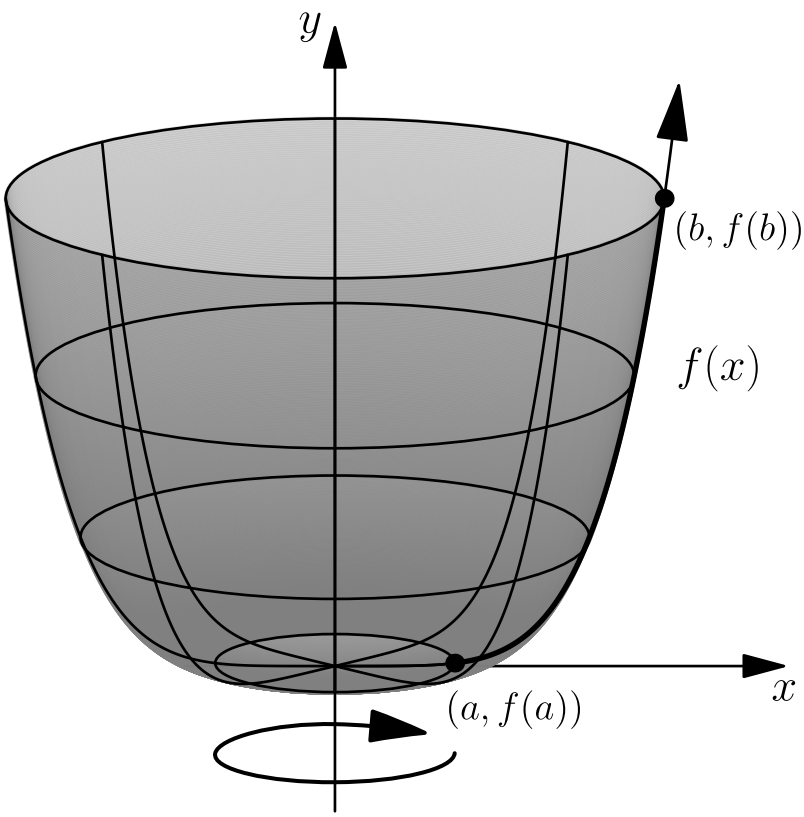
In this section, we will look at a pair of formulas built out of integrals. Let's get them both down here, and we'll play with them and explain them as we go!

1. **Arc Length.** The length of the graph of a function $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by $L = \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x$.



2. **Surface Area of a Surface of Revolution.** If the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is revolved around the y-axis, the surface area is given by

$$SA = \int_{x=a}^{x=b} 2\pi x \sqrt{1 + (f'(x))^2}\, \mathtt{d}x.$$



Exercise 2.4.1. Comparing the Formulas ☕

Look at the two formulas given above. How do those integrands compare to one another? Look back at the formulas for area under a curve vs cylindrical shells; how do those two integrands compare to one another?

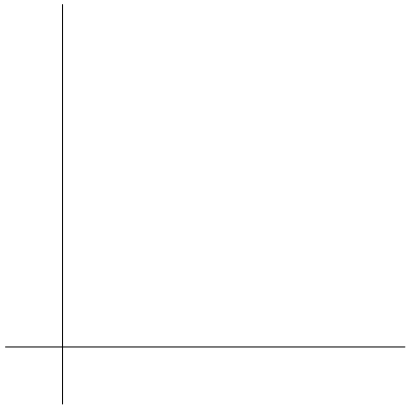
Let's now do a little example of each and throughout analyzing the example, convince ourselves that each formula is correct.

The Arc Length Formula

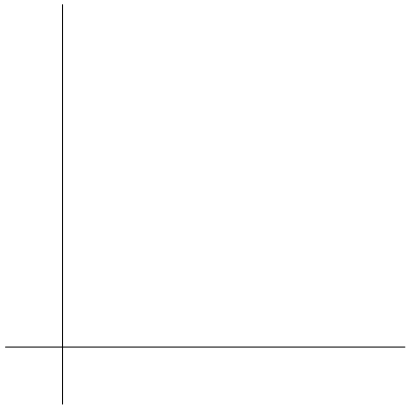
The arc length formula is in essence the following idea: to approximate the length of a curve, let's split it up into line segments, compute each of the line segment lengths using the Pythagorean Theorem, and then take the limit as the number of line segments goes to infinity. Time to carry this out on a good old vanilla parabola!

Exercise 2.4.2. Approximating the Length of a Parabola with Line Segments ☕☕

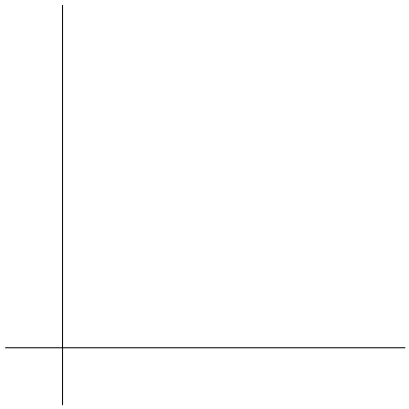
- On the axes below, graph the function $f(x) = x^2$ from the point $A = (0, 0)$ to $E = (1, 1)$. Estimate the length of this arc by just connecting those two endpoints with a straight line and calculating its length via the Pythagorean Theorem.



- Again graph the function $f(x)$ from the point $A = (0, 0)$ to $E = (1, 1)$, but on this graph also include a label for the point $C = (\frac{1}{2}, \frac{1}{4})$. This time lets estimate the length of this arc using two line segments. Specifically, calculate the lengths of $\overline{AC}$ and $\overline{CE}$ and add their lengths to estimate the arc length of the parabola. Did your estimate go up or down by using two segments instead of just one? Does this make sense?



- Define A, C, and E as before. Define B to be the point on the parabola with x-coordinate one-fourth and D to be the point on the parabola with x-coordinate three-fourths. Again estimate the length of the curve by adding the lengths of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$. What happens to the estimate? Did it increase or decrease? Did it get more or less accurate as compared to the true arc length?



Ok, at this point we get the idea. More line segments will result in a more accurate approximation, and the limit of these approximations as the number of line segments goes to infinity will produce the exact result. Here is a derivation that shows this process will result in exactly the right formula.

Let $x_0, x_1, x_2, \ldots, x_n$ be equally spaced points along the x-axis from a to b. That is, $x_0 = a$, $x_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta x = x_{i+1} - x_i = \frac{b-a}{n}$.

With this setup, if we want the length of a line segment connecting points x_{i+1} and x_i, we would use the Pythagorean Theorem to obtain:

$$\sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}$$

as the length.

$$\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{1 + \frac{(f(x_{i+1}) - f(x_i))^2}{(\Delta x)^2}} \sqrt{(\Delta x)^2} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{\Delta x}\right)^2} \Delta x \\
&= \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x
\end{aligned}$$

And thus we have our formula for arc length!

Formula 2.4.3. Arc Length Formula

The graph of the function $f(x)$ from $x = a$ to $x = b$ has length

$$L = \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x.$$

Exercise 2.4.4. Exact Arc Length of the Parabola ☕☕☕

- We now apply this integral formula to calculate the exact length of our parabolic segment. Plugging in the formula $f(x) = x^2$ into the arc length integral, we obtain:

$$L = \int_{x=0}^{x=1} \sqrt{1 + (2x)^2}\, \mathtt{d}x$$

Finish the evaluation of this integral. (**Hint:** The arc length integral will be quite difficult!

See Example 1.2.12 for help.)

- How does the exact value of the arc length compare to the approximations?

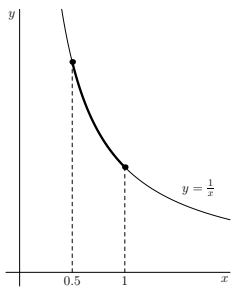
Example 2.4.5. Arc Length of a Hyperbola

Suppose we wish to find the arc length L of the hyperbola

$$xy = 1$$

between the points $\left(\frac{1}{2}, 2\right)$ and $(1, 1)$. We begin by solving for the y coordinate in order to express this section of the graph as a function. Dividing both sides by x produces

$$y = f(x) = \frac{1}{x}.$$



We wish to find the length of the graph of this function between $x = \frac{1}{2}$ and $x = 1$. We plug into the arc length formula to obtain

$$L = \int_{x=1/2}^{x=1} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x = \int_{x=1/2}^{x=1} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, \mathtt{d}x = \int_{x=1/2}^{x=1} \frac{\sqrt{1 + x^4}}{x^2}\, \mathtt{d}x.$$

Though we have demonstrated how to set up the arc length integral, it turns out that this integral is quite difficult to evaluate! In particular, the integrand has no closed form antiderivative. We will revisit this example in Section 4.3 where we'll have extra tools.

Exercise 2.4.6. Checking the Example ☕

In the above example...

- Verify the algebra on the last step, where the integrand is simplified from $\sqrt{1+\left(-\frac{1}{x^2}\right)^2}$ to $\frac{\sqrt{1+x^4}}{x^2}$.

- Try evaluating the integral via the trigonometric substitution $x = \sqrt{\tan(\theta)}$. It may look promising at first! However, where do you get stuck?

Circumference of a Circle

We now use the arc length integral to compute the circumference of a circle. Though one can take the formula for the circumference of a circle simply as the definition of π, it is still nice to go through this as proof of concept.

Exercise 2.4.7. Circumference of a Circle ☕☕

- Use the arc length formula to calculate the circumference of a circle with radius r.

- Take the derivative of your formula for area of a circle with respect to r. How does it relate to your formula for the circumference of a circle? Draw a picture and indicate why this

freakish coincidence actually geometrically makes sense.

Lengths of Some Other Fun Arcs

Exercise 2.4.8. Other Arcs ☕☕☕

Draw the graph and compute the length of each of the following arcs using our arc length formula.

- Line segment from a point (x_0, y_0) to a point (x_1, y_1). (**Hint:** To get your function, use the point-slope form of a line and then solve for y.) How does this compare to the length the Pythagorean Theorem would give you?

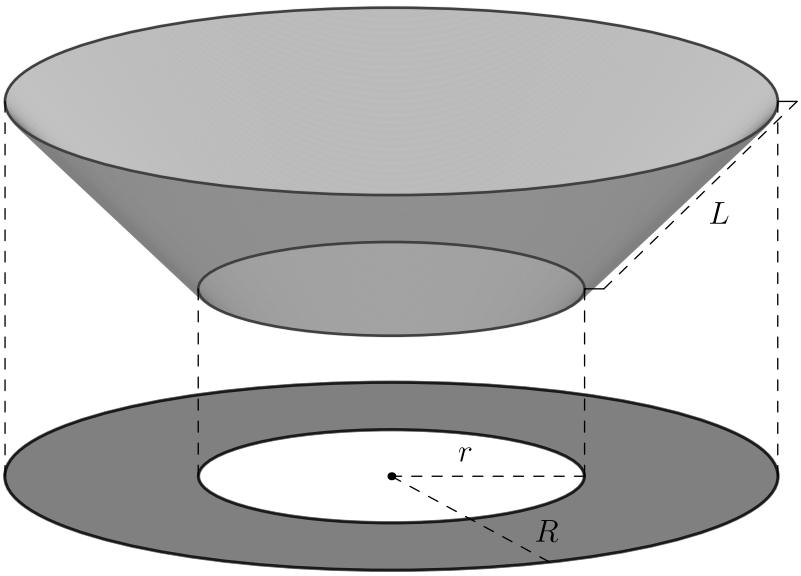
- The graph of $f(x) = \ln(x)$ from $x = 1$ to $x = e$. Take an approximation using two line

segments, and then get the exact length using an arc length integral. How do they compare?

- The graph of $f(x) = e^x$ from $x = 0$ to $x = 1$.

Surface Area

For the surface area of a surface of revolution, we use the frustum of a right circular cone as the object that approximates the surface, much as we used line segments for arc length. Thus, to get off the ground, we must figure out what the surface area of a frustum is. Suppose it has inner radius r and outer radius R and let L be the diagonal length, top to bottom, of the frustum. Project the shape onto a flat plane sitting under it to produce a washer whose outer radius is R and inner radius is r.



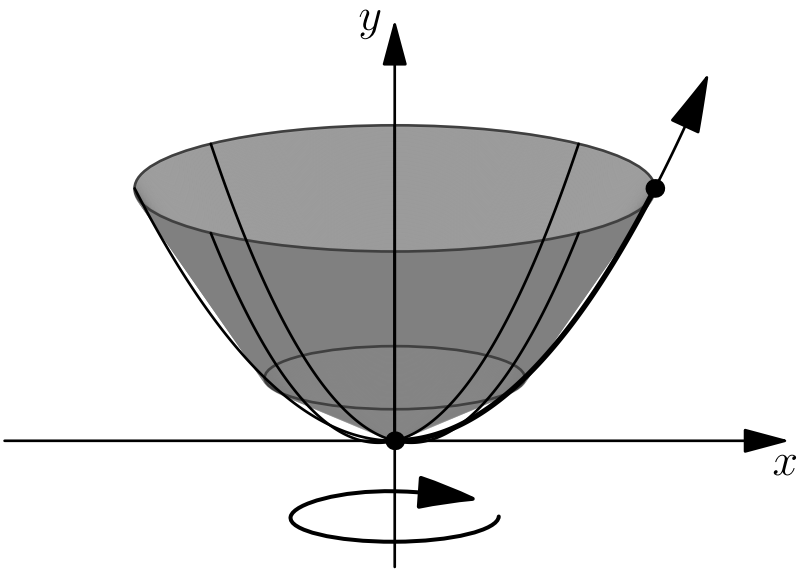
Notice that the washer in the plane has area $\pi(R^2 - r^2)$. Also, the radial lengths got uniformly scaled by a factor of $\frac{L}{R-r}$, since every segment of length $R - r$ got stretched into a segment of length L. Thus we have that the area of the frustum is $\frac{L}{R-r} \cdot \pi(R^2 - r^2)$, which simplifies to

$$SA = \pi L(R + r).$$

Exercise 2.4.9. Surface Area of a Paraboloid ☕☕☕

To test this out, consider the parabola $y = x^2$ from $x = 0$ to $x = 1$. Create a surface by revolving this curve around the y-axis.

- Approximate the surface area of this region using two frusta, one that goes from $x = 0$ to $x = 1/2$, and one that goes from $x = 1/2$ to $x = 1$.



- As in the previous cases, we can see that chopping it up into more approximating regions will increase the accuracy of our approximation as the number of these goes to infinity. However, actually adding up the surface areas by hand would become frusta-ting, so instead we just set up a limit of a summation. Draw a diagram and set up the limit of a sum of surface areas of frusta, similarly to how we did for the arc length formula. Fill in the middle part of the derivation below:

$$\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \pi \sqrt{(x_{i+1}-x_i)^2 + (f(x_{i+1}) - f(x_i))^2}\,(x_{i+1}+x_i) \\
&= \\
&= \\
&= \\
&= \\
&= \\
&= \int_{x=a}^{x=b} 2\pi x \sqrt{1+(f'(x))^2}\,\mathrm{d}x
\end{aligned}$$

- Use this integration formula to find the exact surface area of the parabolic bowl. How does it compare to your approximation?

Exercise 2.4.10. Surface Area of a Sphere ☕☕

Notice that a sphere is a surface of revolution created by a circle centered at the origin. Since the right-hand side of a circle is not the graph of a function, we will just use the top right quarter circle to revolve about the y-axis and then multiply by 2 to pick up the bottom half of the sphere.

- In particular, use the function $f(x) = \sqrt{r^2 - x^2}$ for $x = 0$ to $x = r$ in our surface area

integral to find the formula for the surface area for a sphere of radius r.

- Take the derivative of the formula for sphere volume with respect to r. How does it relate to the formula for surface area? Again, draw a diagram explaining this geometrically.

2.5 Center of Mass

Center of Mass Defined by Moment Integrals

Let R be a region bounded above by $f(x)$, below by $g(x)$, on the left by $x = a$, and on the right by $x = b$. Let $(\bar{x}, \bar{y})$ be the center of mass of R, a region of mass m. The formulas for the x- and y- coordinates of the center of mass of a 2D plate in the shape of R of uniform density ρ are as follows:

$$\bar{x} = \frac{1}{m}\int_{x=a}^{x=b} \rho \cdot x\,(f(x) - g(x))\,\mathtt{d}x$$
$$\bar{y} = \frac{1}{m}\int_{x=a}^{x=b} \rho \cdot \frac{1}{2}\,(f(x) + g(x))\,(f(x) - g(x))\,\mathtt{d}x$$

The integrals above are often referred to as *moments*, a term physicists use to describe the turning effect of a force. Thus, the definition is often stated in terms of these moment integrals M_y and M_x. We define the following abbreviations:

$$m = \int_{x=a}^{x=b} \rho \cdot (f(x) - g(x))\,\mathtt{d}x$$
$$M_y = \int_{x=a}^{x=b} \rho \cdot x\,(f(x) - g(x))\,\mathtt{d}x$$
$$M_x = \int_{x=a}^{x=b} \rho \cdot \frac{1}{2}\,(f(x) + g(x))\,(f(x) - g(x))\,\mathtt{d}x$$

This gives a short formula for the center of mass of the region R.

Formula 2.5.1. Center of Mass

The center of mass of R is the point

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

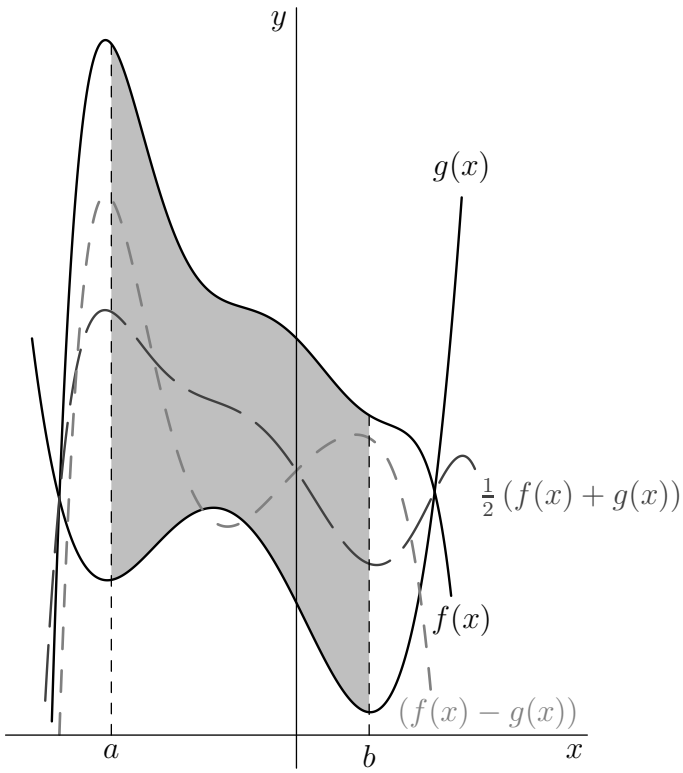
The physical interpretation of the center of mass is quite nice; it is the point where we could theoretically balance our 2D plate on a narrow post. Also, in many physics or engineering applications, one can replace the entire figure with a point mass located at the center of mass. This simplifies many otherwise difficult problems. If you are ever going to take the *Fundamentals of Engineering* exam, this is a huge trick people use!

For the remainder of this section, we make the simplifying assumption that the unit of mass we are using is 1 unit of mass per unit of area, or equivalently, that the density constant $\rho = 1$. In the context of a center of mass calculation, this is a harmless assumption to make, because ρ would show up in both the numerator and denominator of the formula for $\bar{x}$ and cancel thusly:

$$\bar{x} = \frac{M_y}{m} = \frac{\int_{x=a}^{x=b} \rho \cdot x\,(f(x) - g(x))\,\mathtt{d}x}{\int_{x=a}^{x=b} \rho \cdot (f(x) - g(x))\,\mathtt{d}x} = \frac{\rho \int_{x=a}^{x=b} \cdot x\,(f(x) - g(x))\,\mathtt{d}x}{\rho \int_{x=a}^{x=b} (f(x) - g(x))\,\mathtt{d}x} = \frac{\int_{x=a}^{x=b} x\,(f(x) - g(x))\,\mathtt{d}x}{\int_{x=a}^{x=b} (f(x) - g(x))\,\mathtt{d}x}.$$

Exercise 2.5.2. Seeing y Mass Equals Area ☕

Perform the analogous calculation for the formula for $\bar{y}$, and confirm that one can also take density $\rho = 1$ in that formula.



These formulas are plausible, as in some sense we are measuring the tendency to rotate about an axis. We can see the first integral is accumulating x (the length of the torque arm) times $(f(x) - g(x))$, which is measuring the height of the figure at location x. The second integral is much less intuitive, though we can see it again as accumulating the heights times the average of the y coordinate of the boundaries, which in aggregate gives us a measure of the central tendency of the vertical component of the figure. These formulas will be proven in Calc III via double-integration.

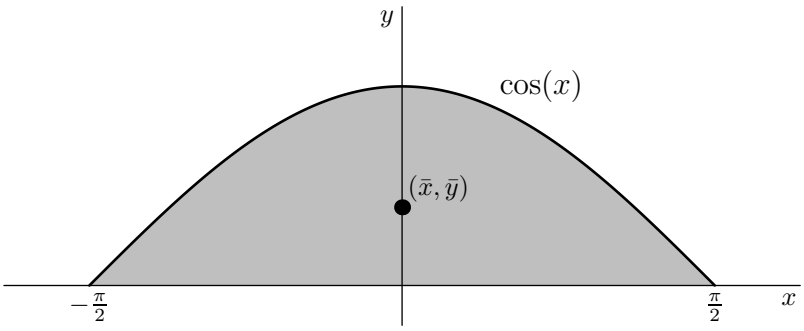
Example 2.5.3. Center of Mass of a Cosine Gumdrop

Consider the region R bounded by the graphs of $y = 0$ and $y = \cos(x)$ between $x = -\pi/2$ and $x = \pi/2$.
To find the center of mass, we have three quantities we need to compute: the area m, the y-axis moment M_y, and the x-axis moment M_x. We compute each now, using $f(x) = \cos(x)$ and $g(x) = 0$:

- $m = \int_{x=-\pi/2}^{x=\pi/2} (\cos(x) - 0)\,\mathrm{d}x = \sin(x)]_{x=-\pi/2}^{x=\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) = 1 - (-1) = 2$
- $M_y = \int_{x=-\pi/2}^{x=\pi/2} x\,(\cos(x) - 0)\,\mathrm{d}x = x\sin(x) + \cos(x)]_{x=-\pi/2}^{x=\pi/2} = \frac{\pi}{2} - \left(-\frac{\pi}{2}(-1)\right) = 0$
- $M_x = \int_{x=-\pi/2}^{x=\pi/2} \frac{(\cos(x)+0)}{2}\,(\cos(x) - 0)\,\mathrm{d}x = \frac{1}{2}\left(\frac{x}{2} - \frac{1}{4}\sin(2x)\right)]_{x=-\pi/2}^{x=\pi/2} = \frac{\pi}{4}$

Thus, the center of mass of R is

$$(\bar{x}, \bar{y}) = (M_y/m, M_x/m) = (0/2, (\pi/4)/2) = (0, \pi/8).$$



We now gather a little geometric intuition on the above example (do we ... reflect?) as well as check the details of the integration.

Exercise 2.5.4. Integration and Intuition ☕

In the above example...

- ... how could we have predicted that $\bar{x}$ would be zero just from the shape of the region?

- ... how could we have predicted that $\bar{y}$ should be a bit less than one half? Specifically, think about how tall the region R is and then ask if the region is more top heavy or more bottom heavy? Verify that $\pi/8$ is in fact a bit less than one half.

- ... the M_y integral required use of Integration by Parts. Show the details of this antidifferentiation.

- ... the M_x integral required use of a trig identity. Which identity was used? Show the details of this antidifferentiation.

Exercise 2.5.5. A Sine Gumdrop ☕☕

Consider the region R bounded by the graphs of $y = 0$ and $y = \sin(x)$ between $x = 0$ and $x = \pi$.

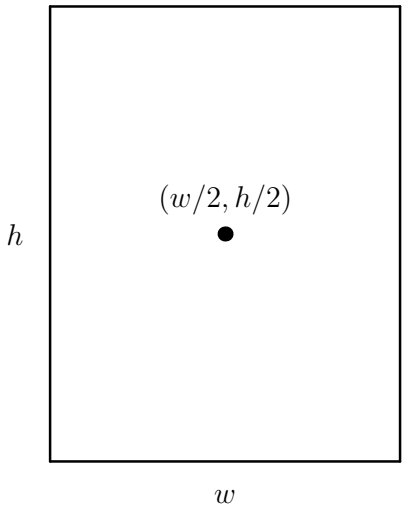
- Based on the center of mass computed in Example 2.5.3, what would you suspect this new center of mass to be?

- Verify or refute your suspicion by computing the center of mass using moment integrals.

We now play with center of mass for a few of our common shapes.

The Rectangle

Suppose we have a rectangle of width w and height h. If we coordinatize the rectangle by placing the lower-left corner at the origin and the bottom side on the positive x-axis, we would expect that the center of mass should come out to be the point $(w/2, h/2)$ since that intuitively is the "middle" of our rectangle. Let's try out the integrals and verify this is indeed what we get. Specifically:

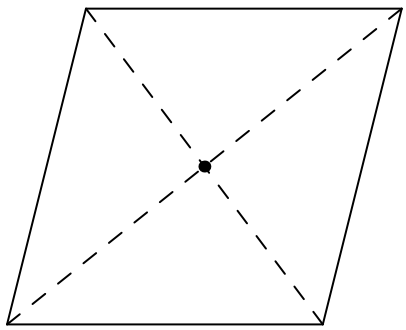


Exercise 2.5.6. A Rectangle ☕☕

- Let $f(x) = h$ the constant function whose graph is a horizontal line at height h, be the top of the rectangle. Let $g(x) = 0$, the constant function whose graph is a horizontal line at height zero, be the bottom of the rectangle. Let the lines $x = 0$ and $x = w$ be the left and right boundaries of the rectangle. Sketch this figure below.

- Use our center of mass formulas to compute $(\bar{x}, \bar{y})$ and verify that it is in fact $(w/2, h/2)$.

The Parallelogram



Again by intuition, we would imagine the center of mass of a parallelogram should be at the intersection of the diagonals. Let's see if this is in fact where the center of mass of a parallelogram must lie. We will do this in three steps:

Exercise 2.5.7. Parallelogram

- Coordinatize the parallelogram. Choosing "nice" coordinates for our figure is the first step. Without loss of generality, we can make one corner of our parallelogram to be the origin and choose one side to lie on the positive y-axis, much like the rectangle. The beauty of this is then the parallelogram is fully determined by only three arbitrary parameters, the horizontal and vertical coordinates of the upper right corner, and the vertical coordinate of

the upper left corner. Call these a, b, and c respectively.

- Compute linear equations for the diagonals in terms of a, b, and c. Solve a simultaneous system of linear equations to find the coordinates of the intersection of these lines.

- Use the integrals for center of mass to compute the actual center of mass. See if it agrees with the coordinates of the point of intersection computed above.

Ok nice, so it did work out to be the same! Well it kind of had to there, right? I mean what else could the center of mass of a parallelogram have been?

The Triangle

And now for a case where the end of the story is far less predictable! Consider the triangle. For the triangle there are *four* completely reasonable geometric guesses as to what the "center" of a triangle could be.

Exercise 2.5.8. Different Notions of Center of a Triangle ☕

Sketch corresponding diagrams for four beautiful theorems of Euclidean geometry.

- The altitudes of a triangle intersect in a point. (This point is called the *orthocenter*.)

- The medians of a triangle intersect in a point. (This point is called the *barycenter*.)

- The perpendicular bisectors of a triangle intersect in a point. (This point is called the

circumcenter.)

- The angle bisectors of a triangle intersect in a point. (This point is called the *incenter*.)

Of these different notions of "center" of a triangle, which one is actually the center of mass? Or is it something different entirely that is not on our list of guesses above? Well, let's figure it out. The steps for determining this are essentially the same as for the parallelogram.

Exercise 2.5.9. The Triangle ☕☕☕

- Coordinatize the triangle. Choosing "nice" coordinates for our figure is the first step. Without loss of generality, we can make one corner of our triangle to be the origin and choose one side to lie on the positive y-axis, much like the rectangle and parallelogram. Now the triangle is fully determined by only three arbitrary parameters, the horizontal and vertical coordinates of the only corner not on the y-axis, and the y-coordinate of the point on the y-axis but not at the origin. Similar to the parallelogram, respectively call these a,b, and c. Note how helpful this is; a generic triangle in the plane would be determined by six

parameters! (two per corner)

- Note that for extreme cases, the circumcenter and orthocenter can actually lie outside of the triangle. This means these are likely to be incorrect guesses, as we would intuitively think the center of mass of the triangle should always lie inside the triangle itself. Thus, we won't expend effort trying to find coordinates for the orthocenter or circumcenter. To confirm this, draw one example of a triangle below that has orthocenter outside and one example of a triangle that has circumcenter outside.

- Going down the list, let's find the barycenter. The process is the same as above. We find equations for two of the medians and find their point of intersection. (Note that thanks to Euclid's theorem that the three medians intersect in a point, it is not necessary to use the equation for the third median as it is guaranteed to pass through the same point of

intersection.)

- Use the integrals for center of mass to compute the actual center of mass. See if it agrees

with the coordinates of the point of intersection computed above.

- Explain why at this point we do not need to try the incenter.

The Half Disc

Ok, now for the half disc, where there isn't even a reasonable geometric basis for a guess!

Exercise 2.5.10. The Half Disc

Find the center of mass of an upper-half circle of radius r. Use $f(x) = \sqrt{r^2 - x^2}$ as your upper boundary and $g(x) = 0$ as the lower. Sketch your figure and its center of mass below. (**Hint:** You can determine one coordinate of the center of mass by symmetry. So you only need to compute

one of the moments.)

Exercise 2.5.11. Center Outside ☕☕☕☕

Can you come up with a shape whose center of mass lies outside the shape? Find such an example, or explain why this is not possible.

2.6 Work and Hydrostatic Pressure

While center of mass is a concept that can be thought of purely geometrically in many cases, there are many important concepts from physics and engineering that are hard to even *define* without an integral. Here we see two such important examples, work and hydrostatic pressure.

Work

Exercise 2.6.1. Fairly Unfair

Consider the dialogue between two pirates, Rho and Arg, shown below.

- **Rho.** *Arg! We've found the spot of the buried treasure!*
- **Arg.** *Aye Rho. Legend says it be buried 12 feet deep, and the chest be in the shape of a 3 foot by 3 foot square. Goin' to be hard work diggin' it out through this heavy dirt after a hard day of sailin'.*
- **Rho.** *Alright, we be havin' only one shovel, so I'll dig three feet down and then we'll switch, and you dig the next three while I swill some grog for the both of us.*
- **Arg.** *But that not be fair! If I dig the deeper part of the hole, it be harder on me, lifting the dirt higher to get it out of the hole!*
- **Rho.** *No, we be diggin' the same amount of dirt if we switch halfway. That is where the fair point to switch is, ya' swindling bag o' bones!*

What do you think? Who is right, Rho or Arg? Where would you guess the fair switching point is?

The key idea behind Arg's objection above is a notion from physics known as **work**. It is defined as force applied across a distance. This is often written using the notation shown below.

Definition 2.6.2. Work

Work, represented by W, is a force F applied across a distance D, so

$$W = F \cdot D.$$

Notice that the units on W are a product of the respective units for F and for D. Thus, foot-pounds (in imperial units) and Joules (in metric, which are equivalent to the product of Newtons and meters) are both commonly used units of work. Note that for human-scale situations, one can expect answers containing large orders of magnitude in Joules, as a single Joule is actually quite tiny!

Example 2.6.3. Hoisting a piano

Suppose a 1200 lb grand piano is hoisted vertically on a pulley from ground level up to the balcony

of a tenth-story apartment. Each story is 15 feet. So, in this case, the amount of work done was

$$W = 1200 \text{ lbs} \cdot 10 \text{ stories} \cdot 15 \frac{\text{feet}}{\text{story}} = 180,000 \text{ ft-lbs}.$$

That is, one hundred and eighty thousand foot-pounds of work was done.

Note that in the above example (and in fact in all of the examples of this section), we are not travelling so far from the center of Earth as to have a noticeable change in the strength of gravity, which would change the force as we move the object. Such a change would render the scenario more complicated, and will be discussed when you cover line integrals in your Calculus 3 course!

Exercise 2.6.4. Comparing Work ☕

In a large apartment building, one neighbor carried their 14 lbs of groceries up 3 flights of stairs. The other carried their 10 lbs of groceries up 4 flights of stairs. Who did more work carrying groceries? (Assume equal height to each flight of stairs.)

Of course, situations can be far more complicated. In the above example, the 14 lbs of force exerted by the grocery bag was constant throughout the bag's journey upward. However, one can imagine cases in which the force changes throughout the journey! Let us consider one such case now.

Suppose we revisit Example 2.6.3, but this time instead of imagining a magical massless cord that is being used to hoist it, we have the piano attached to a nice strong 150 foot length of chain that weighs 4 lbs per foot. Six-hundred pounds of chain –half the weight of the piano itself– is probably not entirely insignificant with regards to the work our pulley system is doing! But here is the catch, the chain is getting wound throughout the journey, so there is continuously less chain hanging off the balcony to be pulled upward. Thus, at every distance in the piano's journey, the force will be different. Let us think about how we could calculate the total work in an example.

Example 2.6.5. Approximating the work, hoisting a piano with a strong chain

Imagine we split the piano's 150-foot vertical journey into 3 different 50-foot increments.

- For the first 50 feet, we will estimate that the full 600 pounds of chain is unwound, so the load is 1800 pounds.
- For the second 50 feet, we will estimate that the first third of the chain is gone, so we have just 400 pounds of chain to wind up, and the load is now only 1600 pounds.
- For the last 50 feet, we will estimate that one-third of the chain remains to be wound, so we have just 200 pounds of chain to wind up, and the load is now 1400 pounds.

Adding up the amount of work done on each of the three legs of the journey, we have

$$50 \text{ feet} \cdot 1800 \text{ lbs} + 50 \text{ feet} \cdot 1600 \text{ lbs} + 50 \text{ feet} \cdot 1400 \text{ lbs} = 240,000 \text{ foot-lbs}.$$

Exercise 2.6.6. Refining an estimate

As mentioned in the example above, what we calculated was only an approximation.

- Our estimate in the above example, was it an overestimate or an underestimate? Explain why, in words.
- Divide the same journey into five segments of 30 feet each to get a more accurate approximation. Did the estimate go up or down, and does this make sense?
- Divide the same journey into ten segments of 15 feet each to get an even more accurate approximation. Did the estimate go up or down, and does this make sense?
- Think back to your first-semester Calculus course. What process does this remind you of, in which one subdivides a calculation into smaller and smaller intervals to obtain more and more accurate approximations of a quantity which is difficult to calculate?

To continue the story, we must spoil the answer to the above exercise. In essence, if $F(y)$ is the force exerted on the pulley once the piano has been hauled to a height of y, what we are doing with those estimates above is calculating Riemann sums on the force function. For example, the estimate with 3 different 50-foot increments could be thought of as

$$F(0)\Delta y + F(50)\Delta y + F(100)\Delta y = 240,000 \text{ foot-lbs},$$

where $\Delta y = 50$ feet, the length of each segment of our journey.

Exercise 2.6.7. Back to the well

What does one do in order to take a Riemann sum and get the exact answer instead of an approximation?

Putting these ideas together, we have a formula for work with variable force!

Formula 2.6.8. Work with Variable Force

If an object is moved from $y = a$ to $y = b$, with a force of $F(y)$ exerted at location y, the total work done is

$$W = \int_{y=a}^{y=b} F(y)dy.$$

We now apply this to get the "true" answer to the piano problem.

Example 2.6.9. Determining the force function

The tricky bit here is determining the force function, $F(y)$. We can best accomplish this by carefully writing down everything we know about it.

- At height $y = 0$, the total weight of the piano plus the chain is 1800 pounds. Said more

concisely, $F(0) = 1800$.

- At height $y = 150$, the total weight is only that of the piano, so 1200 pounds, since all the chain is wound at that point. Thus, $F(150) = 0$.
- The function $F(y)$ should be a linear function, since the loss in force will be proportional to the length of chain that is wound.

Thus, we have a linear function, which written in slope-intercept form is $F(y) = my + b$. Also, we know two points on the graph of this function, as stated above. Since two points uniquely determine a line, we have enough information to now fully determine the function $F(y)$.

Exercise 2.6.10. Finding the unknowns ☕

Use the two given values of F to determine the unknowns m and b, and find the force function $F(y)$.

Exercise 2.6.11. Find the precise amount of work ☕☕

At long last, let's determine precisely the amount of work done in hoisting the piano with the strong chain. Take your force function from the previous exercise and calculate the total work done using Formula 2.6.8. How does it compare to the approximations? Does this comparison make sense?

Exercise 2.6.12. Back to the well, again! ☕☕☕

A 20 kg bucket of water is getting hoisted up from the bottom of a 10 meter well. There is a bit of wobbling and splashing that happens, so no matter how carefully one does it, the bucket consistently loses 1 kg of water per meter of hoisting. How much work is done in hoisting the bucket out of the well? Express your answer in Joules. (Recall that the gravitational constant $g = 9.8m/s^2$ can convert kilograms as a unit of mass to a force exerted by gravity on that mass.)

Here is a different way in which the story can be complicated. Suppose the amount of force is not changing, but rather we have infinitely many different objects which are being lifted across infinitely many distances! Rather than a single object (like a piano or a bucket of water) being carted a single distance, what if different parts are being moved different distances (like in the pirate situation, in which the dirt closer to the surface vs deeper down in the hole)? For this situation, we employ a technique very similar to that of Section 2.3. Specifically, we visit each horizontal layer of the object and multiply the amount of material found there by the distance it was raised, and then integrate across all horizontal

cross sections. We state this idea more formulaically below.

Formula 2.6.13. Integral Formula for Work

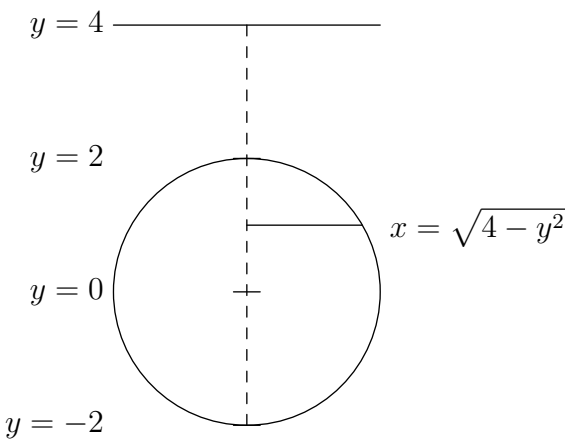
Suppose a 3D solid begins at height $y = a$ and ends at height $y = b$, and $A(y)$ represents the area of the cross section at height y, while $D(y)$ represents the distance that cross section is to be moved. Then, the total work done in moving this solid is

$$W = \int_{y=a}^{y=b} \rho \cdot g \cdot A(y) \cdot D(y) dy,$$

where ρ is the density of the solid and g represents the gravitational constant.

Example 2.6.14. Pumping gas

Suppose a gas station has a full, 4 meter diameter, spherical tank of gasoline, set so that the center of the tank is buried 4 meters below the surface. We wish to find how much work would it take to pump all the gasoline up to ground level. We will spot ourselves the following conversion constants: 800 kilograms per cubic meter as the density of gasoline and the force due to gravity on 1 kilogram as 9.8 Newtons.



Above, we see a diagram showing ground level and the tank below it. Notice that there is no rule that says we have to call ground level $y = 0$. In particular, we choose $y = 0$ to be the center of the tank, and then label everything else accordingly, making the bottom and tops of the tank land on $y = -2$ and $y = 2$, respectively, with ground level at $y = 4$.
Thus, we want to integrate the amount of work done in pumping the gas out at every level from $y = -2$ to $y = 2$, the bounds of the tank. So, our work integral is going to look something like

$$W = \int_{y=-2}^{y=2} 800 \cdot 9.8 \cdot A(y) \cdot D(y) dy,$$

where $A(y)$ represents the area of the cross section of gasoline at height y, and $D(y)$ represents the distance the gas at height y will be pumped, and the constants convert our units to force.
By far, the easier function to determine is $D(y)$. Since the ground level is at height 4, then at an arbitrary height y, the distance the gas has to be pumped is

$$D(y) = 4 - y.$$

For example, the gas at height $y = 1.5$ has to be pumped $4 - 1.5 = 2.5$ meters to reach ground level.
The trickier function is $A(y)$. What is the amount of gasoline at height y? Well, since the tank is spherical, there will be a circular cross section. If we could determine the radius, then we could determine the area of that cross section. Fortunately, we can do this! The equation of a circle of radius 2, centered at the origin, is

$$x^2 + y^2 = 4.$$

Solving for x, taking the positive root, produces

$$x = \sqrt{4 - y^2},$$

which as one can see from the diagram, the x coordinates along the circle themselves determine the radii of the cross sections.
It now only remains to find the area, using $A = \pi r^2$, the area formula for a circle, with the radius that we found above. Thus we have

$$A(y) = \pi \left(\sqrt{4 - y^2}\right)^2 = \pi \left(4 - y^2\right).$$

At last, we can plug in the formulas we have found for $A(y)$ and $D(y)$ and evaluate the integral:

$$\begin{aligned}
\int_{y=-2}^{y=2} 800 \cdot 9.8 \cdot A(y) \cdot D(y) dy &= 7840 \int_{y=-2}^{y=2} \pi \left(4 - y^2\right)(4 - y)\, dy \\
&= 7840\pi \int_{y=-2}^{y=2} \left(16 - 4y - 4y^2 + y^3\right) dy \\
&= 7840\pi \left(16y - 2y^2 - \frac{4}{3}y^3 + \frac{1}{4}y^4\right)\Big|_{y=-2}^{y=2} \\
&= 7840\pi \left(32 - 8 - \frac{32}{3} + 4\right) - 7840\pi \left(-32 - 8 + \frac{32}{3} + 4\right) \\
&= 2 \cdot 7840\pi \left(32 - \frac{32}{3}\right) \\
&= 2 \cdot 7840\pi \left(\frac{64}{3}\right) \\
&= \frac{1003520\pi}{3}.
\end{aligned}$$

Converting to a decimal, we have the total work done as roughly 1050883.68, or a little over a million Joules.

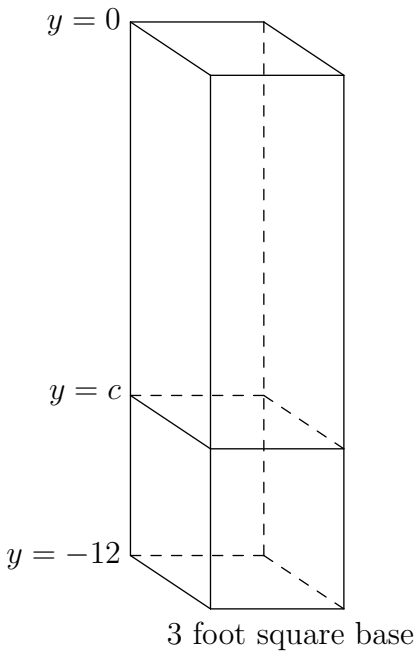
Exercise 2.6.15. A Cylindrical Tank ☕☕☕

Consider a cylindrical tank of radius 2 meters and length 4 meters, laying on its side, set so the top of the tank is 2 meters below the ground. That is, the cylinder is oriented so that the circular cross sections run up and down, and rectangular cross sections are parallel to the ground. Note that this implies that the same diagram from the previous problem may be used, except the cross sections will be rectangular instead of spherical.

- Calculate the total work done in draining all the gasoline from this tank to ground level.
- How does your answer compare to the answer obtained for the spherical tank in the previous problem? Does this comparison make sense?

Example 2.6.16. Settling the pirate dispute!

Let us settle once and for all the argument between Arg and Rho at the beginning of this section! Assume that the dirt has uniform density the whole way down. Let us set up the work integrals that will help us find that perfect trade-off point for handing off the shovel.



The pit that must be dug is a square prism, which we choose to index by y coordinates going from $y = 0$ at ground level down to $y = -6$ at the level of the buried treasure. Next, call $y = c$ the unknown point where the two pirates should trade. Note that in this case, the density of the dirt is irrelevant (assuming it's uniform the whole way down) so we can ignore the utterly exhausting unit conversion that plagued us in the gasoline tank problems.
Thus, to solve the problem, we must set up two integrals, one for Rho's work, and one for Arg's work. If we equate the two, we will have an equation which can then be solved for c. Proceeding, we have Rho's work as

$$W_{\text{Rho}} = \int_{y=c}^{y=0} A(y) \cdot D(y)dy = \int_{y=c}^{y=0} 9 \cdot (-y)dy,$$

while Arg's work would be

$$W_{\text{Arg}} = \int_{y=-12}^{y=c} A(y) \cdot D(y)dy = \int_{y=-12}^{y=c} 9 \cdot (-y)dy.$$

Exercise 2.6.17. A Life on the c ☕☕

- In the above example, explain why $A(y) = 9$ and $D(y) = -y$ were used for the area and distance functions.
- Calculate the two work integrals that were set up in the previous example. Note that the answer to each will not be a numerical value, but rather a formula involving c.
- Set the two equal to each other to solve for c. Tell the pirates when they should switch!

Exercise 2.6.18. c Marks the Spot ☕☕☕

- Suppose the treasure was in a circular chest instead of square, so that the pirates needed to dig a cylindrical chest instead of square. Rework the above problem (i.e., find at what point the pirates should now switch shoveling). How does the new result compare to the original?
- Suppose the density of the dirt was proportional to the depth of the dirt (perhaps the ground gets more wet as you go deeper down). That is, assume that dirt found y meters below ground has density equal to y. Rework the above problem (i.e., find at what point the pirates should now switch shoveling). How does the new result compare to the original?
- Suppose the pirates instead want to trade twice. Rho will take a shift on the easy dirt in the beginning, digging for c meters, after which Arg will take the middle $12 - 2c$ meters, after which Rho will take the brutal final c meters. (And suppose we are back in the uniform density, square pit case.) Now where are their switching points?

Hydrostatic Force

Besides work, another important quantity in physics involving force is *pressure*, a force applied per square unit of area. That is,

$$P = F/A$$

where P is pressure, F is force, and A is area. The standard metric unit for pressure is the pascal (Pa), which is defined as one Newton per square meter, which is represented as $\text{Pa} = \text{N/m}^2$. In the imperial system, the most commonly used unit of pressure is psi, which is short for pounds per square inch.

An interesting fact about water is that it exerts pressure on an object that is proportional to the depth of the object underwater. However, imagine a window is submerged vertically underwater; now we have different amount of pressure at different points on the window! Note how reminiscent this is of the situation in the work problems about draining tanks or digging pits, in which each level has a different distance it must travel. Similarly, in this case, we can calculate the total force the water exerts on the

object by integrating the total pressure at each level of the object.

Formula 2.6.19. Total force exerted on a region submerged in a fluid

Given a region whose top and bottom are respectively at heights $y = a$ and $y = b$, the total force exerted on the window is

$$F = \int_{y=a}^{y=b} \rho \cdot g \cdot d(y) \cdot w(y) dy,$$

where ρ is the density of the fluid, g represents the gravitational constant, $d(y)$ represents the depth of the fluid at height y, and $w(y)$ represents the width of the region at height y.

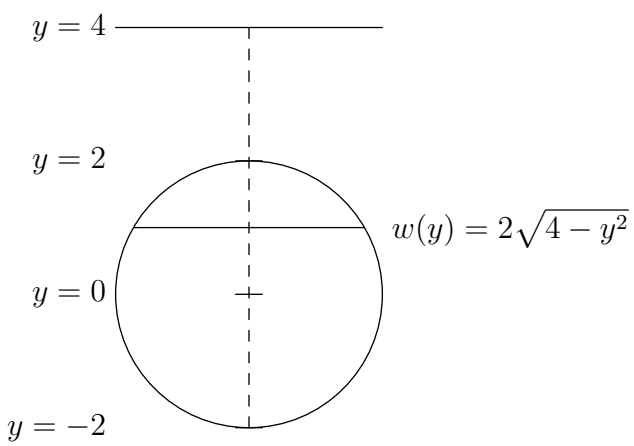
In the rest of this section, we use the metric system constants $\rho = 1000$ kg/m^3 for water and $g = 9.8$ m/s^2 for gravity, so the above formula becomes

$$F = \int_{y=a}^{y=b} 9800 \cdot d(y) \cdot w(y) dy,$$

which will produce an answer in Newtons.

Example 2.6.20. Force on a circular window

Suppose we have a circular window with a diameter of 4 meters whose top is set 2 meters below the surface of the water. Then the width at height y is just double what we found in Example 2.6.14, as can be seen in the diagram below.



The depth underwater in this situation is exactly the same as the distance function from Example 2.6.14, so $d(y) = 4 - y$. We now set up the integral and then break it into two for sake of easier calculation:

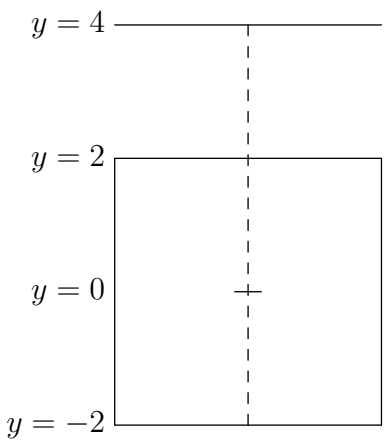
$$\begin{aligned} F &= \int_{y=-2}^{y=2} 9800 \cdot d(y) \cdot w(y) dy \\ &= \int_{y=-2}^{y=2} 9800 \cdot (4 - y) \cdot \sqrt{4 - y^2} dy \\ &= 19600 \left(4 \int_{y=-2}^{y=2} \sqrt{4 - y^2} dy - \int_{y=-2}^{y=2} y\sqrt{4 - y^2} dy \right). \end{aligned}$$

Exercise 2.6.21. The window might Bowie under pressure ☕☕

Finish the evaluation of the total pressure exerted on the circular window in the example above by calculating the two integrals. **Hint!** The first integral does not need to be done with an antiderivative, but rather can be found by using the formula for the area of a circle. The second integral can be calculated with the substitution $u = 4 - y^2$.

Exercise 2.6.22. Other windows! ☕☕

- If the same window from the previous problem were square instead of circular (see diagram below), would you expect the total pressure exerted to be greater or less? Confirm your expectation by calculating the total force with an integral.



- If the original circular window were set one meter closer to the surface, would you expect the total force on the window to be greater or less? Confirm your expectation by calculating the total force with an integral.

2.7 Chapter Summary

In this chapter, we explored how integrals can inform geometric properties of our shapes. There were four primary quantities we studied.

1. **Length:** We defined the **arc length** of the graph of a function $f(x)$ as

$$L = \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x.$$

2. **Area:**

   (a) The **area between curves** given by the graphs of functions $f(x)$ and $g(x)$ is

$$L = \int_{x=a}^{x=b} f(x) - g(x)\, \mathtt{d}x.$$

   (b) If the region of interest is unbounded horizontally or vertically, the corresponding integral is called **improper**. To evaluate, we create a new bound c and take a limit as c approaches the trouble spot. To evaluate the limits that arise in this context, we often need **LHR**.

   (c) If the graph of a function $f(x)$ is revolved about the y-axis, the **surface area** of the resulting shape can be computed as

$$SA = \int_{x=a}^{x=b} 2\pi x \sqrt{1 + (f'(x))^2}\, \mathtt{d}x.$$

   Notice that this integrand is just $2\pi x$ (representing circumference of a circle of radius x) times the arc length integrand.

3. **Volume:**

   (a) The **volume by cylindrical shells** of a 3D solid can be computed if the solid has rotational symmetry about an axis. Without loss of generality we assume this axis is the y-axis, in which case the volume is

$$V = \int_{x=a}^{x=b} 2\pi x f(x)\, \mathtt{d}x.$$

   Notice this integrand is just $2\pi x$ (representing circumference of a circle of radius x) times the integrand for area between $f(x)$ and 0.

   (b) The **volume by cross-sections** of a 3D solid can be computed by slicing it into 2D regions of area $A(x)$ at location x and then integrating the areas. Specifically,

$$V = \int_{x=a}^{x=b} A(x)\, \mathtt{d}x.$$

   (c) The **volume by washers** of a 3D solid generated by revolving the region between $f(x)$ and $g(x)$ about the x-axis can be computed as

$$V = \pi \int_{x=a}^{x=b} (f(x))^2 - (g(x))^2\, \mathtt{d}x.$$

4. **Center of Mass:** The **center of mass** of a region R is the point

$$(\bar{x}, \bar{y}) = (M_y/m, M_x/m)$$

where m is the area of R and M_y and M_x are the **moment integrals** with respect to the y and x axes, respectively. These are computed as follows:

$$M_y = \int_{x=a}^{x=b} x\,(f(x) - g(x))\,\mathtt{d}x$$
$$M_x = \int_{x=a}^{x=b} \frac{1}{2}\,(f(x) + g(x))\,(f(x) - g(x))\,\mathtt{d}x.$$

5. **Work:** The total work done to move an object from a to b is

$$W = \int_{x=a}^{x=b} F(x)\,\mathtt{d}x$$

where $F(x)$ is the force applied to the object in terms of its displacement x.

2.8 Mixed Practice

Exercise 2.8.1. ☕

a.) Compare the growth orders of x^2 and e^x.

b.) Compare the growth orders of x^3 and e^x.

c.) Compare the growth orders of x^4 and e^x.

d.) Let $p(x)$ be a generic polynomial of degree n where n is a natural number. What can you say about the growth order of $p(x)$ versus the growth order of e^x?

Exercise 2.8.2. ☕☕

Suppose a pyramid has height 3 and has an equilateral triangle with sides of length 2 for its base. Find the volume V of the pyramid using cross sections.

Exercise 2.8.3. ☕☕

Let $f(x) = \sqrt{4-(x-1)^2}$, and let R be the region between the graph and the x-axis. Find the volume V of the solid obtained by revolving R about the y-axis.

Exercise 2.8.4. ☕☕

a.) Sketch the region bounded by the following equations:

$$y = 0$$

$$x = 0$$

$$x = 3$$

$$y = \frac{1}{(x+1)(x-3)^2}$$

b.) Use an improper integral to find the area of the region.

Exercise 2.8.5. ☕

Demonstrate that the volume of a sphere is $V = \frac{4}{3}\pi r^3$ by cylindrical shells.

Exercise 2.8.6. ☕

Demonstrate that the volume of a sphere is $V = \frac{4}{3}\pi r^3$ by cross sections.

Exercise 2.8.7. ☕☕

Calculate the center of mass of the region between the graphs of $f(x) = 3x$ and $g(x) = x^2$ on the interval $[0, 3]$.

Exercise 2.8.8. ☕☕☕

Consider the following function:

$$f(x) = \frac{\ln x}{\sqrt{x}}$$

a.) What is $\lim_{x\to\infty} f(x)$?

b.) What is $\lim_{x\to 0^+} f(x)$?

c.) Sketch the graph of f. Include the information gathered in parts a) and b), as well as the point $(1, f(1))$.

d.) Compute $\int_0^1 f(x)dx$. Interpret the result on your graph.

Exercise 2.8.9. ☕☕☕☕

1. Differentiate the function

$$f(x) = \frac{1}{2}(x-1)\sqrt{x^2-2x} - \ln\left(\sqrt{x-2} + \sqrt{x}\right)$$

and verify that

$$f'(x) = \sqrt{x^2-2x}.$$

2. Find the length of $f(x)$ from $x = 2$ to $x = 4$.

Exercise 2.8.10. ☕☕☕

1. Explain in words the difference between finding volume integrating via cylindrical shells vs via cross sections.
2. Consider the tetrahedron with vertices at (0,0,0), (1,0,0), (0,1,0), and (0,0,1). To find its volume, which of the above two methods would you use, and why?
3. Find its volume using an integral.

Exercise 2.8.11. ☕☕☕

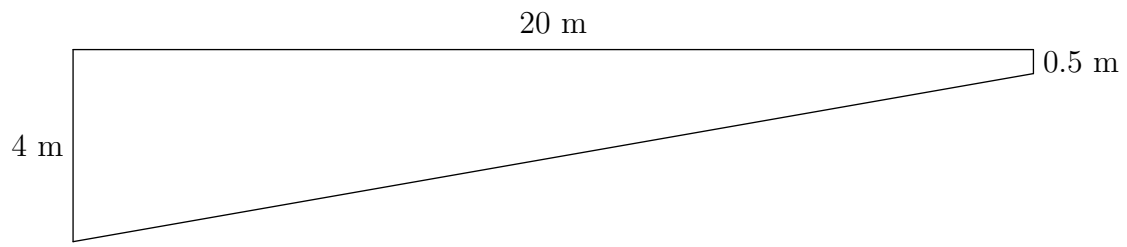
a.) Use integrals to find the center of mass of the first quadrant region bounded by $y = x$ and $y = x^n$, where $n \geq 1$. Call this point $(\overline{x}_n, \overline{y}_n)$.

b.) As $n \to \infty$, what point does $(\overline{x}_n, \overline{y}_n)$ approach?

c.) Use integrals to find the center of mass the triangle whose vertices are (0,0), (1,0), and (1,1). How does this relate to your answer above?

Exercise 2.8.12. Draining a Swimming Pool

A swimming pool is to be drained for winter cleaning. From the top, the pool looks like a 20 meter square. The pool goes from a depth of 4 meters on the deep end to a depth of 0.5 meters on the shallow, as shown in the diagram of a vertical cross-section of the pool.



Given the density of water as 1000 kilograms per cubic meter and the force exerted due to gravity on one kilogram as 9.8 Newtons, calculate how many Joules of work it will take to drain the pool. **Hint!** You'll have to set up two integrals, one for the part of the pool in which the width is changing, and one for which it is not!

Exercise 2.8.13. Digging a Pit ☕☕☕

1. How much work does it take to dig a 6-foot diameter cylindrical pit to a depth of 3 feet in gravel weighing 62.4 lbs per cubic foot? Express your answer in foot-pounds.
2. Suppose we are digging in loose gravel, so we can't dig straight down because the side walls will collapse. Instead we need to dig down in a conical fashion at a 45 degree angle from the surface. How much work is involved in digging a conical pit of the same radius and depth in gravel of the same density?
3. What is the proportion of the work involved in digging the conical pit vs digging the corresponding cylindrical pit?
4. Does that proportion depend on the density of the gravel, the depth, or the radius? That is, if you dig a cylindrical pit vs digging a 45 degree conical pit of the same depth and radius, does your work always go up by the same factor? Explain.

Part II

Sequences and Series

Chapter 3

Sequences and Series: Commas and Plus Signs Run Amok

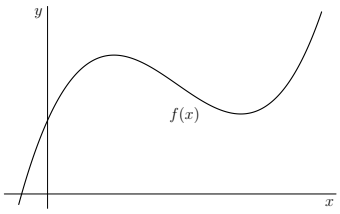
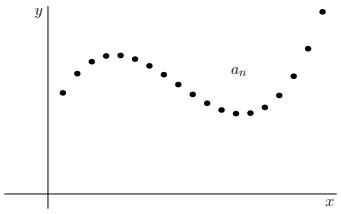
3.1 Sequences

Definition 3.1.1. Sequence

A *sequence* is a function whose domain is a subset of $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers.

Typically for $n \in \mathbb{N}$, we write a_n as the output corresponding to the input n. The output could technically be an object of any type, but in this course we usually use real numbers or complex numbers as our outputs. Technically the sequence itself is the map $n \mapsto a_n$, though since this is a bit cumbersome to write, we often write just a_n to refer to the entire sequence, similar to how we write $f(x)$ for a function on the real numbers. When the outputs are real numbers, we can graph sequences as a collection of points of the form (input,output) just as we would for functions on the real numbers.

The chart below compares sequences (functions whose domain is the natural numbers) and functions whose domain is the real numbers.

Trait or Notation	**Function on Reals**	**Sequence**
Default Independent Variable	x	n
Default Formula Notation	$f(x)$	a_n
Domain D	Subset of $\mathbb{R}$	Subset of $\mathbb{N}$
Graph	$\{(x, f(x)) : x \in D\}$ 	$\{(n, a_n) : n \in D\}$ 

Less formally, a sequence is simply a list of objects. The correspondence between sequences as maps and sequences as lists of objects is that the kth object in the list is the output corresponding to $k-1$ under the map. That is, the map $n \mapsto a_n$ corresponds to the list $a_0, a_1, a_2, a_3, \ldots$.

Example 3.1.2. The Sequence of Even Natural Numbers

Consider the list of nonnegative even numbers: $0, 2, 4, 6, 8, 10, \ldots$. We can view this as the map $n \mapsto 2n$, which we can see as a function on the naturals with the following inputs and outputs:

$$\begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 2 \\ 2 &\mapsto 4 \\ 3 &\mapsto 6 \\ 4 &\mapsto 8 \\ &\vdots \end{aligned}$$

Notation for Sequences

There are two main ways that we define sequences: as explicit formulas and as recursive formulas. The next two subsections describe these methods.

Explicit Formulas

Often times we define a sequence via what is called an *explicit formula* or a *closed formula*. This is a formula given in terms of n that shows explicitly how to compute the output corresponding to an input n in finitely many steps expressed in our usual language of algebraic and transcendental functions.

Example 3.1.3. The Sequence of Even Natural Numbers: Explicit Formula

The sequence of even natural numbers defined above has

$$a_n = 2n$$

as its explicit formula.

Exercise 3.1.4. Practice with Explicit Formulas ☕☕

Find an explicit formula for the sequence of...

- ...odd natural numbers.

$$1, 3, 5, 7, \ldots$$

- ...even integers starting at -4 and counting upwards, two at a time.

$$-4, -2, 0, 2, \ldots$$

- ...all multiples of 5, starting from 20 and counting downwards.

$$20, 15, 10, 5, 0, -5, \ldots$$

- ...all natural numbers that are one more than a multiple of 3.

$$1, 4, 7, 10, \ldots$$

- ...consecutive powers of 2, starting from 1.

$$1, 2, 4, 8, \ldots$$

- ...terms that alternate forever between positive and negative one.

$$1, -1, 1, -1, \ldots$$

- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.

$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

Recursive Formulas

Recursion is a beautiful, powerful, and useful concept. It is the idea of defining a structure in terms of smaller instances of that same type of structure. In the case of sequences, we want to define a later term a_n as a formula given in terms of a_k for some k values strictly less than n. This definition of later terms built out of previous terms is called the *recursion*. Additionally, a recursive definition requires the definition of some initial term or terms to get the process rolling. These early terms are called the *base cases* or *initial terms*.

Returning to our favorite little example once again, we ask how we can find a recursive formula for the sequence of even numbers. Notice how the later terms relate to the earlier terms; each term is exactly two more than the previous term. We build a recursive formula out of this observation.

Example 3.1.5. The Sequence of Even Natural Numbers: Recursive Formula

$$\begin{aligned} a_0 &= 0 \\ a_n &= 2 + a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

Exercise 3.1.6. Absorbing the Language ☕

In the recursive formula above, which expression is the base case? Which part is the recursion?

Exercise 3.1.7. Practice with Recursive Formulas ☕☕

Find a recursive formula for the sequence of...

- ...odd natural numbers.

$$1, 3, 5, 7, \ldots$$

- ...even integers starting at -4 and counting upwards, two at a time.

$$-4, -2, 0, 2, \ldots$$

- ...all multiples of 5, starting from 20 and counting downwards.

$$20, 15, 10, 5, 0, -5, \ldots$$

- ...all natural numbers that are one more than a multiple of 3.

$$1, 4, 7, 10, \ldots$$

- ...consecutive powers of 2, starting from 1.

$$1, 2, 4, 8, \ldots$$

- ...terms that alternate forever between positive and negative one.

$$1, -1, 1, -1, \ldots$$

- ...fractions whose numerators are all even natural numbers starting from zero and whose denominators are all odd natural numbers starting from 1.

$$\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$$

Factorials

Some sequences have no simple explicit formula and are most easily thought of recursively. The sequence of factorials is a famous example of this type.

Example 3.1.8. Factorials, Defined Recursively

Consider the following recursively defined sequence:

$$\begin{aligned} a_0 &= 1 \\ a_n &= n \cdot a_{n-1} \text{ for } n \geq 1 \end{aligned}$$

We can unwind this recursion a bit to obtain a more accessible expression for factorials. Observe the following calculations based on the base case and recursion given above:

$$\begin{aligned} a_0 &= 1 = 1 \\ a_1 &= 1 \cdot a_0 = 1 \\ a_2 &= 2 \cdot a_1 = 2 \cdot 1 = 2 \\ a_3 &= 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 6 \\ a_4 &= 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ a_5 &= 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

This sequence comes up so frequently that we give it its own symbol, the exclamation point! Since the factorial of n always amounts to the product of all natural numbers greater than or equal to 1 but less than or equal to n, we write the following:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Note that almost any expression involving a shady "$\cdots$" is truthfully a recursion in disguise!

Exercise 3.1.9. Why is the Factorial of Zero Equal to One? ☕

Looking carefully at the above definition, you will notice that

$$0! = 1$$

It is a common mistake to compute 0! as 0 instead. Here is one way to see why it should in fact be 1.

- If you compute 2^2, how many numbers are you multiplying together?
- If you compute 2^1, how many numbers are you multiplying together?
- If you compute 2^0, how many numbers are you multiplying together?
- Right, zero numbers are being multiplied together. A product like this is called an *empty product* and is always defined to be one, since that is the multiplicative identity.
- If you compute 3! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 3, how many numbers are you multiplying together?
- If you compute 2! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 2, how many numbers are you multiplying together?

- If you compute 1! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 1, how many numbers are you multiplying together?
- If you compute 0! as the product of all natural numbers greater than or equal to 1 and but less than or equal to 0, how many numbers are you multiplying together?
- Since 0! is also an empty product (much like 2^0), what should we define it to be?

The following type of simplification will occur frequently throughout the our adventures in infinite series and power series. They all follow directly from the recursive definition of factorials.

Example 3.1.10. Simplifying Factorials

Let n represent a positive natural number. Consider the expression $\frac{(n+2)!}{n!}$. The numerator represents the product of all natural numbers between $n+2$ and 1, inclusive. The denominator represents the product of all natural numbers between n and 1, inclusive. We expand out these products and then cancel whatever factors they have in common.

$$\begin{aligned}\frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}\\ &= (n+2)(n+1)\end{aligned}$$

Thus, that ratio of factorials cleans up to just a polynomial!

Exercise 3.1.11. Simplifying Factorials

Let n be a natural number greater than or equal to 1. Reduce the following fractions! (Are they fractorials?)

- $\frac{n!}{(n+1)!}$
- $\frac{(n+1)!}{n!}$
- $\frac{(n+2)!}{n!}$
- $\frac{(2n+2)!}{(2n)!}$

Explicit Formula for Factorials: The Gamma Function

The gamma function, denoted Γ, is the most common smooth interpolation of the factorial function on the positive integers. The gamma function is defined using improper integrals.

Definition 3.1.12. The Gamma Function

For $n \in \mathbb{R}$, define

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, \mathtt{d}x$$

That is, $\Gamma(n)$ is the area under the graph of $x^{n-1}e^{-x}$ in the first quadrant.

Exercise 3.1.13. Computing Values of Gamma ☕☕☕

- To see the manner in which the gamma function provides a continuous analog of the factorial function, fill out the values in the following table:

n	$n!$	$\Gamma(n)$
1		
2		
3		
4		

  Computing the values of the gamma function will require quite a bit of work. You don't have to show all details of the integrals above, but make sure you are comfortable doing such manipulations by hand.

- Given the table above, conjecture an explicit formula for the factorials. It won't be a nice little algebraic formula, but express it in terms of an improper integral. Write the conjecture below.

$$n! = \int_0^\infty \qquad\qquad\qquad\qquad \mathtt{d}x$$

Observe that you were able to use the smaller instances of the gamma function to help you compute the larger instances! That is, when you apply integration by parts to compute $\Gamma(n)$, it will produce an expression that involves the integral you computed for $\Gamma(n-1)$.

It turns out this relationship is exactly what shows that the Γ function will *always* match the values of the factorial function. For factorial, we have:

$$n! = n \cdot (n-1)!$$

Exercise 3.1.14. Gamma Recursion ☕☕☕

What is the corresponding relationship for the Gamma function? Specifically, how does $\Gamma(n)$

relate to $\Gamma(n-1)$? Write your answer below.

An Approximation for the Factorials

As mentioned above, there does not exist a simple algebraic explicit formula for the factorial function. However **Stirling's Formula** gives a very nice explicit asymptotic formula.

Formula 3.1.15. Stirling's Formula

For large values of n,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

As n goes to infinity, the error between the two quantities approaches zero.

That is to say, as n approaches infinity, $n!$ approaches $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. We don't have the tools to fully prove Stirling's Formula in this course, though it will be occasionally helpful to have a rough measure of the growth order of a factorial function.

Arithmetic and Geometric Sequences

Here we provide the definitions for two particularly famous families of sequences, arithmetic and geometric.

Definition 3.1.16. Arithmetic Sequence

A sequence a_n is called *arithmetic* if and only if there exists some real constant d such that $a_{n+1} - a_n = d$ for all natural numbers n. In such a sequence, the number d is called the *common difference*.

Definition 3.1.17. Geometric Sequence

A sequence a_n is called *geometric* if and only if there exists some real constant r such that $a_{n+1}/a_n = r$ for all natural numbers n. In such a sequence, the number r is called the *common ratio*.

Exercise 3.1.18. Playing with the Definition ☕☕

- Return to Exercise 3.1.4. Which of those sequences are arithmetic? For those that are, what

is the common difference?

- Return again to Exercise 3.1.4. Which of those sequences are geometric? For those that are, what is the common ratio?

- Give an informal definition of an arithmetic sequence. (Think of what you would say if you had to explain what it was to a fifth grader).

- Give an informal definition of a geometric sequence. (Think again of what you would say if you had to explain what it was to a fifth grader).

- Give an example of a sequence that is arithmetic but not geometric.

- Give an example of a sequence that is geometric but not arithmetic.

- Can a sequence simultaneously be both arithmetic and geometric? If it is possible, give an example of such a sequence. If it is not possible, explain why it is not possible.

Exercise 3.1.19. Converting Between Recursive and Explicit Definitions ☕☕

- Write a sentence that explains the difference between defining a sequence recursively vs defining a sequence explicitly.

- Consider the following recursively-defined sequence:

$$a_0 = 5$$
$$a_n = 2 \cdot a_{n-1}$$

  Write out the first five terms of this sequence. Can you find an explicit formula?

- Consider the following explicitly-defined sequence:

$$a_n = 3n - 2$$

  Write out the first five terms of this sequence. Can you find an recursive formula?

Explicit Formulae for Arithmetic and Geometric Sequences

Example 3.1.20. Explicit Formula for Arithmetic Sequences

Since every arithmetic sequence starts with some initial term a_0 and then adds the same number d each time to get from term to term, we can say the explicit formula will always have the same

form. In particular, we have the terms of the sequence as follows:

$$a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \cdots .$$

Thus, a generic term of the sequence looks like

$$a_n = a_0 + dn$$

where a_0 is the initial term and d is the common difference.

Exercise 3.1.21. Explicit Formula for Geometric Sequences ☕☕

Repeat the process of the above example to demonstrate that every geometric sequence has explicit formula

$$a_n = a_0 r^n$$

where a_0 is the initial term and r is the common ratio.

3.2 Convergence of Sequences

Intuitive and Formal Definitions

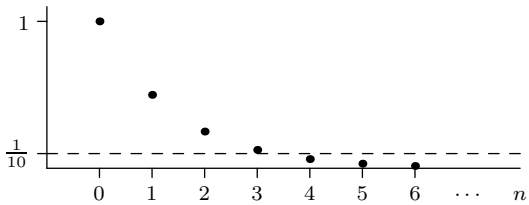
Consider the sequence $a_n = \frac{1}{2^n}$. Listing out a few terms, we see that a_n looks like:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

We would like a way to describe the long-term behavior of such a sequence. Intuitively, we see that the numbers are becoming arbitrarily close to zero.

Exercise 3.2.1. The Idea of Convergence

- Label the y coordinates on the graph of the sequence below.



- How far into the sequence would you have to travel to find only terms that are no more than one-tenth from zero? That is to say, how large does n have to be to guarantee that a_n is between one-tenth and zero?

- How far into the sequence would you have to travel to find only terms that are no more than one-hundredth from zero?

- How far into the sequence would you have to travel to find only terms that are no more than one-thousandth from zero?

No matter how small of a measurement we choose (one-tenth, one-hundredth, one-thousandth, etc), we could always find that after a certain point, all of our sequence terms are no further than that measurement from zero. This is exactly the notion we will reformulate in a more formal manner to define sequential convergence.

Recall the mathematical shorthands often used to help concisely state messy definitions: the symbol "$\forall$" means "for all" and the symbol "$\exists$" means "there exists". See Chapter 0 for more detail. Also recall that for any real numbers a and b, the distance between a and b can be written as $|a-b|$. Using these shorthands, we define the limit of a sequence!

Definition 3.2.2. Convergence of a Sequence

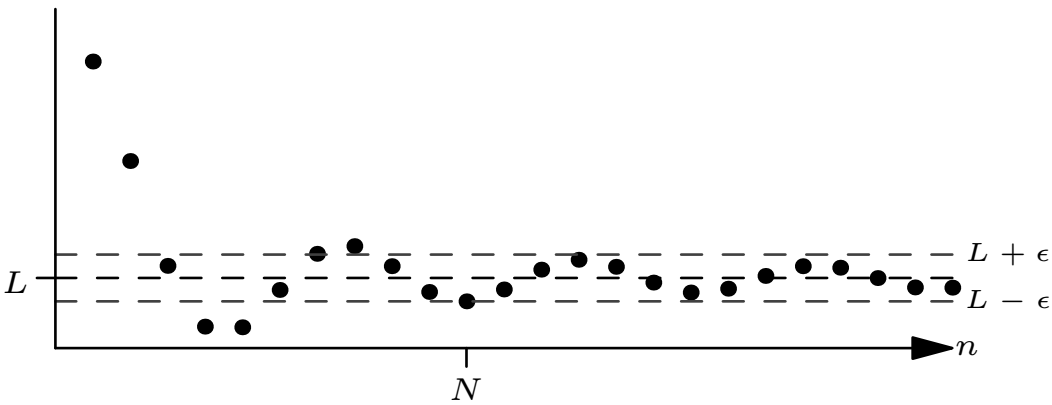
We say the sequence a_n converges to a limit $L \in \mathbb{R}$ and write

$$\lim_{n\to\infty} a_n = L$$

if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon$$

If no such L exists, we say the sequence *diverges*.



Exercise 3.2.3. Digesting the Definition ☕☕

- In the definition of convergence, what role does ϵ play? Specifically, what is it bounding the distance between?

- In the definition of convergence, what role does N play? What role does n play?

- Restate the formal definition of sequential convergence in words rather than symbols. The statement

$$\lim_{n\to\infty} a_n = L$$

means...

Exercise 3.2.4. The Fibonacci Numbers ☕☕

Define the sequence of Fibonacci numbers F_n via the following recursive formula:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

- Compute the first eight terms of the Fibonacci sequence using the above recursion. That is, compute F_0 through F_7.

- Compute the following quantities:

$$\begin{aligned} F_2/F_1 &= \\ F_3/F_2 &= \\ F_4/F_3 &= \\ F_5/F_4 &= \\ F_6/F_5 &= \\ F_7/F_6 &= \end{aligned}$$

- What would you conjecture about

$$\lim_{n\to\infty} F_{n+1}/F_n$$

Does it seem to be going to infinity, zero, or stabilizing at something inbetween?

It is difficult to tell exactly what that limit of ratios is without knowing an explicit formula for the Fibonacci numbers. Stay tuned, as we will find this in a later chapter!

Exercise 3.2.5. Comparing Growth Orders of Sequences ☕☕

Rank the following functions in growth order from smallest to largest:

$$a_n = n^n$$
$$b_n = e^n$$
$$c_n = n^2$$
$$d_n = n!$$

Note that for sequences we can compare growth orders in the same manner as we did in Subsection 2.1. To compare the growth orders of two sequences a_n and b_n, we compute $\lim_{n\to\infty} \frac{a_n}{b_n}$ and conclude that a_n has larger growth order if the limit is infinity, b_n has larger growth order if the limit is zero, and the growth orders are the same if the limit is a nonzero constant. Here you will need Stirling's Formula from Subsection 3.1.

$N - \epsilon$ Proofs

This complicated definition can be unwound into a to-do list for what one must do to prove that a sequence converges to a particular limit. In particular, to show that the limit of a_n is equal to a number L, one must:

- Let ϵ be an arbitrary positive real number.
- Choose N, typically defined as a function of ϵ, since smaller values of ϵ will usually require a larger N to be chosen.

- Let n represent an arbitrary natural number greater than N.
- Using the definition of N and the assumption that $n > N$, prove that any corresponding a_n satisfies $|a_n - L| < \epsilon$.

Figuring out exactly what N should be in terms of ϵ usually requires a bit of algebra before the proof is written up. If the formula for a_n is clean enough, you might be able to just work backwards from the inequality $|a_n - L| < \epsilon$. If you solve it for n, you will find an expression that n must be larger than. Note here we are essentially just finding an inverse function for a_n.

Example 3.2.6. Solving for N

Let us solve for N with regards to our sequence $a_n = \frac{1}{2^n}$. Since here we suspect $L = 0$, we solve for n in the following inequality:

$$\begin{aligned}
\left|\frac{1}{2^n} - 0\right| &< \epsilon \\
\frac{1}{2^n} &< \epsilon \\
\frac{1}{\epsilon} &< 2^n \\
\ln\left(\frac{1}{\epsilon}\right) &< \ln\left(2^n\right) \\
\ln\left(\frac{1}{\epsilon}\right) &< n\ln(2) \\
\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)} &< n
\end{aligned}$$

Thus we determined our choice of N, namely

$$N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}.$$

Exercise 3.2.7. Justifying Our Work ☕

In words, annotate the above example to indicate why each line follows from the previous.

Now that we found our value for N, we are ready to follow the steps described above and construct our proof.

Example 3.2.8. Writing an $N - \epsilon$ Proof

Prove that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Proof. Let ϵ be an arbitrary positive real number. Choose $N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}$. Let n be a natural number such that $n > N$. Under these circumstances, we wish to show that a corresponding a_n will be less than ϵ away from 0. Proceeding:

$$\begin{aligned}
\left|\frac{1}{2^n} - 0\right| &= \frac{1}{2^n} \\
&< \frac{1}{2^N} \\
&= \frac{1}{2^{\left(\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}\right)}} \\
&= \frac{1}{2^{\left(\log_2\left(\frac{1}{\epsilon}\right)\right)}} \\
&= \frac{1}{\frac{1}{\epsilon}} \\
&= \epsilon.
\end{aligned}$$

Thus, for indices n that are larger than our choice of N, the corresponding terms in our sequence are less than ϵ away from zero as desired. □

Exercise 3.2.9. Justifying Our Work ☕

Once again in words, annotate the above example to indicate why each line follows from the previous. Pay particular attention to identify where we used the starting assumption that $n > N$.

As this course does not go through a general treatment of what constitutes a proof or how to come up with one, the example above could be taken as a template for how an $N - \epsilon$ proof should be written. In a more in-depth study of analysis, you will encounter more complicated situations where the above template may be too simplistic. It will be expanded upon when you have the right tools! For now, follow the above proof template for the following exercises:

Exercise 3.2.10. Verifying a Limit ☕☕☕

Consider the sequence given by the following explicit formula:

$$a_n = \frac{2n}{n+1}$$

- List the terms of the sequence corresponding to $n = 1$, $n = 10$, $n = 100$, and $n = 1000$. What do the terms appear to be converging to as n goes to ∞?

- If you choose $\epsilon = 0.1$, what could the corresponding N be?

- If you choose $\epsilon = 0.01$, what could the corresponding N be?

- Write an $N - \epsilon$ proof that verifies your guess above is correct.

Exercise 3.2.11. Writing $N - \epsilon$ Proofs ☕☕☕

Write $N - \epsilon$ proofs for each of the following limits:

- $\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$

- $\lim_{n\to\infty} \sqrt{9+1/n} = 3$

3.3 Series

While a sequence is a list of numbers, a series is a sum of a list of numbers. That is, a sequence is a list of numbers with commas inbetween; a series is a list of numbers with plus signs inbetween. We often use the very compact *sigma notation* to represent series.

Definition 3.3.1. Sigma Notation for Series

If a_n is a sequence and j, k are both natural numbers, then we define the series:

$$\sum_{n=j}^{k} a_n = a_j + a_{j+1} + a_{j+2} + \cdots + a_k$$

That is, we add up all consecutive terms of the sequence a_n, starting at index j and stopping at index k.

Notice that the above summation has $k - j + 1$ terms in it, not $k - j$ as one might quickly guess. One way to see this is to rewrite the sum slightly as

$$a_j + a_{j+1} + a_{j+2} + \cdots + a_k = \underbrace{a_{j+0}}_{\text{One extra for term 0...}} + \underbrace{a_{j+1} + a_{j+2} + \cdots + a_k}_{\text{...then count terms 1,2,...}k}.$$

This subtlety is often called an *off by one error* or *fencepost problem*, since one can view it as if the plus signs were sections of fence, and the terms in the sequence were posts holding up those sections. To support $k - j$ sections of fence, we need $k - j + 1$ posts, since each section has a post to the right of it, but the very first section of fence also has a post to the left which is not to the right of any section.

If the starting index is greater than the stopping index, we consider the sum to be empty. Since it has no terms, we define the total to be zero. Thus, an *empty sum* is the additive identity zero, just like an empty product is the multiplicative identity one.

The sequence a_n that is being totaled is called the *summand*, much as the function $f(x)$ is referred to as the integrand in the expression $\int f(x)\,\mathrm{d}x$.

Example 3.3.2. Evaluating a Summation

Consider the sum of all even numbers between six and fourteen. Of course we don't need sigma notation to evaluate such a sum, but just for proof of concept, let's write this sum in sigma notation, expand, and evaluate.

- First we discuss the summand. The sequence of all even numbers has the explicit formula $a_n = 2n$, so $2n$ will be our summand.
- We want the first term to be six, so we set $n = 3$ as the starting index.
- We want the last term to be fourteen, so we set $n = 7$ as the stopping index.

Thus our summation is

$$\begin{aligned}\sum_{n=3}^{7} 2n &= \underbrace{6}_{n=3} + \underbrace{8}_{n=4} + \underbrace{10}_{n=5} + \underbrace{12}_{n=6} + \underbrace{14}_{n=7} \\ &= 60.\end{aligned}$$

Exercise 3.3.3. Not Crashing Into That Extra Fencepost ☕

The summation in the above example has starting index 3 and stopping index 7. So, does the sum have $7 - 3 = 4$ terms, or does it have $7 - 3 + 1 = 5$ terms?

Exercise 3.3.4. Sigma Notation ☕

Evaluate the following sums:

- $\sum_{n=0}^{3} 2n$
- $\sum_{n=0}^{3} (-1)^n n^2$
- $\sum_{n=0}^{3} 2^n$

Exercise 3.3.5. Properties of Summations ☕☕☕

Let c be an arbitrary real number, j and k natural numbers with $j < k$, and a_n and b_n be arbitrary sequences. For each of the following properties, explain why it is true, or come up with a counterexample that shows it is not.

- $\sum_{n=j}^{k} c \cdot a_n = c \sum_{n=j}^{k} a_n$
- $\sum_{n=j}^{k} (a_n + b_n) = \left(\sum_{n=j}^{k} a_n \right) + \left(\sum_{n=j}^{k} b_n \right)$

- $\sum_{n=j}^{k} (a_n \cdot b_n) = \left(\sum_{n=j}^{k} a_n\right) \cdot \left(\sum_{n=j}^{k} b_n\right)$

- $\sum_{n=0}^{k} a_n = \sum_{n=1}^{k+1} a_{n-1}$

- $\sum_{n=0}^{k} c = ck$

- $\sum_{n=1}^{k} c = ck$

Arithmetic Series

If the summand a_n is an arithmetic sequence, the summation is called an *arithmetic series*. In this case, we have a nice formula for the sum!

Theorem 3.3.6. Arithmetic Series Formula

Let a_n be an arithmetic sequence with initial term a_0 and common difference d. Then the following sum has closed form

$$\sum_{n=0}^{N} a_n = (N+1) \cdot \frac{a_0 + (a_0 + Nd)}{2}.$$

A nice short way to state the arithmetic series formula is as follows:

The sum of an arithmetic series is equal to the number of terms times the average of the first term and the last term.

Exercise 3.3.7. Lining Up the Formal and the Informal ☕

In the more formal statement of the arithmetic series formula, what expression represents...

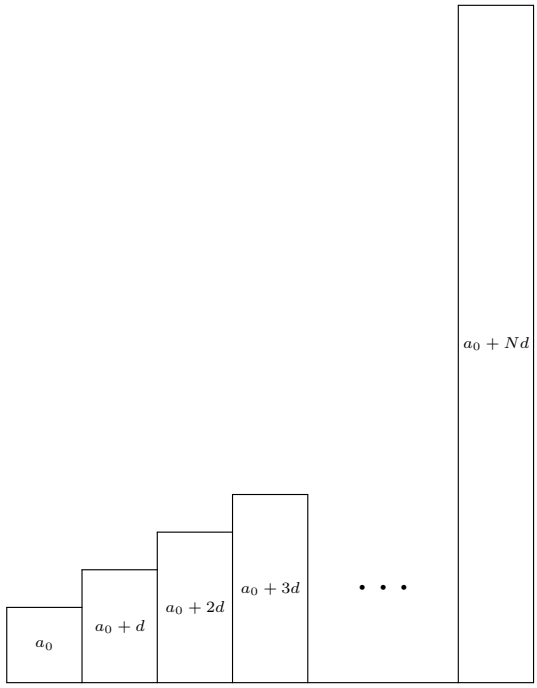
- ...“number of terms”?
- ...“first term”?
- ...“last term”?
- ...“average”?

Exercise 3.3.8. A Visual Argument for the Arithmetic Series Formula ☕☕☕

Here we draw a diagram to show why the Arithmetic Series Formula works. Consider the arithmetic sum

$$(a_0) + (a_0 + d) + (a_0 + 2d) + \cdots + (a_0 + Nd)\,.$$

- For each term in the sum, we draw a corresponding rectangle. Specifically, a one by a_0 rectangle represents the first term, a one by $a_0 + d$ rectangle represents the second term, and so on. These rectangles are stacked in order in the first quadrant, next to each other on the x-axis, with sides of width one all on the x-axis. Explain why the area of the region is equal to the sum.



- Duplicate the entire region in the opposite order to build one giant rectangle. Draw a one by $a_0 + Nd$ rectangle on top of the leftmost, then a one by $a_0 + (N-1)d$ rectangle on top of the second, and so on until the last rectangle gets topped with a one by a_0 rectangle. In this new giant rectangle that is formed...
  - ...what is the width?
  - ...what is the height?
  - ...what is the total area?
- Explain why the total area of that rectangle must be exactly double the value of the arithmetic sum.
- Divide the total area by two to arrive at the arithmetic series formula!

Note that you have seen something similar in Calculus I in the context of evaluating Riemann Sums. In particular, Gauss's Formula states

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}.$$

Exercise 3.3.9. Gauss's Formula as an Arithmetic Series ☕☕

Pretend for a second (or thirty) that you do not know Gauss's Formula. Evaluate $\sum_{n=1}^{N} n$ using the Arithmetic Series Formula. Verify this produces the right-hand side of Gauss's Formula.

Example 3.3.10. Adding Multiples of Six

Suppose we wish to find the sum of all multiples of six between 1000 and 2000. We notice that neither 1000 nor 2000 are divisible by six. However, multiples of six can never be too far away. In particular, $1002 = 6 \cdot 167$ and $1998 = 6 \cdot 333$. Thus, the summation we wish to evaluate is

$$1002 + 1008 + 1014 + \cdots + 1998$$

which can also be written as

$$6 \cdot 167 + 6 \cdot 168 + 6 \cdot 169 + \cdots + 6 \cdot 333.$$

From the above forms, we now have all the information we need to apply the Arithmetic Series Formula.

- First Term: 1002
- Last Term: 1998
- Number of Terms, Remembering the Fencepost: $333 - 167 + 1 = 167$

We now evaluate the summation using the Arithmetic Series Formula as follows:

$$\begin{aligned} 1002 + 1008 + 1014 + \cdots + 1998 &= (\text{Number of Terms})\,(\text{Average of First and Last}) \\ &= (167)\left(\frac{1002 + 1998}{2}\right) \\ &= 167 \cdot 1500 \\ &= 250,500. \end{aligned}$$

Exercise 3.3.11. Practice with Arithmetic Series ☕☕

- Add up all the whole numbers from 1 to 1000 inclusive.

- Add up all the whole numbers from 1000 to 2000 inclusive.

- What is the sum of all multiples of seven between 1000 and 2000?

- Compute the following summation using the Arithmetic Series Formula:

$$\sum_{n=4}^{13}(3n-1).$$

Geometric Series

If the summand a_n is a geometric sequence, the summation is called an *geometric series*. In this case, we again have a nice formula for the sum!

Theorem 3.3.12. Finite Geometric Series Formula

Let a_n be a geometric sequence with initial term a_0 and common ratio r. Then the following sum has closed form

$$\sum_{n=0}^{N} a_n = a_0 \cdot \frac{1-r^{N+1}}{1-r}.$$

In words, you can state the geometric series formula as follows:

The sum of a geometric series is equal to the first term times one minus the common ratio raised to the number of terms, divided by one minus the common ratio.

Exercise 3.3.13. An Algebraic Argument for the Geometric Series Formula ☕☕☕

Here we use algebra to demonstrate why the Geometric Series Formula is valid. Consider the following geometric series and call it S for sum:

$$S = (a_0) + (a_0 r) + \left(a_0 r^2\right) + \cdots + \left(a_0 r^N\right)$$

- Explain why the following equality holds:

$$rS = (a_0 r) + \left(a_0 r^2\right) + \left(a_0 r^3\right) + \cdots + \left(a_0 r^{N+1}\right)$$

- Subtract the two above equations. Fill in the right hand side below.

$$S - rS =$$

- Solve for S in the equation above to construct the Geometric Series Formula!

Exercise 3.3.14. Trying Out the Geometric Series Formula ☕

Consider the summation $1 + 10 + 10^2 + 10^3 + 10^4 + 10^5$.

- Find the total by just doing the arithmetic. Evaluate the powers of ten and then add them up.

- Find the total by using the Geometric Series Formula. Verify that your answers match!

Exercise 3.3.15. Powers of Two ☕☕

Consider the summation $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5$.

- Find the total by just doing the arithmetic. Evaluate the powers of two and then add them up.

- Find the total by using the Geometric Series Formula. Verify that your answers match!

- Use the Geometric Series Formula to evaluate

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^N.$$

- Write in words the answer to the following: "*A finite sum of consecutive powers of two, starting at one, is equal to...* "

Example 3.3.16. Difference of Two Quartics Formula

Here we show how the geometric series formula can be used to obtain a factorization formula! In particular, let us evaluate the summation

$$A^3 + A^2B + AB^2 + B^3.$$

We follow the little remark above, noting that the first term is $\left(A^3\right)$ and the common ratio is (B/A). We now evaluate the sum and then clean up the resulting compound fraction:

$$\begin{aligned} A^3 + A^2B + AB^2 + B^3 &= A^3 \frac{1 - \left(\frac{B}{A}\right)^4}{1 - \left(\frac{B}{A}\right)} \\ &= A^3 \frac{A - B^4/A^3}{A - B} \\ &= \frac{A^4 - B^4}{A - B}. \end{aligned}$$

Multiplying both sides by $A - B$, we have the difference of two quartics factorization as follows:

$$A^4 - B^4 = (A - B) \cdot \left(A^3 + A^2B + AB^2 + B^3\right).$$

Exercise 3.3.17. Difference of Two Cubes

- In the same manner, use the geometric series formula to build the more familiar difference of two cubes formula:

$$A^3 - B^3 = (A - B) \cdot \left(A^2 + AB + B^2\right).$$

- Again using the same technique, figure out a formula for factoring $A^n - B^n$ for an arbitrary

natural number n.

Exercise 3.3.18. Alternate Factorization of a Difference of Quartics

Notice that we could also factor a difference of two quartics by using the difference of two squares formula. In particular,

$$A^4 - B^4 = \left(A^2\right)^2 - \left(B^2\right)^2 = \left(A^2 - B^2\right)\left(A^2 + B^2\right).$$

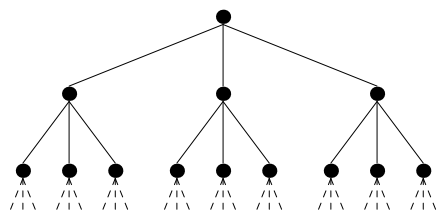
Is this factorization compatible with the one we found via the geometric series formula? Explain.

Applications of Geometric Series

Geometric series come up very frequently in almost any quantitative discipline. Here we give a few such examples.

Example 3.3.19. A Population Model

Suppose we have a mama critter who is pregnant with three baby critters. Each generation, each critter will give birth to three new critters. Let's find a formula for the number of descendants up to and including n generations of descendants. To get started, we make a table listing the population each generation.



Year	Summation	Description	Total
0	1	Just the mama	1
1	1+3	The mama and her three children	4
2	$1+3+3^2$	The mama, three children, and nine grandchildren	13
3	$1+3+3^2+3^3$	All previous plus twenty-seven great-grandchildren	40
$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$\sum_{n=0}^{N} 3^n$	N generations of descendants	GSF

where GSF stands for Geometric Series Formula. We now use it to evaluate the general summation.

$$\sum_{n=0}^{N} 3^n = (1)\left(\frac{1-3^{N+1}}{1-3}\right) = \left(\frac{1-3^{N+1}}{-2}\right) = \frac{1}{2}\left(3^{N+1}-1\right)$$

Thus, the model predicts that after N generations, there are $\frac{1}{2}\left(3^{N+1}-1\right)$ total critters in the family tree.

Exercise 3.3.20. Meeting Critter-ia for a Geometric Series ☕

- In the above example, why was it valid to use the Geometric Series Formula?
- In the above example, plug in $N = 0, 1, 2,$ and 3 into the formula $\frac{1}{2}\left(3^{N+1}-1\right)$ and verify that it produces the correct totals.
- The above example demonstrates that the quantity $3^{N+1}-1$ is always divisible by 2. Why is this the case?

Economists often use Geometric Series when studying economic activity. The exercise below is related to the idea of the *velocity of money*, a measurement of how quickly money gets re-spent as it is received.

Exercise 3.3.21. Velocity of Money ☕☕

Let us a consider a simple model of income and spending. People on average spend 80% of what income they receive. For example, say a contractor earns \$100,000 for a building job. He then

spends 80% of this on a fancy automobile. Collectively, the car salesman, dealership, and auto manufacturer receive \$80,000. They in turn go spend 80% of that \$80,000 on going out to dinner, reinvestment in their business, or other goods and services that are then received as income by others. Thus, these others will receive \$64,000 which they in turn will again go out and spend 80% of.
Suppose also that on average, money changes hands roughly once a month. That is, there is a one-month delay between receiving money as income and going out to spend it.
The government invests \$5 billion in public infrastructure. How much total economic activity is actually generated by this investment in one year? Use the Geometric Series Formula to evaluate your answer!

Here is an example that will initially look out of place in this section. We will later see why this is in fact geometric series in disguise!

Exercise 3.3.22. Partial Sums of Fibonacci Numbers ☕☕☕

Recall F_n, the sequence of Fibonacci numbers.

- Compute the following quantities:

$$\begin{aligned} F_0 &= \\ F_0 + F_1 &= \\ F_0 + F_1 + F_2 &= \\ F_0 + F_1 + F_2 + F_3 &= \\ F_0 + F_1 + F_2 + F_3 + F_4 &= \\ F_0 + F_1 + F_2 + F_3 + F_4 + F_5 &= \\ F_0 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6 &= \end{aligned}$$

- Explain why $F_0 + F_1 + \cdots + F_N$ cannot be evaluated by directly applying the Arithmetic or Geometric Series Formula.

- Do you notice any patterns in the sums above? In particular, see if you can express the sum of Fibonacci numbers $\sum_{i=0}^{n} F_i$ in terms of just a single Fibonacci number.

We don't have the tools at the moment to prove that formula is correct, but we will revisit this example in Chapter 4.8.

3.4 The Sequence of Partial Sums

Adding Terms in a Sequence: Integration for Sequences

Given a sequence a_n, we build a new sequence A_N called the *sequence of partial sums* by keeping a running total of all terms in a_n from 0 to N. We state this definition more formally.

Definition 3.4.1. Sequence of Partial Sums

Let a_n be a sequence. Define A_N, the *sequence of partial sums* of a_n to be

$$A_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + a_2 + \cdots + a_N.$$

When studying a sequence and its partial sums, it can be helpful to organize your data in a table.

Example 3.4.2. From a Sequence to Partial Sums

Consider the sequence of odd natural numbers $a_n = 2n + 1$. We compute a few partial sums and see if we can notice a pattern.

n	a_n	A_n	Total
0	1	1	1
1	3	1+3	4
2	5	1+3+5	9
3	7	1+3+5+7	16
4	9	1+3+5+7+9	25
5	11	1+3+5+7+9+11	36

We notice the column of totals contains all perfect squares. In particular, the number in row n is always exactly $(n+1)^2$. Thus, the pattern suggests that

$$A_N = \sum_{n=0}^{N} (2n+1) = (N+1)^2$$

Exercise 3.4.3. Computing a Partial Sum with the Arithmetic Series Formula ☕☕

Notice the sum A_N above is in fact an arithmetic series! Use the arithmetic series formula to evaluate

$$A_N = \sum_{n=0}^{N} (2n+1)$$

and confirm it matches our conjectured formula from the table.

Notice in Example 3.3.19, a_n represents the number of critters in generation n, whereas A_N represents

the total number of descendants up to and including generation N. This is a good way to think of the relationship between a sequence a_n and the corresponding partial sums A_N; the quantities A_N keep running totals of all a_n we have encountered up to and including index N.

Exercise 3.4.4. Add-Ups

Suppose you start a push-up routine on day 0, where you do a_0 push-ups. On day 1, you do a_1 push-ups. On day 2, you do a_2 push-ups, and so on. In this context, what does the sequence of partial sums A_N represent?

Discrete Derivatives: Derivatives for Sequences

Given a sequence of partial sums, we can uncover the sequence from which it came. The difference of two consecutive partial sums will be a single term in the sequence, since

$$\begin{aligned} A_N - A_{N-1} &= \sum_{n=0}^{N} a_n - \sum_{n=0}^{N-1} a_n \\ &= (a_0 + a_1 + a_2 + \cdots + a_N) - (a_0 + a_1 + a_2 + \cdots + a_{N-1}) \\ &= a_N. \end{aligned}$$

Example 3.4.5. From Partial Sums to a Sequence

Let's try to undo Example 3.4.2. Suppose we start with $A_N = (N+1)^2$. We draw a table to see what terms a_n would have been added together to obtain those totals.

N	A_N	a_N	Difference
0	1	$1-0$	1
1	4	$4-1$	3
2	9	$9-4$	5
3	16	$16-9$	7
4	25	$25-16$	9
5	36	$36-25$	11

We see that sure enough, the last column is the sequence of odd numbers and is always one more than twice N. Thus, we have that $a_N = 2N + 1$.

Exercise 3.4.6. Taking a Difference of Partial Sums

Using the formula $A_N = (N+1)^2$, try taking the difference $A_N - A_{N-1}$ and verify you get the same a_N. That is, simplify the right-hand side of the following expression:

$$A_N - A_{N-1} = (N+1)^2 - ((N-1)+1)^2$$

Note that we have two different indices, as we are taking the convention that n indexes the sequence a_n and N indexes the partial sums A_N. Thus, depending on which we start with, it looks like we have the "wrong" index for the other (A_n vs A_N or a_n vs a_N). This is nothing to worry about, as the sequence is really just the mapping from the natural numbers to the reals. This is similar to how $f(x) = x^2$ and $f(t) = t^2$ are the same function on the reals, but just listed with different independent variables.

Example 3.4.7. Revisiting Our Critters

In Example 3.3.19, we had computed the formula for a sequence of partial sums, even though at the time we didn't call it that. In particular, given the sequence

$$a_n = 3^n$$

we found a closed formula for the sequence of partial sums as

$$A_N = \sum_{n=0}^{N} a_n = \frac{1}{2}\left(3^{N+1} - 1\right)$$

by using the Geometric Series Formula. Here we demonstrate that the difference of consecutive partial sums will reproduce the original summand.

$$\begin{aligned} A_N - A_{N-1} &= \frac{1}{2}\left(3^{N+1} - 1\right) - \frac{1}{2}\left(3^{(N-1)+1} - 1\right) \\ &= \frac{1}{2}\left(3^{N+1} - 1 - 3^N + 1\right) \\ &= \frac{1}{2}\left(3^{N+1} - 3^N\right) \\ &= \frac{3^N}{2}\left(3^1 - 1\right) \\ &= 3^N \end{aligned}$$

Sure enough, $a_n = 3^n$ was our original summand!

Taking a Sequence to Partial Sum and Back Again

Here we summarize a bit of what happened above.

- Given a sequence a_n, we can define an associated sequences of partial sums $A_N = \sum_{n=0}^{N} a_n$.
- If a_n is an arithmetic or geometric sequence, we can find a formula for A_N using the Arithmetic

Series Formula or Geometric Series Formula. If it a_n is not an arithmetic or geometric sequence, then writing out a table of partial sums and looking for a pattern can be a good strategy.

- Given A_N, we can recover a_n by taking differences of consecutive partial sums $A_N - A_{N-1}$.

Notice this is very similar to what happened in Calculus I or even in the first part of this course. In those sections, you could start with a function $f(x)$ and find its antiderivative $F(x)$. If you then took the derivative of this $F(x)$, you would end up with the original $f(x)$.

Exercise 3.4.8. Converting Back and Forth ☕☕☕

For each of the following sequences a_n, compute the corresponding sequence of partial sums A_N. Once you have A_N, then compute the difference of consecutive partial sums $A_N - A_{N-1}$ and verify that the original sequence comes back!

- $a_n = \frac{5}{2^n}$

- $a_n = \frac{2}{3^{2n+1}}$

- $a_n = 5 - n$

- $a_n = 3n^2 + 3n + 1$

- $a_n = (-1)^n$

- $a_n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$

Often the study of the real numbers and related objects is called *continuous* mathematics while the study of the natural numbers and related objects is called *discrete* mathematics. In this course, we encounter many interesting parallels between the two pursuits!

Exercise 3.4.9. Discrete/Continuous Analogy ☕☕☕☕

- In what ways is taking the partial sums of a sequence similar to taking the integral of a function over the real numbers? (**Hint:** Plot your sequence and draw rectangles with height a_n and width one.)

- In what ways is taking the difference of consecutive terms in a sequence $(a_n - a_{n-1})$ similar to taking the derivative of a function over the real numbers? (**Hint:** Plot your sequence and think of how you could obtain $a_n - a_{n-1}$ as the slope of a secant line.)

- Suppose you start with a sequence a_n. You add up terms to create the sequence of partial sums A_N. You then take the difference of consecutive terms in the partial sums and find that $A_N - A_{N-1} = a_N$. What theorem of calculus is this analogous to and why?

3.5 Infinite Series

Well here's an interesting question.

What does it mean to add up infinitely many numbers?
-Lots of people

We provide the most commonly used modern definition.

Definition 3.5.1. Infinite Series, Convergence, and Divergence

Let a_n be a sequence of real numbers. Then the *infinite sum* of all terms of a_n is defined to be the limit of partial sums A_N. That is,

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} A_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n.$$

If the limit exists, we say the infinite series *converges* to the value of the limit. If the limit is infinity or does not exist, then we say the infinite series *diverges*.

The idea is simple; if you want to add up infinitely many numbers, a good place to start is by just adding up finitely many of them. However, if you only add up finitely many, your answer has some error to it. If you want that error to go down, add up more and more of them! The limit of the values of these partial sums will be the exact answer.

Exercise 3.5.2. The Return of the Discrete/Continuous Analogy ☕☕☕

In what way is the definition of an infinite series analogous to the definition of a horizontally unbounded improper integral?

Exercise 3.5.3. The Definitions in Words ☕☕

We have defined three very important interconnected structures:

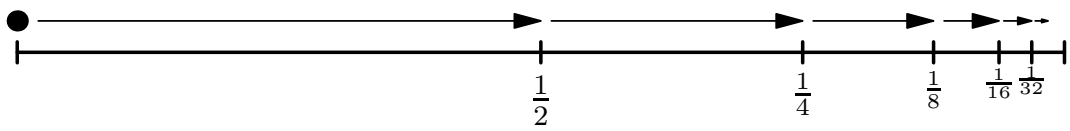
- A sequence a_n.
- A sequence of partial sums A_N.
- An infinite series $\sum_{n=0}^{\infty} a_n$.

Describe in words how the three structures are related and are built from one another.

Zeno's Paradox, Resolution, and Consequences

The next example is traceable back to the writings of Aristotle in the third century BC! Specifically, he states Zeno's Paradox of *Dichotomy* as:

> *That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*



This was meant to be a "proof" that an object (say an arrow in flight) could never reach its target. This paradox is resolved with our notion of infinite series.

Exercise 3.5.4. A Classic Infinite Series

Consider the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$. This can be interpreted as the sequence of distances the arrow must travel in the Dichotomy paradox (supposing it was fired one meter from its target and all lengths are measured in meters). Since it were fired from one meter away, we expect that the total distance traveled is one.

- Find an explicit formula a_n that describes the sequence above.

- Compute the corresponding sequence of partial sums A_N.

- Evaluate the infinite sum
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$
by taking the limit of the sequence of partial sums. Verify the total is in fact one.

In the above case, we were able to provide a resolution for the paradox and verify the total with our geometric series formula, but the answer was not particularly surprising. Here is a more interesting example!

Exercise 3.5.5. An Alternating Geometric Series ☕☕☕

Suppose a bug moves forward half a meter. It then moves backwards one-fourth of a meter. It then moves forward one-eighth of a meter. It then moves backwards one-sixteenth of a meter. This pattern of moving forwards, then backwards, by half the previous distance each time, continues forever. At the end of time, where does the bug end up?

To solve this problem, we notice that it is equivalent to adding up all terms in the sequence $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots$. Following the method of Example 3.1.21, this sequence can be expressed as a geometric sequence with initial term $a_0 = \frac{1}{2}$ and common ratio $r = -\frac{1}{2}$ as follows:

$$a_n = \frac{1}{2}\left(-\frac{1}{2}\right)^n.$$

- Let $A_N = \sum_{n=0}^{N} a_n$ be the sequence of partial sums. Find a formula for A_N.

- Compute $\sum_{n=0}^{5} a_n$.

- Compute $\sum_{n=0}^{10} a_n$.

- Compute $\sum_{n=0}^{\infty} a_n$ from the definition of an infinite series.

So, where does the bug end up?

Infinite Geometric Series

The notion of an infinite series can be used to give a rigorous interpretation to the infinite decimal expansions as well!

Exercise 3.5.6. Repeating Decimal Expansion

- Notice that the decimal expansion 0.333 can be written as a geometric series with three terms and common ratio 1/10 using the definition of place value. In particular,

$$0.333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}.$$

  Compute its value via the finite geometric series formula.

- Write 0.3333 as a geometric series with four terms and common ratio 1/10. Compute its value via the finite geometric series formula.

- Write 0.33333 as a geometric series with five terms and common ratio 1/10. Compute its value via the finite geometric series formula.

- Write

$$\underbrace{0.3333\ldots3}_{n \text{ threes}}$$

  as a geometric series with n terms and common ratio 1/10. Compute its value in terms of n via the finite geometric series formula.

- Take the limit as n approaches infinity of your formula from the previous part to prove that "point three repeating" really does equal one-third.

We can generalize the previous examples. Notice in all cases, the geometric series formula let us calculate an explicit formula for the sequence of partial sums. As long as the common ratio $|r| < 1$, the limit as $n \to \infty$ will exist, as the r^{N+1} term will go to zero and we will be left with just $a_0 \frac{1}{1-r}$. This brings us to the *infinite geometric series* formula (also sometimes just referred to as the geometric series formula).

Theorem 3.5.7. Infinite Geometric Series Formula

If a and r are real numbers and $|r| < 1$, then

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Notice in the above formula, the number a represents the first term of the series and r represents the common ratio.

Example 3.5.8. A Messier Repeating Decimal Expansion

Suppose we wish to write the repeating decimal

$$1.\overline{615384}$$

as a fraction. We repeat (heh) the method of Exercise 3.5.6, where we use base-ten place value to express the decimals as sums of terms with common ratio equal to a negative power of ten.

$$\begin{aligned} 1.\overline{615384} &= 1.615384615384615384\ldots \\ &= 1 + \frac{615384}{10^6} + \frac{615384}{10^{12}} + \frac{615384}{10^{18}} + \cdots \end{aligned}$$

The very first term, 1, clearly does not fit the pattern given by the rest of the terms. So, we won't worry about that term, and instead just work on evaluating the rest while we leave the 1 out front. The rest of the terms form an infinite geometric series with initial term $a = \frac{615384}{10^6}$ and common ratio $r = \frac{1}{10^6}$. We now apply the infinite geometric series formula, noting that r, being one over a million, is comfortably between -1 and 1 as required. Note that we resolve the compound fraction

below by multiplying the top and bottom by 10^6.

$$\begin{aligned}
1.\overline{615384} &= 1 + \frac{\frac{615384}{10^6}}{1 - \frac{1}{10^6}} \\
&= 1 + \frac{615384}{10^6 - 1} \\
&= 1 + \frac{615384}{999999} \\
&= 1 + \frac{8}{13} \\
&= \frac{21}{13}
\end{aligned}$$

Exercise 3.5.9. Using the Geometric Series Formula ☕☕

Consider the following series:

$$\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$$

- Write out the first few terms of the above series. That is, expand the sigma notation by plugging in $n = 5, 6, 7, 8, \ldots$ and evaluating the summand in each case.

- Is the above series geometric? Explain why or why not. If so, what is the common ratio r? What is the first term a?

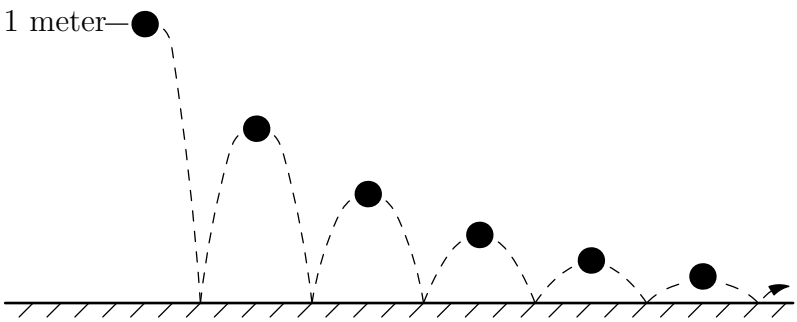
- Find the value of the above series.

Exercise 3.5.10. Not Using the Geometric Series Formula ☕

Explain why the following calculation is not valid according to our definition of infinite series:

$$\begin{aligned} 1 + 2 + 2^2 + 2^3 + 2^4 + \cdots &= \frac{1}{1-2} \\ &= -1 \end{aligned}$$

Exercise 3.5.11. The Bouncing Ball ☕☕



A magical bouncy ball is bounced from a height of 1 meter. On each bounce, it always rebounds to exactly five-eighths of the height it fell from. What is the ball's total vertical distance traveled from now until the end of time?

Exercise 3.5.12. Evaluating Another Infinite Series ☕☕

Consider the constant sequence $a_n = 2$. Now consider the corresponding infinite sum:

$$\sum_{n=0}^{\infty} a_n$$

- Write out the first five terms of the sequence a_n. Also write out the first five terms of the corresponding sequence of partial sums.

- Find an explicit formula for the sequence of partial sums.

- Does the infinite series converge? If so, what value does it converge to?

Exercise 3.5.13. A Telescoping Sum ☕☕☕

Consider the following sequence:

$$a_n = \frac{2}{n^2 + 5n + 6}$$

- Compute the first five terms of the sequence.

- Compute the first five partial sums of the sequence.

- Based on your data, conjecture a formula for

$$A_N = \sum_{n=0}^{N} \frac{2}{n^2 + 5n + 6}$$

- Prove your answer is correct via a partial fraction decomposition. Specifically, perform a PFD on $a_n = \frac{2}{n^2+5n+6}$ and then notice that when you add the terms in a partial sum, all but two terms cancel! (This lucky happening is what is referred to as a series *telescoping*,

as it is collapsing in on itself much like a retractable telescope would.)

- Use your formula for the partial sums and the definition of an infinite series to write an $N - \epsilon$ proof for the value of
$$\sum_{n=0}^{\infty} \frac{2}{n^2 + 5n + 6}.$$

Exercise 3.5.14. Practice with Infinite Series ☕☕☕

For each of the following sequences a_n, carry out the following steps:

- Write out the first five terms of the sequence a_n. Also write out the first five terms of the

sequence of partial sums A_N for the corresponding series.

- Find a formula for the sequence of partial sums $A_N = \sum_{n=0}^{N} a_n$.
- Does the infinite series $\sum_{n=0}^{\infty} a_n$ appear to converge? If so, what value does it appear to converge to?

And now, the sequences:

- The sequence defined by
$$a_n = 2n$$

- The sequence defined by
$$a_n = 2^n$$

- The sequence defined by
$$a_n = \left(\frac{2}{3}\right)^n$$

- The sequence defined by
$$a_n = \left(\frac{-1}{2}\right)^n$$

- The sequence defined by

$$a_n = (-1)^n$$

- The sequence defined by

$$\begin{aligned} a_0 &= 3 \\ a_n &= \frac{-1}{3} a_{n-1} \end{aligned}$$

- The sequence defined by

$$\begin{aligned} a_0 &= 5 \\ a_n &= a_{n-1} + 1 \end{aligned}$$

- The sequence defined by

$$a_n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise} \end{cases}$$

Absolute Convergence and Rearrangements

When we add up finitely many numbers, we take properties like commutativity and associativity for granted. We add up numbers in whatever order is most convenient. With infinite series, we cannot be quite so cavalier!

Example 3.5.15. Did You Know that Zero Equals One?

Here we use the fact that 0=-1+1.

$$\begin{aligned} 1 &= 1 + 0 + 0 + 0 + 0 + \cdots \\ &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots \\ &= (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + (1 + -1) + \cdots \\ &= 0 + 0 + 0 + 0 + 0 + \cdots \\ &= 0 \end{aligned}$$

Exercise 3.5.16. What? ☕☕☕

Let us now correctly analyze the infinite sum

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$$

- Consider the sequence $a_n = (-1)^n$. Use the Geometric Series Formula to find the corresponding sequence of partial sums A_N.

- What is the limit of the sequence of partial sums?

- Thus, what is the correct value of the infinite series $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$?

It turns out that the key lies in the distinction between a series being convergent vs being *absolutely convergent*, a stronger type of convergence.

Definition 3.5.17. Absolute Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if and only if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Absolute convergence is the idea that it wasn't just some sort of cancellation of positive and negative terms that let the partial sums stabilize. Rather, the magnitudes of the terms were going to zero quickly enough. To test this, we just take the term-by-term absolute value of the series and see if the resulting series still converges.

Example 3.5.18. An Absolutely Convergent Series

The infinite geometric series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

is absolutely convergent because

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{4}\right| + \left|-\frac{1}{8}\right| + \left|\frac{1}{16}\right| + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

The term-by-term absolute value of the series still converges, so the original series is declared absolutely convergent.

Contrast this concept with the following definition, a weaker form of convergence called conditional.

Definition 3.5.19. Conditional Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *conditionally convergent* if and only if it converges but $\sum_{n=0}^{\infty} |a_n|$ diverges.

Example 3.5.20. A Conditionally Convergent Series

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

is conditionally convergent because

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

diverges. In particular, it is the harmonic series which totals to ∞ (as we will see in Example 3.7.1).

Exercise 3.5.21. A Maybe Absolutely Convergent Series ☕☕

Is the infinite geometric series

$$0.1 - 0.02 + 0.004 - 0.0008 + 0.00016 - \cdots$$

absolutely convergent or conditionally convergent? Explain.

It turns out that for absolutely convergent series, rearranging of terms and any sort of normal algebraic manipulation is fine. This is a theorem that is rather difficult to prove and will be saved for a later mathematical adventure. For this course, we will just use it!

3.6 Convergence Tests

It is often too difficult to determine the exact value of an infinite series (if it converges at all). Thus, we usually settle for the knowledge *that* a series converges (or diverges) as opposed to finding *what* number it converges to. By "settling" as such, we are not actually giving up too much. If we can guarantee that a series converges, it means it is safe to approximate it by just taking a partial sum with lots of terms.

Exercise 3.6.1. Why Convergence is So Critical ☕☕

Why would it not make sense to approximate a divergent infinite series using a partial sum with lots of terms?

We should also mention that in essence, our study of infinite geometric series could be viewed as a convergence test! In particular, use our work from Section 3.5 to answer the following question.

Exercise 3.6.2. Convergence Criterion for Geometric Series ☕

Let a be a nonzero real number. Then, for what real numbers r does the infinite geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

converge? For what values of r does it diverge?

The big pro of the above test is that in the case that the series converges, we also know the value it converges to. The big con is how limited the scope of the test is; the vast majority of series are not geometric! Thus, to broaden the scope of series whose convergence/divergence can be determined, we need more tests. Below, we list the names of the eight convergence tests we will present in this section.

1. No Hope Test
2. Alternating Series Test
3. Integral Test
4. p-series Test
5. Direct Comparison Test
6. Limit Comparison Test
7. Ratio Test
8. Root Test

3.7 No Hope Test and Alternating Series Test

No Hope Test and the Harmonic Series

The No Hope Test is sometimes also referred to as the Divergence Test or the nth Term Test. Intuitively, this test says that if the terms of the sequence a_n do not go to zero, then their sum has no hope of converging. In other words, for a series to have any hope of converging, the numbers that are being added together must become arbitrarily small. We give a more formal statement here.

Theorem 3.7.1. No Hope Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Example 3.7.2. A Very Divergent Series

Consider the series

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}+1}.$$

One can apply the No Hope Test to the above series, because the limit of the summand is

$$\lim_{n\to\infty} \frac{2^n}{2^{n+1}+1} \overset{\text{LHR}}{=} \lim_{n\to\infty} \frac{2^n \ln(2)}{2^{n+1}\ln(2)} = \frac{1}{2}.$$

The limit of the summand was not zero, and thus the series diverges.

Intuitively, the above example was just saying that our sequence $a_n = \frac{2^n}{2^{n+1}+1}$ approached $1/2$ as n got large, so in essence the infinite series was adding the number one-half to itself infinitely many times. Said a bit more formulaically, one could write

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}+1} &= \frac{1}{3} + \frac{2}{5} + \frac{4}{9} + \frac{8}{17} + \frac{16}{33} + \cdots \\ &\approx \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,\end{aligned}$$

and this sum of infinitely many one-halves is clearly infinity.

Exercise 3.7.3. Believing the Approximation Above ☕

Verify the approximation

$$\frac{1}{3} + \frac{2}{5} + \frac{4}{9} + \frac{8}{17} + \frac{16}{33} + \cdots \approx \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

that we wrote above by crunching out decimal approximations for all the terms on the left-hand side. Seeing those, do you believe it?

Pro tip: listing out the first few terms and looking at decimal approximations for them, as well as their partial sums, is always a good starting place for getting a sense of what an infinite series is doing!

Another pro tip: the No Hope Test does *not* say the following:

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then } \sum_{n=0}^{\infty} a_n \text{ converges.} \tag{☹}$$

This is a fallacy and a *very* common mistake. If the terms of the sequence go to zero, then the series has some chance of converging, but it is no guarantee. Summing this all up in two bullets, we have the following:

- If $\lim_{n\to\infty} a_n$ is nonzero, you can conclude the corresponding infinite series $\sum_{n=0}^{\infty} a_n$ diverges by the No Hope Test.
- If $\lim_{n\to\infty} a_n$ is zero, you cannot conclude anything about the corresponding infinite series $\sum_{n=0}^{\infty} a_n$. One simply moves on and tries some other test.

Example 3.7.4. Where NHT Applies and Does Not Apply

- Consider the infinite series
$$\sum_{n=0}^{\infty} \arctan(n).$$
We notice that $\lim_{n\to\infty} \arctan(n) = \pi/2 \neq 0$. Thus, the series diverges by NHT, since we are essentially adding up infinitely many copies of $\pi/2$ to itself.
- Consider the infinite series
$$\sum_{n=0}^{\infty} \arctan\left(\frac{1}{n}\right).$$
Since $\lim_{n\to\infty} \arctan\left(\frac{1}{n}\right) = 0$, we cannot make any conclusion about this series via NHT. The summand approaches zero, so there is some hope of the series converging, but further work would be required to determine this.

One particularly important infinite series is the *harmonic series*, written as follows

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.$$

We will revisit this series again and again!

Exercise 3.7.5. First Encounter ☕

What does the No Hope Test tell you about the convergence/divergence of the harmonic series?

Now try a few on your own!

Exercise 3.7.6. Practice with the No Hope Test ☕☕

- Use the No Hope Test to prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+2}}{n}$ diverges.

- Use the No Hope Test to prove that the series $\displaystyle\sum_{n=1}^{\infty} \cos(1/n)$ diverges.

- What does the No Hope Test tell you about the convergence/divergence of the series $\sum_{n=1}^{\infty} \sin(1/n)$?

Alternating Series Test and Error Bound

Theorem 3.7.7. Alternating Series Test (AST)

Let a_n be a decreasing sequence of positive numbers that approaches zero as n approaches infinity. The summation

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

Intuitively, this test says that the positives and negatives will cancel each other out and the partial sum pendulum will eventually stabilize at some well-defined limit. The following exercise will help us visualize this cancellation.

Exercise 3.7.8. Literal Exercise ☕

- Get up from your chair. Go ahead, I'm waiting. Really. Get up.
- Jump forwards by some amount a_0.
- Jump backwards by some amount a_1, but hey, we're a little tired from that first jump, so jump backwards by less than the distance you jumped forwards.
- Jump forwards by an even lesser amount a_2.
- Jump backwards by an even lesser amount a_3.

- Jump forwards by an even lesser amount a_4.
- Iterate this forever.

Where do you end up? Does your position go to infinity, minus infinity, oscillate wildly, or does it converge upon some particular location?

Example 3.7.9. Using AST

Consider the series $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$. This series can be rewritten as $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n$. Since the sequence $\left(\frac{1}{2}\right)^n$ is positive and decreasing to zero, AST tells us that it converges.

Exercise 3.7.10. Conditional vs Absolute ☕

Did the above series converge conditionally or absolutely? Explain.

Notice that the above series also converges by the infinite geometric series formula. It is very common that more than one convergence test will apply to the same series.

So far, we know nothing about the convergence/divergence of the harmonic series,

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots.$$

However, the AST would apply to that series if only the signs were alternating. Accordingly, define the *alternating harmonic series* to be

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots.$$

Exercise 3.7.11. Alternating Harmonic Series ☕

Use the AST to prove that the alternating harmonic series converges.

Exercise 3.7.12. The AST in Action ☕☕

For each of the series below, explain why it converges by AST or explain why AST does not apply to that series.

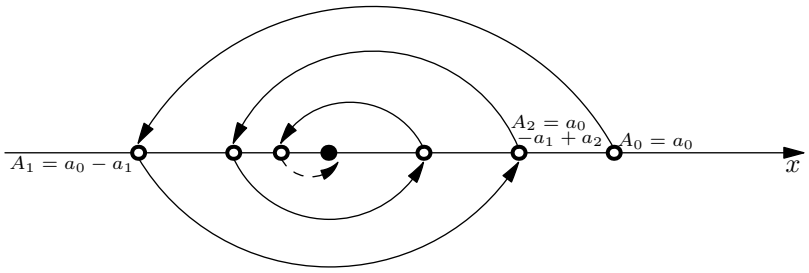
- $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$

- $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \arctan(n)\right)^n$

- $\displaystyle\sum_{n=1}^{\infty} \left(\arctan(n) - \frac{\pi}{2}\right)^n$

In a convergent alternating series, the partial sums always "leapfrog" back and forth over the limiting value the series converges to. This implies that the value of the infinite series is always no further away than the next unused term in any partial sum.



Theorem 3.7.13. Alternating Series Error Bound

Let a_n be a sequence of positive decreasing terms. Let $\sum_{n=0}^{\infty}(-1)^n a_n$ be a convergent series. Then the error in a partial sum is always less than the first unused term in that partial sum. That is,

$$\left|\sum_{n=0}^{\infty}(-1)^n a_n - \sum_{n=0}^{N}(-1)^n a_n\right| < |a_{N+1}|.$$

Example 3.7.14. Alternating Series Error Bug

Consider again the bug described in Exercise 3.5.5. Suppose we wish to know how close the bug is to its ending destination after it has reversed course two times. Since the series is a convergent alternating series, we can use the Alternating Series Error Bound to determine how close it is to its final location. In particular, the Alternating Series Error Bound implies

$$\left|\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\cdots\right)-\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}\right)\right| < \frac{1}{16}.$$

To independently verify this claim, notice that we had already concluded in Exercise 3.5.5 that the infinite series totaled to $\frac{1}{3}$. If we plug in this value, we obtain

$$\begin{aligned}\left|\left(\frac{1}{3}\right)-\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{8}\right)\right| &= \left|\frac{1}{3}-\left(\frac{3}{8}\right)\right| \\ &= \left|\frac{8}{24}-\frac{9}{24}\right| \\ &= \left|-\frac{1}{24}\right|\end{aligned}$$

which is indeed less than one-sixteenth.

Exercise 3.7.15. Alternating Series Error Bug ☕☕

- After the bug has reversed course three times, how close does the Alternating Series Error Bound guarantee the bug is to its final location?

- How many times would the bug have to reverse course for the Alternating Series Error Bound to guarantee the bug is within one one-thousandth of its final location? After you figure this

out, compute the corresponding partial sum and verify that it is within 0.001 of one-third.

3.8 Integral Tests

Integral Test

Here we use integrals to test convergence of infinite series! Intuitively this test says the following:

An infinite series converges if and only if the corresponding improper integral converges.

Theorem 3.8.1. Integral Test

Let $a \in \mathbb{N}$ and $f(n)$ be a decreasing function on $[a, \infty)$.

- If $\displaystyle\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x$ converges, then $\displaystyle\sum_{n=a}^{\infty} f(n)$ converges as well.
- Conversely, if $\displaystyle\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x$ diverges, then $\displaystyle\sum_{n=a}^{\infty} f(n)$ diverges as well.

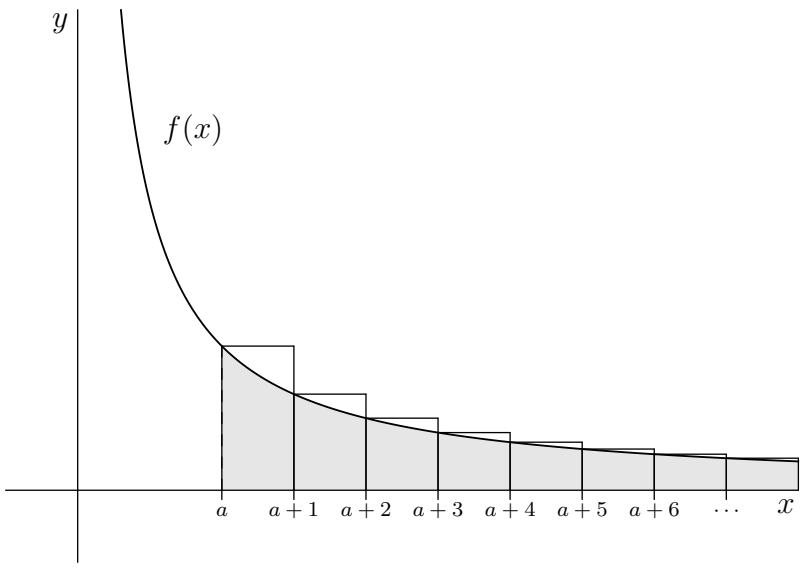
Here is a "proof by picture" to justify the Integral Test.

Exercise 3.8.2. Explaining the Integral Test ☕☕☕

Study the following diagrams and use them to determine why the series converges if and only if the series converges. Specifically:

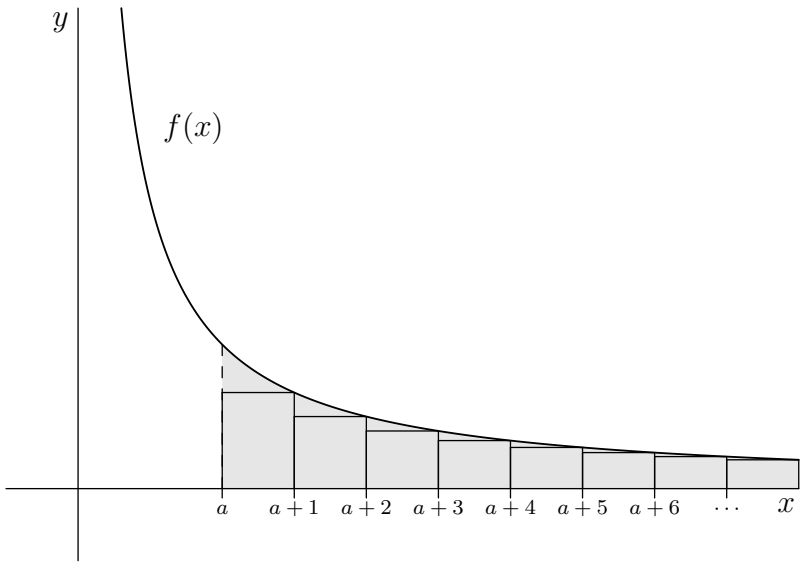
- Explain why the diagram below justifies the inequality

$$\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x \leq \sum_{n=a}^{\infty} f(n).$$



- Explain why the diagram below justifies the inequality

$$\sum_{n=a+1}^{\infty} f(n) \leq \int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x.$$



- Add $f(a)$ to both sides of the previous inequality to conclude

$$\sum_{n=a}^{\infty} f(n) \leq f(a) + \int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x.$$

- Putting both inequalities together, we now have that

$$\int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x \leq \sum_{n=a}^{\infty} f(n) \leq f(a) + \int_{x=a}^{x=\infty} f(x)\,\mathtt{d}x. \qquad (\bowtie)$$

Explain why this inequality shows that the infinite series converges if and only if the corresponding improper integral converges.

Example 3.8.3. Using the Integral Test

- The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because the corresponding improper integral is

$$\begin{aligned}\int_{x=1}^{x=\infty} \frac{1}{x^2}\,\mathtt{d}x &= \lim_{c\to\infty} -\frac{1}{x}\Big]_{x=1}^{x=c} \\ &= \lim_{c\to\infty} -\frac{1}{c} - -\frac{1}{1} \\ &= 1.\end{aligned}$$

  Since the improper integral converges to a finite value, the Integral Test says that the corresponding infinite series converges as well.

- The infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because the corresponding improper integral is

$$\begin{aligned}\int_{x=1}^{x=\infty} \frac{1}{\sqrt{x}}\,\mathtt{d}x &= \lim_{c\to\infty} 2\sqrt{x}\Big]_{x=1}^{x=c} \\ &= \lim_{c\to\infty} 2\sqrt{c} - 2\sqrt{1} \\ &= \infty.\end{aligned}$$

  Since the improper integral diverges, the Integral Test says that the corresponding infinite series diverges as well.

Exercise 3.8.4. Practice with the Integral Test ☕☕

Use the Integral Test to decide if the following infinite series converge or diverge.

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\mathtt{ln}(n)}$

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\mathtt{ln}(n))^2}$

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+1}$

The next example shows why the assumption of $f(x)$ being a decreasing function is necessary for the Integral Test.

Exercise 3.8.5. An Interesting Example ☕☕☕

Consider the function

$$f(x) = |\sin(\pi x)|$$

- Graph the function $f(x)$ over the positive x-axis.

- Explain why the integral $\displaystyle\int_{x=0}^{x=\infty} f(x)\,\mathtt{d}x$ is equal to infinity.

- Compute the infinite sum $\sum_{n=0}^{\infty} f(n)$.

- In this case, the integral diverged, while the infinite sum converged. Why does this example not contradict the Integral Test?

p-series Test

The next result is just a special case of the Integral Test. However, it comes up often enough that it is worth stating on its own.

Definition 3.8.6. p-series

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for some real number p is called a p-series.

One of these is famous enough to have its own name: the series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

a p-series with $p = 1$, is called the *harmonic series.*

Theorem 3.8.7. The p-series Test

Let p be a real number. Then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

Exercise 3.8.8. Justifying the p-series Test Using the Integral Test ☕

- Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in the case where $p > 1$. Show the corre-

sponding improper integral converges, and thus the series does as well.

- Apply the Integral Test to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ in the case where $p < 1$. Show the corresponding improper integral diverges, and thus the series does as well.

- What does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ do when $p = 1$? Again, write down the corresponding improper integral and calculate it!

3.9 Comparison Tests

Direct Comparison Test

We begin this section with a very famous argument that serves as a counterexample to the converse of the No Hope Test (the statement labeled with (☺) in that section). The following divergence proof by Nicole Oresme dates all the way back to the mid-fourteenth century!

Example 3.9.1. Divergence of the Harmonic Series

Here we show that the harmonic series diverges, that is,

$$\sum_{n=0}^{\infty} \frac{1}{n} = \infty$$

We accomplish this by showing the partial sums will exceed any sum of one-half added to itself again and again. Proceeding:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \infty
\end{aligned}$$

Since the harmonic series is greater than a sum of one-half added to itself infinitely many times, it is infinite.

Exercise 3.9.2. Justifying the Steps ☕☕

Annotate the above proof with a short comment justifying each line of equality or inequality.

Exercise 3.9.3. Revisiting a Convergent Series ☕☕☕

The infinite geometric series formula shows that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1/2}{1-1/2}=1.$$

If you attempt to use Oresme's argument to show that it diverges, where does it break down? Why can't you just group terms together into batches that are at least size one-half?

Exercise 3.9.4. No Hope Test Backwards ☕

Explain why the Harmonic Series is a counterexample to the claim

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ then } \sum_{n=0}^{\infty} a_n \text{ converges.} \qquad (☹)$$

The key idea above is something quite intuitive: a quantity greater than infinity must also be infinity. Conversely, a positive quantity less than a finite number must also be finite. This key idea, stated a bit more formally, is the Direct Comparison Test (DCT).

Theorem 3.9.5. Direct Comparison Test

Let a_n and b_n be nonnegative sequences. If for all natural numbers n, $a_n \le b_n$, then

- $\sum_{n=0}^{\infty} b_n$ converges implies that $\sum_{n=0}^{\infty} a_n$ also converges.
- $\sum_{n=0}^{\infty} a_n$ diverges implies that $\sum_{n=0}^{\infty} b_n$ also diverges.

A short intuitive way to state DCT is as follows:

A series with smaller terms than the terms of a convergent series must also be a convergent series. A series with terms larger than the terms of a series summing to infinity must also sum to infinity.

The trick when using DCT is to pick an easy series to compare to. For example, we often can compare to a p-series.

Example 3.9.6. Redirect the Trig Function

Here we analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(3+\sin(n))}$. Notice that $-1 < \sin(n) < 1$ for all $n \in \mathbb{N}$. Adding three to all sides of that inequality produces

$$2 < 3 + \sin(n) < 4.$$

Thus, we can bound our summand as

$$\frac{1}{n\,(4)} < \frac{1}{n\,(3+\sin(n))} < \frac{1}{n\,(2)}.$$

The right-hand side bound gives no information, but on the left-hand side we have something useful! In particular, the series $\sum_{n=1}^{\infty} \frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Since our series in question is larger than a series that totals to infinity, it must also diverge to infinity.

Exercise 3.9.7. Practice with DCT ☕☕

Use the Direct Comparison Test to prove the following series converge or diverge.

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos(n)}{n}$

Limit Comparison Test

The Direct Comparison Test is an incredibly elegant logical argument, but it's often inconvenient. While it's clear that if a series is *smaller* than a known convergent series, that series must itself converge, what if it is very very slightly larger than a known convergent series? Or very slightly smaller than a known divergent series? For example, we saw that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges, since } \frac{1}{n^2+1} \leq \frac{1}{n^2} \text{ and } \sum_{1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

On the other hand, what about

$$\sum_{n=1}^{\infty} \frac{1}{n^2-2}?$$

The Limit Comparison Test is a stronger version of the direct comparison test. This test says that if two sequences have the same growth order, then the corresponding infinite series either both converge or both diverge. Intuitively, this works because if two sequences have the same growth order, then in the long term they are just a nonzero constant factor apart, and multiplying by a nonzero constant factor cannot change convergence or divergence. Furthermore, having larger magnitude terms than a divergent

series implies divergence, and smaller magnitude terms than a convergent series implies convergence.

Theorem 3.9.8. Limit Comparison Test (LCT)

Let a_n and b_n be sequences of nonnegative terms with the same growth order. That is,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for some nonzero real number c (i.e., the limit of their ratios is neither infinity nor zero). Then, their infinite sums, $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, have the same behavior; they either both converge or both diverge.

A short intuitive way to state LCT is as follows:

A series with the same growth order as or smaller growth order than a convergent series must also be convergent. A series with the same growth order as or larger growth order than a divergent series must also be divergent.

The Limit Comparison Test is particularly useful if a_n is expressed as a fraction with the numerator and denominator both algebraic (expressed just with polynomials and radicals). In this case, one can build a comparison sequence by taking the ratio of leading terms from the numerator and denominator.

Example 3.9.9. Using Leading Terms from the Numerator and Denominator

Suppose we wish to determine the convergence/divergence of

$$\sum_{n=1}^{\infty} \frac{7n+3}{n\sqrt{n^2+n+1}}.$$

Call the summand $a_n = \frac{7n+3}{n\sqrt{n^2+n+1}}$. The idea is that in the numerator, the "plus three" is insignificant as n approaches infinity. Thus, we keep only the $7n$ in the numerator. In the denominator, we observe that the $n+1$ is insignificant compared to the n^2 it is being added to. Again, we keep only the $n\sqrt{n^2} = n \cdot n = n^2$, the leading term of the denominator. We have built our comparison function

$$b_n = \frac{7n}{n^2} = \frac{7}{n}.$$

Sometimes it is nice to visualize this as just crossing out all lower order terms. For large n,

$$\frac{7n+\cancel{3}}{n\sqrt{n^2+\cancel{n}+\cancel{1}}} \approx \frac{7n}{n^2} = \frac{7}{n}.$$

Next, we verify that a_n and b_n have the same growth order. Proceeding with the limit, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{7n+3}{n\sqrt{n^2+n+1}}\frac{n}{7}\\
&= \lim_{n\to\infty}\frac{7n+3}{7\sqrt{n^2+n+1}}\\
&= \lim_{n\to\infty}\frac{7n^2+3n}{7n\sqrt{n^2+n+1}}\\
&= \lim_{n\to\infty}\frac{7n^2+3n}{7\sqrt{n^2}\sqrt{n^2+n+1}}\\
&= \lim_{n\to\infty}\frac{7n^2+3n}{7\sqrt{n^4+n^3+n^2}}\\
&= \lim_{n\to\infty}\frac{7n^2+3n}{7\sqrt{n^4+n^3+n^2}}\frac{\frac{1}{n^2}}{\frac{1}{\sqrt{n^4}}}\\
&= \lim_{n\to\infty}\frac{7+\frac{3}{n}}{7\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}}\\
&= \frac{7}{7\sqrt{1}}\\
&= 1.
\end{aligned}$$

Thus, a_n and b_n have the same growth order, since their ratio is a nonzero constant. Furthermore,

$$\sum_{n=1}^{\infty}\frac{7}{n} = 7\sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges by a p-series test with $p=1$. By LCT, $\sum_{n=1}^{\infty}\frac{7n+3}{n\sqrt{n^2+n+1}}$ diverges as well.

Exercise 3.9.10. Practice with LCT

Use LCT to determine whether the series converge or diverge.

- $\displaystyle\sum_{n=1}^{\infty}\frac{n+2}{n^3+1}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{n\sqrt{n^7+3n+1}}$

Note that one can attempt to apply LCT to essentially any function, not just algebraic functions. However, when transcendental functions are involved, it can be more difficult to identify the choice of comparison function.

Example 3.9.11. Cosine of Reciprocals

Consider the series $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n}\right)\right)$. We admittedly pull a rabbit out of a hat and decide to compare to the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. We demonstrate the summands have the same growth order by taking a limit of their ratios. In particular,

$$\begin{aligned}
\lim_{n\to\infty}\frac{1/n^2}{1-\cos\left(\frac{1}{n}\right)} &= \lim_{n\to\infty}\frac{-2/n^3}{\sin\left(\frac{1}{n}\right)(-1/n^2)} \\
&= \lim_{n\to\infty}\frac{2/n}{\sin\left(\frac{1}{n}\right)} \\
&= \lim_{n\to\infty}\frac{-2/n^2}{\cos\left(\frac{1}{n}\right)(-1/n^2)} \\
&= \lim_{n\to\infty}\frac{2}{\cos\left(\frac{1}{n}\right)} \\
&= \frac{2}{1} \\
&= 2.
\end{aligned}$$

Since the limit is a nonzero constant, we conclude that $\frac{1}{n^2}$ and $\left(1-\cos\left(\frac{1}{n}\right)\right)$ have the same growth order. Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we use LCT to conclude that $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n}\right)\right)$ also converges.

The above example may be a bit unsatisfying in the sense that we gave no indication where the choice of comparison function came from. Hang in there though! Once we have techniques of power series in our toolbox, we will revisit such examples and see why this was in fact a very natural choice of comparison.

Exercise 3.9.12. In and Out of L'Hospital ☕

- In the above example, LHR was applied twice. Identify which two lines it was used on. In each case, make a brief note as to why it was valid to use LHR.
- In the above example, we compared to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. How did we know that series was convergent?

Exercise 3.9.13. Sine of Reciprocals ☕☕☕

Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

Ratio Test

This test is essentially a LCT against a geometric series.

Exercise 3.9.14. Ratio of Consecutive Terms ☕

If a_n is a geometric sequence, what is a_{n+1}/a_n?

For a sequence that is not geometric, there may not be a constant ratio of consecutive terms, but we

can still look at the limit of ratios of successive terms!

Theorem 3.9.15. Ratio Test

Consider the series

$$\sum_{n=0}^{\infty} a_n.$$

- If $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$ then the series converges absolutely.
- If $\lim_{n\to\infty} |a_{n+1}/a_n| > 1$ then the series diverges.
- If $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$ then the ratio test gives no information.

Exercise 3.9.16. The "No Info" Case ☕☕

To see why the ratio test gives no info in the case where the ratio of consecutive terms converges to 1, we only need two examples: one in which a convergent series has ratio 1 and one in which a divergent series has ratio 1. Find one of each!

The Ratio Test is particularly helpful in analyzing series involving factorials, since so much cancellation will occur when computing the ratio of consecutive terms.

Example 3.9.17. A Series with Factorials ☕☕

Here we analyze the convergence of the series

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Call $a_n = \frac{2^n}{n!}$. We now compute the limit of the ratio of consecutive terms. Proceeding, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{2^{(n+1)}}{(n+1)!}}{\frac{2^n}{n!}} \\
&= \lim_{n\to\infty} \frac{2^{(n+1)}}{2^n} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} 2 \frac{n\cdots 3\cdot 2\cdot 1}{(n+1)\cdot n\cdots 3\cdot 2\cdot 1} \\
&= \lim_{n\to\infty} \frac{2}{n+1} \\
&= 0 \\
&< 1.
\end{aligned}$$

Thus, the series converges by the Ratio Test!

Exercise 3.9.18. Practice with Ratio Test ☕☕

Use the Ratio Test to prove the following series converge or diverge, or explain why the Ratio Test provides no information in that case.

- $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n!}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$

Root Test

This test is extremely similar to the Ratio Test, as it is again simply asking if long term, our series behaves like a convergent geometric series.

We begin with a comment on infinite geometric series. An infinite series is geometric if and only if it can be written as

$$\sum_{n=0}^{\infty} ar^n$$

for some real numbers a and r. Notice that as far as convergence/divergence goes, it is equivalent to study the series

$$\sum_{n=0}^{\infty} r^n,$$

since one could always factor out the constant a. So, let us now look at the convergence/divergence of series of the form $\sum_{n=0}^{\infty} r^n$.

Exercise 3.9.19. nth Root of a Geometric Sequence

Consider the infinite geometric series

$$\sum_{n=0}^{\infty} r^n.$$

- Under what conditions does it converge?
- Under what conditions does it diverge?
- Take the nth root of the summand. What does it produce?

Thus, we can see that for an infinite geometric series with constant term 1, the series converges only when the nth root of the summand is less than 1. If a series is not geometric, we instead ask if in the *long run* (i.e., the limit as n goes to ∞) the nth root of the summand becomes less than 1, in which case

it is behaving like a convergent geometric series and we can conclude that it converges. We state this more formally below.

Theorem 3.9.20. Root Test

Given an infinite series

$$\sum_{n=0}^{\infty} a_n,$$

calculate L, the limit of the nth root of the magnitude of the summand:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then,

- the series converges if $L < 1$,
- the series diverges if $L > 1$,
- and may converge or diverge if $L = 1$.

Example 3.9.21. Clash of the Titans

Here are two functions that have utterly massive growth: $n!$ and n^n. Certainly n^n is bigger, since it is a product of all copies of n whereas $n!$ multiplies successively smaller numbers together (for example, $4^4 = 4 \cdot 4 \cdot 4 \cdot 4$ while $4! = 4 \cdot 3 \cdot 2 \cdot 1$). But does the sum of their ratios converge? Let us see!

Let us apply the Root Test on

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

to see if it converges or diverges. We calculate the limit of the nth root of the summand, splitting the root across the top and bottom in order to simplify:

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}.$$

The numerator is a bit tricky, with that nth root of a factorial! Conveniently, since we are anyhow taking a limit to infinity, we can replace the factorial with Stirling's Formula (see Formula 3.1.15).

$$L = \lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2\pi n}}\frac{n}{e}}{n} = \lim_{n\to\infty} \frac{1\frac{1}{e}}{1} = 1/e.$$

Since the limit is less than 1, we have that the series converges by the Root Test.

Exercise 3.9.22. Filling in the Details of a Wonky Limit ☕☕

Notice that the limit calculation above relied on the fact that

$$\lim_{n\to\infty} \sqrt[n]{\sqrt{2\pi n}} = 1.$$

Use LHR to justify that! (**Hint.** Simplify the roots by rewriting them as fractional exponents. Also, rewrite the expression inside the limit by taking e raised to the natural log of that expression.)

Exercise 3.9.23. Root Test, Occasionally Stronger than the Ratio Test! ☕☕☕

Consider again the infinite series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n},$$

but this time perform the ratio test instead of the root test. What happens?

Exercise 3.9.24. Applying the Root Test ☕☕

Apply the Root Test to each of the following infinite series. In each case, conclude whether the series converges or diverges.

- $\displaystyle\sum_{n=0}^{\infty} 2^n$

- $\displaystyle\sum_{n=0}^{\infty} 5 \cdot 2^n$

- $\displaystyle\sum_{n=0}^{\infty} 5/2^n$

- $\sum_{n=0}^{\infty} \cos(n)/2^n$

- $\sum_{n=0}^{\infty} 1/n!$

- $\sum_{n=1}^{\infty} 1/n$

3.10 Mixed Practice with Convergence Tests

In practice, when you encounter a series in the wild, there are usually many tests that will apply. Try the following, using any valid applicable test you like!

Exercise 3.10.1. Mixed Practice ☕☕☕

Determine if each of the following infinite series converges absolutely, converges conditionally, or diverges. In each case, explain what tests you used and how!

- $\sum_{n=0}^{\infty} \frac{n}{n+2}$

- $\sum_{n=0}^{\infty} \frac{n}{n^2+2}$

- $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+2}$

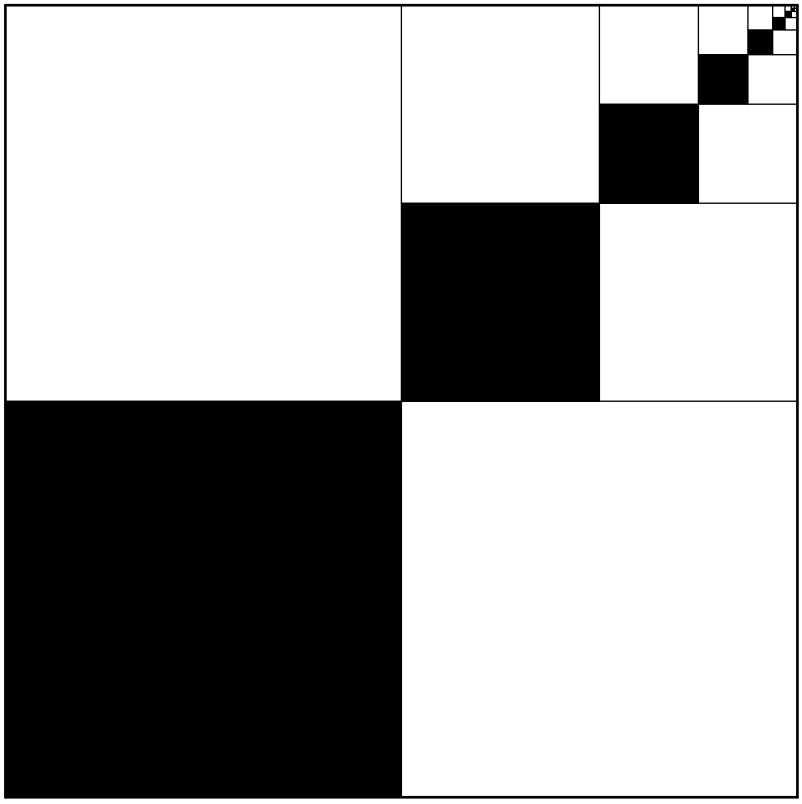
- $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^3+2}$

A Classical Infinite Series

Having played with formalism for far too long at this point, it is time to visit an infinite series coming from the geometry of Archimedes!

Exercise 3.10.2. An Infinite Series of Archimedes ☕☕☕

Consider the following diagram.



Assume the entire square has side length 1 and that each further subdivision into squares uses side lengths that are half the previous.

- What proportion of the whole large square is colored black? As a consequence, what is the total area of all the black squares added up?

- Write the area of each individual black square and build an infinite series that represents the total black area.

- Find the sum of the series using the Geometric Series Formula. Verify it agrees with your

total for the area above.

- Can you show the series converges using the No Hope Test?
- Can you show the series converges using the Geometric Series Test?
- Can you show the series converges using the Ratio Test?
- Can you show the series converges using the Root Test?
- Can you show the series converges using the Alternating Series Test?
- Can you show the series converges using the Limit Comparison Test?
- Can you show that the series converges using the Integral Test?

3.11 Absolute Convergence and Rearrangements

When we add up finitely many numbers, we take properties like commutativity and associativity for granted. We add up numbers in whatever order is most convenient. With infinite series, we cannot be quite so cavalier!

Example 3.11.1. Did You Know that Zero Equals One?

Here we use the fact that 0=-1+1.

$$\begin{aligned}
1 &= 1+0+0+0+0+\cdots \\
&= 1+(-1+1)+(-1+1)+(-1+1)+(-1+1)+\cdots \\
&= 1-1+1-1+1-1+1-1+1-\cdots \\
&= (1-1)+(1-1)+(1-1)+(1-1)+(1+-1)+\cdots \\
&= 0+0+0+0+0+\cdots \\
&= 0
\end{aligned}$$

Exercise 3.11.2. What? ☕☕☕

Let us now correctly analyze the infinite sum

$$1-1+1-1+1-1+1-1+1-\cdots$$

- Consider the sequence $a_n = (-1)^n$. Use the Geometric Series Formula to find the corresponding sequence of partial sums A_N.

- What is the limit of the sequence of partial sums?

- Thus, what is the correct value of the infinite series $1-1+1-1+1-1+1-1+1-\cdots$?

It turns out that the key lies in the distinction between a series being convergent vs being *absolutely*

convergent, a stronger type of convergence.

Definition 3.11.3. Absolute Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if and only if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Absolute convergence is the idea that it wasn't just some sort of cancellation of positive and negative terms that let the partial sums stabilize. Rather, the magnitudes of the terms were going to zero quickly enough. To test this, we just take the term-by-term absolute value of the series and see if the resulting series still converges.

Example 3.11.4. An Absolutely Convergent Series

The infinite geometric series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

is absolutely convergent because

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{4}\right| + \left|-\frac{1}{8}\right| + \left|\frac{1}{16}\right| + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

The term-by-term absolute value of the series still converges, so the original series is declared absolutely convergent.

Contrast this concept with the following definition, a weaker form of convergence called conditional.

Definition 3.11.5. Conditional Convergence

An infinite series $\sum_{n=0}^{\infty} a_n$ is *conditionally convergent* if and only if it converges but $\sum_{n=0}^{\infty} |a_n|$ diverges.

Example 3.11.6. A Conditionally Convergent Series

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$$

is conditionally convergent because

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

diverges. In particular, it is the harmonic series which totals to ∞ (as we will see in Example 3.7.1).

Exercise 3.11.7. A Maybe Absolutely Convergent Series ☕☕

Is the infinite geometric series

$$0.1 - 0.02 + 0.004 - 0.0008 + 0.00016 - \cdots$$

absolutely convergent or conditionally convergent? Explain.

It turns out that for absolutely convergent series, rearranging of terms and any sort of normal algebraic manipulation is fine. This theorem, often called the **Riemann Rearrangement Theorem**, is rather difficult to prove and will be saved for a later mathematical adventure. For this course, we will just use it! In particular, the next chapter on power series will *heavily* use this theorem, as we do all kinds of algebra with infinite series, rearranging infinitely many terms willy-nilly according to the same rules of algebra one would use with polynomials. Just a heads up that this giant theorem underlies these coming manipulations!

3.12 Chapter Σmary

In this chapter, we introduced three main concepts: sequences, series, and infinite series. Under each, we put a list of tasks you want to be able to complete for each by the end of this chapter.

1. **Sequences:** Lists of numbers.
$$a_0, a_1, a_2, a_3, \ldots$$
   (a) **Convert** sequences between our three forms of writing them:
      i. List of terms.
      ii. Explicit formula.
      iii. Recursive formula.
   (b) Identify **geometric** and **arithmetic** sequences and know their explicit and recursive formulas.

	Arithmetic Sequence	Geometric Sequence
Defining Feature	Common difference d	Common ratio r
Recursive Formula	$a_n = a_{n-1} + d$	$a_n = a_{n-1}r$
Explicit Formula	$a_n = a_0 + dn$	$a_n = a_0 r^n$

   (c) Memorize the $N - \epsilon$ **definition of the limit of a sequence**, namely
$$\lim_{n\to\infty} a_n = L$$
   if and only if
$$\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall n \in \mathbb{N}, n > N \implies |a_n - L| < \epsilon.$$
   Be able to use the definition to inform the steps in **writing an** $N - \epsilon$ **proof** of sequential convergence.

2. **Series:** A sum of a finite list of numbers.
$$a_0 + a_1 + a_2 + a_3 + \cdots + a_n$$
   (a) Given a sequence a_n, build a new sequence $A_N = \sum_{n=0}^{N} a_n$ called the **sequence of partial sums**. To find A_N from a_n, we discussed the following three strategies:
      i. If a_n is an arithmetic sequence, use the **arithmetic series formula** to calculate the sequence of partials sums as
$$A_N = (\text{Number of Terms}) \cdot (\text{Average of First and Last}).$$
      ii. If a_n is a geometric sequence, use the **geometric series formula** to calculate the sequence of partials sums as
$$A_N = \text{Initial Term} \cdot \frac{1 - \text{Common Ratio}^{\text{Number of Terms}}}{1 - \text{Common Ratio}}.$$
      iii. If a_n is neither arithmetic nor geometric, write out a table of values of A_N for the first few N values and see if you notice a pattern.
   (b) Given a partial sum A_N, find the sequence a_n from which it came by taking the **difference of consecutive terms**, namely
$$A_n - A_{n-1} = a_n.$$

3. **Infinite Series:** A sum of an infinite list of numbers.
$$a_0 + a_1 + a_2 + a_3 + \cdots$$

(a) Understand Cauchy's **definition of infinite series** as the limit of the sequence of partial sums. More formally, we define

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty}\left(\sum_{n=0}^{N} a_n\right).$$

(b) Use the above definition along with the geometric series formula to build the **infinite geometric series formula**, the fact that

$$\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r}$$

as long as $|r| < 1$.

(c) Understand the distinction between a **conditionally convergent series** and an **absolutely convergent series**, as well as the key reason why we care. You might destroy all of mathematics and the universe as we know it if you perform harmless-looking rearrangements on a conditionally convergent series. Absolutely convergent series, on the other hand, can be rearranged as you would for any finite sum.

(d) Given an infinite series $\sum_{n=0}^{\infty} a_n$, determine if it converges absolutely, converges conditionally, or diverges using one or more of our eight **convergence tests**. We list these tests with short descriptions below. Note these descriptions do not necessarily include every detail and precondition of the test; these are intended only as a short phrase to help remember the essence of the test.

i. **No Hope Test:** If the summand does not approach zero, the series has no hope of converging. If the summand does approach zero, the series has some hope of converging and another test is needed.

ii. **Geometric Series Test:** A geometric series converges if and only if the common ratio is between 1 and -1.

iii. **Integral Test:** A summation converges if and only if the corresponding improper integral converges.

iv. **p-series Test:** A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \in \mathbb{R}$ converges if and only if $p > 1$.

v. **Alternating Series Test :** A summation of terms that approach zero and alternate sign will converge.

vi. **Limit Comparison Test:** Having larger growth order than a divergent series implies divergence. Having smaller growth order than a convergent series implies convergence. If two summands have the same growth order, then either both series converge or both diverge.

vii. **Ratio Test:** If the absolute value of the ratio between consecutive terms in the series approaches a value...

- ...less than 1, the series converges.
- ...greater than 1, the series diverges.
- ...equal to 1, the test gives no information.

Note that this test is just doing LCT against a geometric series and seeing if the given series has the same growth order as a convergent or as a divergent geometric series.

viii. **Direct Comparison Test:** Being greater than a divergent series implies divergence. Being smaller than a convergent series implies convergence.

(e) Be able to apply the **Alternating Series Error Bound** to determine an upper bound for how far an approximation via a partial sum can be from the true value of an infinite alternating series.

3.13 Mixed Practice

Exercise 3.13.1. ☕

Consider the sequence

$$a_n = \frac{1}{2n}$$

and the claim that

$$\lim_{n\to\infty} \frac{1}{2n} = L = 0.$$

a.) For the limit above, find minimum N values for each of the following ϵ. That is, for each ϵ, find the *smallest* value of N such that all $|a_n - L| < \epsilon$ for $n > N$.

- $\epsilon = 0.1$
- $\epsilon = 0.01$
- $\epsilon = 0.001$

b.) Find a formula for N in terms of ϵ. Plug in $\epsilon = 0.01$ and confirm your answer above.

c.) Write an $N - \epsilon$ proof to verify that the above limit is correct.

Exercise 3.13.2. ☕

In the game *Clash of Clans*, a Barbarian King can be upgraded using Dark Elixir. Suppose your king is currently at level 10. To upgrade from level 10 to level 11 requires 40,000 Dark Elixir. Every upgrade past that requires 5,000 more Dark Elixir than what the previous upgrade cost. For example, to upgrade from level 11 to level 12 will require 45,000. To upgrade from level 12 to level 13 requires 50,000, and so on. What is the total amount of Dark Elixir required to upgrade your level 10 king to level 40?

Exercise 3.13.3. ☕☕

A math professor confesses he has a pizza-eating problem. He decides to change his usual policy of "Each minute, I eat all the pizza I see, until it's all gone", to his new rule: "Each minute, I eat 1/4 of all the pizza I see." He orders an 8 slice pan of pizza.

a.) After one minute, how much pizza is left?

b.) After two minutes, how much pizza is left?

c.) After twenty minutes, how much pizza is left?

d.) No matter how long he spends with the pan, how much of the pan will never be eaten? Explain.

Exercise 3.13.4. ☕☕☕

For each of the following infinite series, determine if it converges absolutely, converges conditionally, or diverges. Explain clearly what your reasoning is, citing any tests you use.

a.) $\sum\limits_{n=0}^{\infty} \frac{(-2)^n+n^2}{n!}$

b.) $\sum\limits_{n=0}^{\infty} F_n$

c.) $\sum_{n=0}^{\infty}(-1)^n F_n$

d.) $\sum_{n=0}^{\infty} \sqrt{\frac{2}{n^3+n+1}}$

e.) $\sum\limits_{n=0}^{\infty} \frac{2^n}{2^n+1}$

f.) $\sum\limits_{n=0}^{\infty} \frac{n}{2^n+1}$

g.) $\sum\limits_{n=0}^{\infty} \frac{e^n}{e^{2n}+1}$

Exercise 3.13.5. ☕☕☕

Define the following recursive sequence:

$$a_0 = 2$$
$$a_{n+1} = a_n - \frac{1}{2^n}$$

a.) Compute the first few values of the sequence a_n. Fill them in the table below:

n	0	1	2	3	4	5
a_n						

b.) Define A_N to be the sequence of partial sums of a_n. Find the first few values of the sequence A_N.

N	0	1	2	3	4	5
A_N						

c.) Compute the following infinite sum:

$$\sum_{n=0}^{\infty} a_n$$

How does this quantity relate to your work in part (b)?

Exercise 3.13.6.

a.) Formally define what it means for a sequence a_n to converge to a limit L.

b.) Consider the sequence $a_n = \frac{n}{3n+1}$. What is $\lim_{n\to\infty} a_n$?

c.) Write an $N - \epsilon$ proof of your claim in part b.

Exercise 3.13.7.

Consider the following infinite series;

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

a.) Apply the Divergence Test/No Hope Test to the above series. What does it tell you about its convergence or divergence?

b.) Apply the Alternating Series Test to the above series. What does it tell you about its convergence or divergence?

c.) Apply the Ratio Test to the above series. What does it tell you about its convergence or divergence?

Exercise 3.13.8.

Consider the sequence given by the following recurrence relation:

$$\begin{aligned} a_0 &= 0 \\ a_n &= a_{n-1} + 3n^2 - 3n + 1 \end{aligned}$$

a.) Write out the first five terms of a_n.

b.) Find an explicit formula for a_n.

c.) Does $\sum_{n=0}^{\infty} a_n$ converge or diverge? Explain why, clearly indicating any tests you use in the process.

Exercise 3.13.9. Practice with the Geometric Series ☕☕☕

- Explain why one cannot use the formula $\frac{a}{1-r}$ to evaluate the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

- Explain why the series $18 - 6 + 2 - \frac{2}{3} + \frac{2}{9} - \cdots$ converges absolutely. Can you find what number it converges to?

- Give an example of a geometric series that converges conditionally, or explain why it is not possible to construct such a series.

Exercise 3.13.10. Tests Not Applying ☕

- Explain why the convergence of $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ cannot be analyzed using the No Hope Test.

- Explain why the convergence of $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ cannot be analyzed using the Integral Test.

- Explain why the convergence of $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ cannot be analyzed using the p-Test.

Exercise 3.13.11. Why Not Arithmetic? ☕☕☕

Why is there *not* another result in this chapter called the "Arithmetic Series Test" that classifies the convergence/divergence of infinite series $\sum_{n=0}^{\infty} a_n$ in the case where a_n is an arithmetic sequence?

Chapter 4

Power Series

4.1 Wouldn't It Be Nice If All Functions Were Polynomials?

Think about say, differentiating and antidifferentiating. It becomes difficult when rational functions, trigonometric functions, logarithms, and exponentials are involved. If every function were just polynomial, calculus would be much easier!

Power series is an attempt to make this dream a reality and turn these non-polynomial functions into polynomials! There is just one slight hangup; it is mathematically impossible. For example...

Theorem 4.1.1. Cosine Cannot Be Written as a Polynomial

Cosine cannot be written as a polynomial.

Proof. Let n be a natural number. By the Fundamental Theorem of Algebra, a degree n polynomial has at most n roots. However, $\cos(x)$ has infinitely many roots. Thus, the cosine function cannot be equal to any polynomial, since no polynomial has infinitely many roots. □

Unphased by minor complications like something being impossible, we proceed anyway. If polynomials can only have as many roots as their degree and cosine needs infinitely many roots, maybe we just need to allow polynomials to have infinite degree! This is exactly the definition of a *power series*. A power series is simply a polynomial whose degree is allowed to be infinite. Let's say this again but more formally.

Definition 4.1.2. Power Series

A *power series* is an expression of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

for real or complex numbers a_i.

Many other sources refer to the above as a *Maclaurin Series*, or a *Taylor Series Centered at Zero*. Since it is a sum of powers of x, we stick with the simple descriptive name, power series.

Example 4.1.3. The Power Series for Cosine, One Coefficient at a Time

We return to our original goal! Since we cannot find a finite degree polynomial equal to the cosine function, we instead find an infinite degree polynomial (aka power series) for cosine. We desire a_i

such that

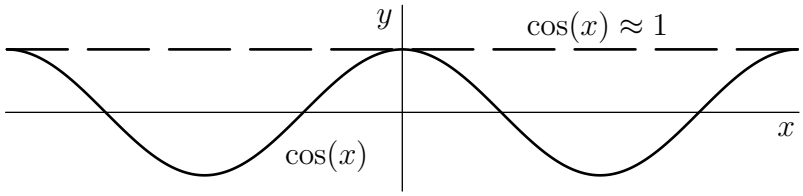
$$\cos(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots .$$

We solve for the coefficients a_i one at a time.

- **Solving for a_0:** We certainly want this formula to be true for $x = 0$, so let's plug $x = 0$ into both sides and see what happens.

$$\begin{aligned} \cos(0) &= a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + a_4 \cdot 0^4 + a_5 \cdot 0^5 + \cdots \\ 1 &= a_0 \end{aligned}$$

  Thus, we found our first coefficient of our power series. Notice that this shows us a degree zero polynomial approximation $P_0(x) = 1$ to cosine! In a certain sense, it is giving us the best horizontal line approximation to the graph of cosine near the origin.



- **Solving for a_1:** To solve for a_1, we can't just plug in $x = 0$ because a_1 will get multiplied by zero. This causes a_1 to disappear, and we can't solve for it. So, we need some operation which will keep the a_1 around but get rid of the x attached to it. Differentiation fits this description perfectly! So, we take the derivative of both sides.

$$\begin{aligned} (\cos(x))' &= \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots\right)' \\ -\sin(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots \end{aligned}$$

  Now if we plug $x = 0$ into both sides, we will be able to solve for a_1, rather than deleting it.

$$\begin{aligned} -\sin(0) &= a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0^2 + 4a_4 \cdot 0^3 + 5a_5 \cdot 0^4 + \cdots \\ -0 &= a_1 \\ a_1 &= 0 \end{aligned}$$

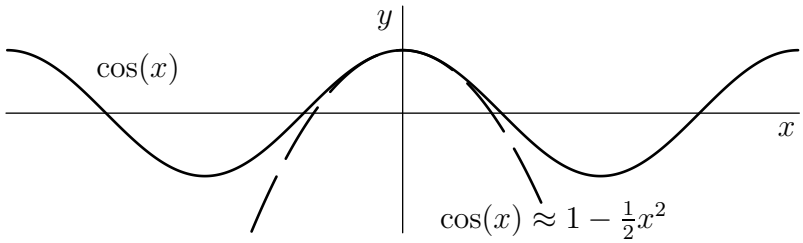
  Thus, the best degree one polynomial approximation to cosine is $P_1(x) = 1 + 0x = 1$. Notice this is no different from the degree zero approximation, and notice this is also identical to our definition of tangent line from Calculus I!

- **Solving for a_2:** To solve for a_2, we take another derivative of both sides, to strip away the x that it was being multiplied by. After differentiating, we plug in $x = 0$.

$$\begin{aligned} (-\sin(x))' &= \left(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots\right)' \\ -\cos(x) &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \cdots \\ -\cos(0) &= 2a_2 + 3 \cdot 2a_3 \cdot 0 + 4 \cdot 3a_4 \cdot 0^2 + 5 \cdot 4a_5 \cdot 0^3 + \cdots \\ -1 &= 2a_2 \\ a_2 &= -\frac{1}{2} \end{aligned}$$

We now have the best degree-two polynomial approximation for cosine near zero!

$$P_2(x) = 1 + 0x - \frac{1}{2}x^2 = 1 - \frac{1}{2}x^2$$



- **Solving for a_3:** Yet again we differentiate and then plug in zero to find the coefficient a_3.

$$\begin{aligned}
(-\cos(x))' &= \left(2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \cdots\right)' \\
\sin(x) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4x + 5 \cdot 4 \cdot 3a_5x^2 + \cdots \\
\sin(0) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 \cdot 0 + 5 \cdot 4 \cdot 3a_5 \cdot 0^2 + \cdots \\
0 &= 3 \cdot 2a_3 \\
a_3 &= 0
\end{aligned}$$

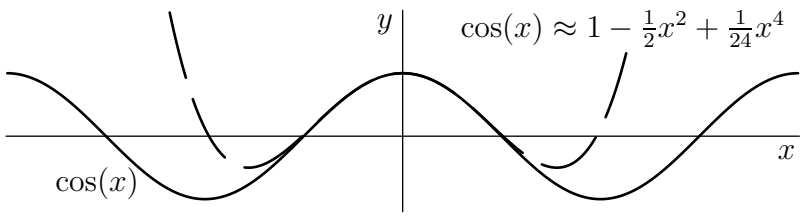
Graphically this makes sense; it is essentially saying that we couldn't represent cosine more accurately with a cubic polynomial function than we could have with a quadratic. Near zero, the graph of cosine looks a lot more like a parabola than it does like the graph of a cubic.

$$P_3(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 = 1 - \frac{1}{2}x^2$$

- **Solving for a_4:** Yet again we differentiate and then plug in zero to find the coefficient a_4.

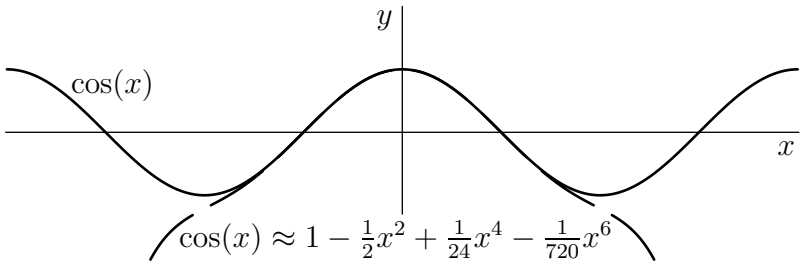
$$\begin{aligned}
(\sin(x))' &= \left(3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4x + 5 \cdot 4 \cdot 3a_5x^2 + \cdots\right)' \\
\cos(x) &= 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5x + \cdots \\
\cos(0) &= 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 \cdot 0 + \cdots \\
1 &= 4 \cdot 3 \cdot 2a_4 \\
a_4 &= \frac{1}{4!}
\end{aligned}$$

$$P_4(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{4!}x^4 = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$



- **Solving for a_5 and a_6:** By a similar argument, we find $a_5 = 0$ and $a_6 = -\frac{1}{6!}$. Thus, we have the best sixth-degree polynomial approximation.

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6$$



From here we see the pattern:

- **Left-hand Side:** The derivatives will continue cycling through the four functions $\cos(x), -\sin(x), -\cos(x)$, and $\sin(x)$. When we plug in $x = 0$, these functions give us the numbers $1, 0, -1$, and 0 respectively.
- **Right-hand Side:** After n derivatives, the only term that will not have a power of x attached to it is of the form $n!a_n$. We then must divide both sides by $n!$ to solve for a_n.

Extrapolating this pattern, we can state the full power series for cosine.

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

Often, we condense our notation by writing the power series in sigma notation rather than in expanded form as follows:

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n}.$$

Ok, you know what has to happen next.

Exercise 4.1.4. Time for Sine ☕☕

Repeat the above process to find the power series for sine!

Exercise 4.1.5. The Exponential Function ☕☕

Repeat the above process to find the power series for the natural exponential function, $f(x) = e^x$.

This method can be considered the brute force method of finding a power series. Later, we will develop

more efficient methods, but this is how we get off the ground! Let us sum up (heh) this method below.

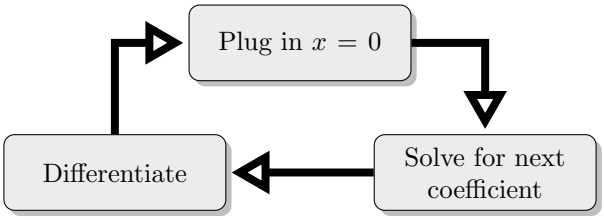
Formula 4.1.6. Brute Forcing a Power Series

To find a power series for a function $f(x)$, carry out the following steps:

- Write down the form of an unknown power series:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots .$$

- Plug in $x = 0$ to solve for a_0.
- Differentiate both sides and plug in $x = 0$ to solve for a_1.
- Differentiate both sides and plug in $x = 0$ to solve for a_2.
- Repeat this until you have as many terms as you need!



Often this method is called *Taylor's Formula.* The key idea is to notice that to solve for the coefficient a_n, we must differentiate exactly n times and then divide by $n!$. We rewrite this below in a more formulaic manner.

Theorem 4.1.7. Taylor's Formula

If $f(x)$ and all of its derivatives are defined at $x = 0$, then

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \end{aligned}$$

for any value of x for which the sum converges.

Exercise 4.1.8. A Familiar Function ☕☕

- Use Taylor's Formula to find the power series for the function

$$f(x) = \frac{1}{1-x}.$$

- How does the above power series relate to the geometric series formula? In particular, what is a? What is r?

While the method above works for many functions, it sometimes fails. In particular, it was no coincidence that the natural logarithm was absent from our examples above!

Exercise 4.1.9. Natural Log ☕☕

Try to find a power series by our brute force method and/or Taylor's Formula for the function $f(x) = \ln(x)$. What goes wrong?

To work around this, we adopt a more flexible view of power series. Rather than attempting to write our function as a sum of powers of x, which requires the function and its derivatives to be defined at $x = 0$, we express it as a sum of powers of $(x - a)$ for some real number a. It is essentially the same process, but to solve for the coefficients, you would set $x = a$ instead of $x = 0$. This produces a power series centered at a. We state this method below.

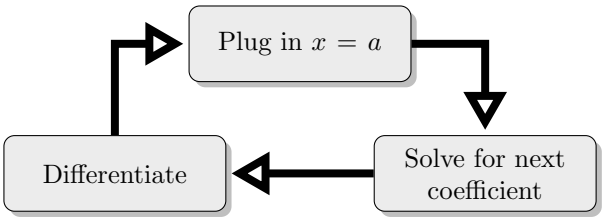
Formula 4.1.10. Brute Forcing a Power Series Centered at a

To find a power series for a function $f(x)$ centered at $x = a$, carry out the following steps:

- Write down the form of an unknown power series:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + a_5(x-a)^5 + \cdots .$$

- Plug in $x = a$ to solve for a_0.
- Differentiate both sides and plug in $x = a$ to solve for a_1.
- Differentiate both sides and plug in $x = a$ to solve for a_2.
- Repeat this until you have as many terms as you need!



Similarly, Taylor's Formula can be generalized to include any center $x = a$.

Theorem 4.1.11. Taylor's Formula, any center

If $f(x)$ and all of its derivatives are defined at $x = a$, then

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

for any value of x for which the sum converges.

Exercise 4.1.12. Natural Log, Again

Use the upgraded brute force method, or Taylor's Formula, centered at 1, to find a power series

for $f(x) = \ln(x)$ centered at 1.

Binomial Series

The next exercise constructs the binomial series, an extremely important series in the fields of combinatorics, probability, and statistics.

Exercise 4.1.13. Binomial Series ☕☕☕

Let $m \in \mathbb{R}$. A function of the form

$$f(x) = (1 + x)^m$$

is called a *binomial*, since it has two terms in the polynomial inside parentheses. Let us run the brute force method on this function to find its power series, the *binomial series*.

- Use the brute force method to find the first five terms in the series.

- Since that is cumbersome to write down, we define the *binomial coefficient* "m choose n" to be the following:

$$\binom{m}{n} = \frac{m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)}{n!}$$

For example, $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$.

Demonstrate that with this notation, the terms you found via brute force are equivalent to the series

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n.$$

Exercise 4.1.14. Understanding Binomial Coefficient Notation

In the formula for the binomial coefficient $\binom{m}{n}$, how many numbers are multiplied together in the numerator?

Note that m was not restricted to being a whole number. Since it could be fractional, the binomial series allows us to rewrite radicals as polynomials!

Example 4.1.15. Power Series of a Cubed Root

Suppose we wish to find a degree three power series for the function

$$f(x) = \sqrt[3]{1+x}.$$

We apply the Binomial Series with $m = 1/3$. Proceeding, we have

$$\begin{aligned}
f(x) &= \sqrt[3]{1+x} \\
&= \sum_{n=0}^{\infty} \binom{1/3}{n} x^n \\
&= \binom{1/3}{0} + \binom{1/3}{1} x + \binom{1/3}{2} x^2 + \binom{1/3}{3} x^3 + \cdots \\
&= \frac{1}{0!} + \frac{(1/3)}{1!} x + \frac{(1/3)(-2/3)}{2!} x^2 + \frac{(1/3)(-2/3)(-5/3)}{3!} x^3 + \cdots \\
&= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \cdots .
\end{aligned}$$

Thus, the best degree three polynomial approximation of $f(x)$ centered at zero is

$$\sqrt[3]{1+x} \approx 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3.$$

4.2 Interval of Convergence

With power series, be warned that not every infinite series will converge for all values of x.

Exercise 4.2.1. A Convergent Series ☕

Let us consider what happens when we evaluate the power series

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

at $x = 1$.
We obtain the claim that $\cos(1) = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots$. Let us test this numerically, remembering that an infinite series is really just a limit of partial sums. Calculate decimal representations of the partial sums below.

Partial Sum	Decimal Approximation
1	
$1 - \frac{1}{2!}$	
$1 - \frac{1}{2!} + \frac{1}{4!}$	
$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}$	
$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!}$	

Does it appear that the partial sums are converging to the true value of $\cos(1)$?

Exercise 4.2.2. A Not-So-Convergent Series ☕

Let us consider what happens when we evaluate the power series

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \cdots$$

at $x = 3$.
We obtain the claim that $\ln(3) = 2 - \frac{1}{2}2^2 + \frac{1}{3}2^3 - \frac{1}{4}2^4 + \frac{1}{5}2^5 - \cdots$. Let us again test this numerically, remembering that an infinite series is really just a limit of partial sums. Calculate decimal representations of the partial sums below.

Partial Sum	Decimal Approximation
2	
$2 - \frac{1}{2}2^2$	
$2 - \frac{1}{2}2^2 + \frac{1}{3}2^3$	
$2 - \frac{1}{2}2^2 + \frac{1}{3}2^3 - \frac{1}{4}2^4$	
$2 - \frac{1}{2}2^2 + \frac{1}{3}2^3 - \frac{1}{4}2^4 + \frac{1}{5}2^5$	

Does it appear that the partial sums are converging to the true value of $\ln(3)$?

Definition of IOC

As you can see, power series expansions may be valid for some values of x but not others. The set of all the "good" x-values is called the interval of convergence.

Definition 4.2.3. Interval of Convergence

Given a power series, the set of all x values for which the infinite sum converges is called the *interval of convergence* (IOC). The midpoint of the interval is called the *center* of the IOC and the distance from the center to endpoints is called the *radius*.

On the interior of the interval, the power series will not only be convergent, but it will be absolutely convergent. This is of enormous convenience, because it means we are free to rearrange terms and do algebra as we like with our power series.

The fact that such a set is always an interval (as opposed to say just scattered points or a union of two disjoint intervals) is not obvious, but it turns out to be true. To find the interval of convergence, it is typically easiest to use the Ratio Test. Since power series always have powers of x, these will cancel nicely when we apply the Ratio Test.

Example 4.2.4. Interval of Convergence for Cosine

Consider again our power series

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$$

Let us apply the Ratio Test to this series. Our goal is to get the absolute value of the ratios of consecutive terms to be less than 1, in which case we are certain the series converges.

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{1}{(2(n+1))!}x^{2(n+1)}}{(-1)^n\frac{1}{(2n)!}x^{2n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^{2n+2}\,(2n)!}{(2n+2))!x^{2n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^2\,(2n)\,(2n-1)\cdots 3\cdot 2\cdot 1}{(2n+2)\,(2n+1)\,(2n)\,(2n-1)\,(2n-2)\cdots 3\cdot 2\cdot 1}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)\,(2n+1)}\right| \\
&= \left|x^2\right| \lim_{n\to\infty}\left|\frac{1}{(2n+2)\,(2n+1)}\right| \\
&= 0\left|x^2\right| \\
&= 0
\end{aligned}
$$

The ratio of consecutive terms, being zero, is always strictly less than one no matter what x is. Since it converges for every x value, the interval of convergence is $(-\infty, \infty)$.

Exercise 4.2.5. Careful with Algebra! ☕

Annotate each line above. Provide a short phrase indicating the reason each simplification was valid.

Exercise 4.2.6. IOC for Natural Log ☕☕☕

Repeat the process above for the series

$$\ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$$

For what x-values is the limit of ratios of consecutive terms less than one?

Notice in the above example, if $x = 2$ or $x = 0$, the limit of ratios of consecutive terms is equal to 1. This is the case in which the ratio test gives us no information. Thus, we must test these series for convergence separately. We can use any of the tests from Section 3.6.

Exercise 4.2.7. Checking Endpoints ☕☕

- Set $x = 2$ in the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$ and test it for convergence/divergence.

- Set $x = 0$ in the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$ and test it for convergence/divergence.

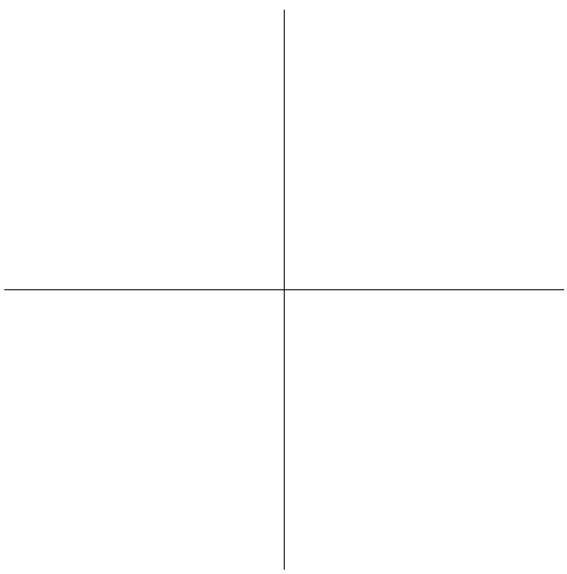
- Explain why the interval of convergence for $\ln(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$ is $(0, 2]$.

Looking at IOC Graphically

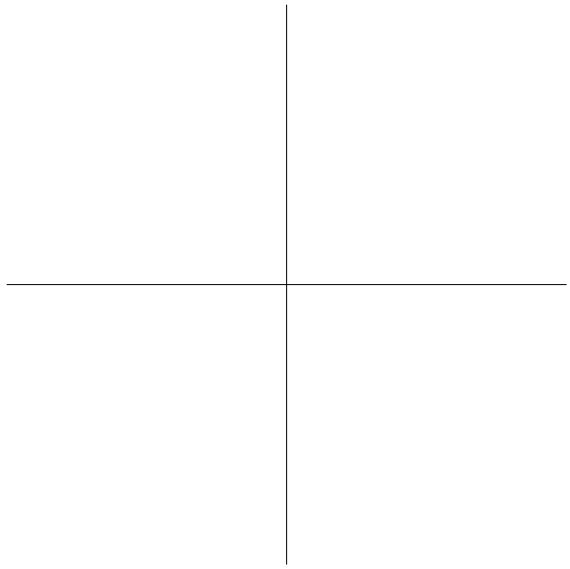
So far we found two IOC's. Our series for cosine had IOC $(-\infty, \infty)$. Our series for the natural logarithm had IOC $(0, 2]$. Let us compare these situations graphically!

Exercise 4.2.8. I See the IOC ☕☕

- Sketch the graph of cosine, and on the same axes plot the graphs of $P_n(x)$ for $n = 0, 1, 2, 3, 4, 5$, and 6. You may use a CAS to help come up with the graphs of the polynomials.



- Sketch the graph of natural log, and on the same axes plot the graphs of $P_n(x)$ for $n = 0, 1, 2, 3, 4, 5$, and 6. You may use a CAS to help come up with the graphs of the polynomials.



- What feature of those graphs shows you the IOC? Explain!

Mixed Practice with IOC

Recall our general framework for power series. To find a power series centered at $x = a$ for a function $f(x)$, we write down an equation of the form

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_2(x-a)^3 + a_2(x-a)^4 + \cdots$$

and then repeatedly plug in $x = a$ and differentiate in order to solve for the coefficients, one at a time. Alternatively, one can write up just the results of this method and skip to the end via the formula often called Taylor's Formula (see Theorem 4.1. 11).

Though this method will work to find the coefficients as long as the function and its derivatives exist at $x = a$, there remains the question: for what x-values will the infinite series on the right actually converge to the corresponding value of $f(x)$? In this activity, we investigate this both numerically (just using a table of values) and theoretically (using the ratio test). We call the set of all x-values for which a given series converges the *interval of convergence*.

Here we work through this framework in a variety of examples.

Exercise 4.2.9. Natural Logarithm

Start with the function $f(x) = \ln(x)$ and do the following:

- Set up a power series centered at $a = 2$ for $\ln(x)$. Solve for the degree 2, degree 3, degree 4,

degree 5, and degree 6 power series approximations for $\ln(x)$ centered at 2. Accomplish this by just repeatedly plugging in $x = 2$ and differentiating both sides, or by applying Taylor's Formula. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.

- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\ln(1/2)$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\ln(2)$.
- Perform the Ratio Test on your series expansion for $\ln(x)$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 4.2.10. Arctangent ☕☕☕

Start with the function $f(x) = \arctan(x)$ and do the following:

- Set up a power series centered at $x = 0$ for $\arctan(x)$. Solve for the degree 3, degree 5, degree 7, and degree 9 power series approximations. Accomplish this by just repeatedly plugging in $x = 0$ and differentiating both sides, or by applying Taylor's Formula. Call these functions $P_3(x), P_5(x), P_7(x)$, and $P_9(x)$, respectively. You may want to use a computer algebra system to help with the messy derivatives that will arise!
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\arctan(1/2)$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\arctan(2)$.
- Perform the Ratio Test on your series expansion for $\arctan(x)$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 4.2.11. Exponential ☕☕☕

Start with the function $f(x) = e^x$ and do the following:

- Set up a power series centered at $x = 0$ for e^x. Solve for the degree 2, degree 3, degree 4, and degree 5 power series approximations. Accomplish this by just repeatedly plugging in $x = 0$ and differentiating both sides, or by applying Taylor's Formula. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $e^{1/2}$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of e^2.
- Perform the Ratio Test on your series expansion for e^x. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 4.2.12. Cosine ☕☕☕

Start with the function $f(x) = \cos(x)$ and do the following:

- Set up a power series centered at $x = \pi$ for $\cos(x)$. Solve for the degree 2, degree 4, degree 6, and degree 8 power series approximations. Accomplish this by just repeatedly plugging in $x = \pi$ and differentiating both sides, or by applying Taylor's Formula. Call these functions $P_2(x), P_4(x), P_6(x)$, and $P_8(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = \pi/2$. Compare this to the true value of $\cos(\pi/2)$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\cos(2)$.
- Perform the Ratio Test on your series expansion for $\cos(x)$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 4.2.13. Square Root ☕☕☕

Start with the function $f(x) = \sqrt{x}$ and do the following:

- Explain why you can't do a power series centered at zero for $\sqrt{x}$. (**Hint:** Try it.)

- Instead, set up a power series centered at $x = 1$ for $\sqrt{x}$. Solve for the degree 2, degree 3, degree 4, degree 5, and degree 6 power series approximations for $\sqrt{x}$ centered at $x = 1$. Accomplish this by just repeatedly plugging in $x = 1$ and differentiating both sides, or by applying Taylor's Formula. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\sqrt{1/2}$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\sqrt{2}$.
- Perform the Ratio Test on your series expansion for $\sqrt{x}$. For what x would you have a ratio less than one?
- How do your results of the Ratio Test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

Exercise 4.2.14. Reciprocal ☕☕☕

Start with the function $f(x) = \frac{1}{1+x}$ and do the following:

- Set up a power series centered at $x = 0$ for $\frac{1}{1+x}$. Solve for the degree 2, degree 3, degree 4, and degree 5 power series approximations. Accomplish this by just repeatedly plugging in $x = 0$ and differentiating both sides, or by applying Taylor's Formula. Call these functions $P_2(x), P_3(x), P_4(x)$, and $P_5(x)$, respectively.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 1/2$. Compare this to the true value of $\frac{1}{1+\frac{1}{2}}$.
- Make a small table of values where you list out the values of each of your P_i evaluated at $x = 2$. Compare this to the true value of $\frac{1}{1+2}$.
- Perform the Ratio Test on your series expansion for $\frac{1}{1+x}$. For what x would you have a ratio less than one?
- How do your results of the ratio test compare to the numerical evidence you found above?
- Notice that there will be two x-values that cause the series to have ratio exactly equal to one when Ratio Test is applied. Since the Ratio Test gives no info in that case, try a different test for each of those two series.
- At last, state the power series you came up with and the Interval of Convergence.

4.3 New Series from Old

Although the brute force method is a great starting point, we don't want to have to do that every time we want a power series for a function. Now that we have a library of known power series to draw from, we wish to manipulate these to construct new series rather than starting from scratch every time. We have five loose categories for finding new series from old:

- Substitution
- Algebraic Operations, including the following...
  - Addition/Subtraction of power series
  - Multiplication of power series
  - Division of power series
  - Partial Fraction Decomposition
- Differentiation and Antidifferentiation

Substitution

Replacing x in a known series by another expression is often a useful way to generate a new power series from a previously known power series with minimal effort!

Example 4.3.1. Variations on a Theme of Euler

Suppose we wish to find the power series for the function $f(x) = e^{-2x}$. We can take our known series for the exponential function,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots,$$

and substitute $-2x$ for each occurrence of x. Proceeding, we have

$$\begin{aligned} e^{-2x} &= 1 + (-2x) + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \cdots \\ &= 1 - 2x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} x^n. \end{aligned}$$

Exercise 4.3.2. Checking Against Brute Force ☕

Use the brute force method, or Taylor's Formula, to find the first four coefficients of the power series for $f(x) = e^{-2x}$. Confirm they match what we obtained above via substitution.

Exercise 4.3.3. Practice with Substitution ☕☕☕

- Find a power series and the IOC for $\sin(x-1)$ centered at 1.

- Find a power series and the IOC for $\sin(2x)$ centered at 0.

Example 4.3.4. Revisiting the Arc Length of a Hyperbola

The binomial series gives us a method by which we can obtain a decimal approximation for the arc length of a hyperbola, as stated in Example 2.4.5. In that example, we wished to compute the arc length of

$$y = f(x) = \frac{1}{x}.$$

between $x = \frac{1}{2}$ and $x = 1$. Unfortunately, the arc length integral came out to the less than

cooperative

$$L = \int_{x=1/2}^{x=1} \sqrt{1 + (f'(x))^2}\, \mathtt{d}x = \int_{x=1/2}^{x=1} \frac{\sqrt{1+x^4}}{x^2}\, \mathtt{d}x.$$

The trick here is to rewrite the function $\sqrt{1+x^4}$ using the Binomial Series. In this case, $m = 1/2$ and the usual x in the Binomial Series has x^4 substituted in. Proceeding, we see that

$$\begin{aligned}
\sqrt{1+x^4} &= \left(1+x^4\right)^{1/2} \\
&= \binom{1/2}{0} + \binom{1/2}{1}\left(x^4\right) + \binom{1/2}{2}\left(x^4\right)^2 + \binom{1/2}{3}\left(x^4\right)^3 + \binom{1/2}{4}\left(x^4\right)^4 + \cdots \\
&= 1 + \frac{\left(\frac{1}{2}\right)}{1!}x^4 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^8 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^{12} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^{16} + \cdots \\
&= 1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \frac{5}{128}x^{16} + \cdots .
\end{aligned}$$

Noticing that the x-values in the integral are all between $\frac{1}{2}$ and 1, the quantity x^4 will also be in that interval. Since the IOC of the Binomial Series at $m = 1/2$ is $[-1, 1]$, it is safe to use the power series as a substitute for $\sqrt{1+x^4}$ in the integral. We now evaluate the integral using the Binomial Series as follows:

$$\begin{aligned}
L &= \int_{x=1/2}^{x=1} \frac{\sqrt{1+x^4}}{x^2}\, \mathtt{d}x \\
&= \int_{x=1/2}^{x=1} \frac{1 + \frac{1}{2}x^4 - \frac{1}{8}x^8 + \frac{1}{16}x^{12} - \frac{5}{128}x^{16} + \cdots}{x^2}\, \mathtt{d}x \\
&= \int_{x=1/2}^{x=1} \frac{1}{x^2} + \frac{1}{2}x^2 - \frac{1}{8}x^6 + \frac{1}{16}x^{10} - \frac{5}{128}x^{14} + \cdots \mathtt{d}x \\
&= -\frac{1}{x} + \frac{1}{2\cdot 3}x^3 - \frac{1}{8\cdot 7}x^7 + \frac{1}{16\cdot 11}x^{11} - \frac{5}{128\cdot 15}x^{15} + \cdots \Big]_{x=1/2}^{x=1} \\
&= \left(-1 + \frac{1}{6} - \frac{1}{56} + \frac{1}{176} - \frac{5}{1920} + \cdots\right) - \left(-2 + \frac{1}{6\cdot 2^3} - \frac{1}{56\cdot 2^7} + \frac{1}{176\cdot 2^{11}} - \frac{5}{1920\cdot 2^{15}} + \cdots\right) \\
&\approx 1.13.
\end{aligned}$$

Note this approximation is computed by just adding the terms written above and throwing away all further terms!

Exercise 4.3.5. Comparing Two Approximations

In the above example, we approximated the arc length of the section of the hyperbola by using finitely many terms from the Binomial Series. How does it compare if you instead approximated

that same arc length via a single line segment?

Often, a desired power series might not seem like a substitution problem, but could become one with a bit of rewriting, applying identities, or juggling constants around.

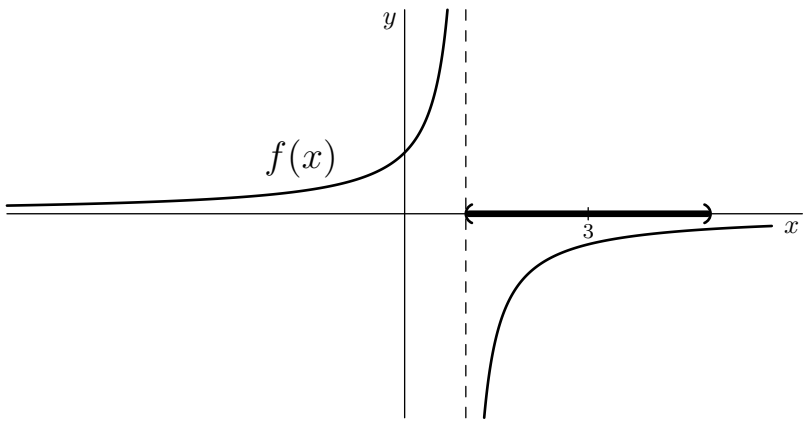
Example 4.3.6. Variation on a Geometric Series

Suppose we wish to find a power series for the function $f(x) = \frac{1}{1-x}$ centered at 3. We begin by adding and subtracting 3 from x, and then proceed to get the geometric series in a form where we can use substitution as follows:

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ &= \frac{1}{1-(x-3+3)} \\ &= \frac{1}{1-(x-3)-3} \\ &= \frac{1}{-2-(x-3)} \\ &= \frac{1}{-2}\frac{1}{1+\frac{(x-3)}{2}} \\ &= -\frac{1}{2}\frac{1}{1-\left(-\frac{(x-3)}{2}\right)}. \end{aligned}$$

The motivation for the steps above was to get the series in a form where we can now use substitution into the standard geometric series. Specifically, we substitute $\left(-\frac{(x-3)}{2}\right)$ in for every occurrence of x, while we retain the constant $-1/2$ out front:

$$\begin{aligned} f(x) &= -\frac{1}{2}\left(1+\left(-\frac{(x-3)}{2}\right)+\left(-\frac{(x-3)}{2}\right)^2+\left(-\frac{(x-3)}{2}\right)^3+\cdots\right) \\ &= -\frac{1}{2}+\frac{1}{2^2}(x-3)-\frac{1}{2^3}(x-3)^2+\frac{1}{2^4}(x-3)^3-\cdots. \end{aligned}$$



Exercise 4.3.7. Checking Against Brute Force ☕

Once again, use the brute force method, or Taylor's Formula, to find the first four coefficients of the power series for $f(x) = \frac{1}{1-x}$ centered at 3. Confirm they match what we obtained above via algebra and substitution.

Exercise 4.3.8. More Practice with Substitution ☕☕

- Find a power series and the IOC for e^x centered at 2. (**Hint.** Replace x by $(x-2)+2$ and then pull out an e^2.)

- Find a power series and the IOC for $\sin^2(x)$ centered at 0. (**Hint:** Apply the sine half-angle

identity!)

- Find a power series and IOC for $\frac{1}{x}$ centered at 5. (**Hint:** Add and subtract 5 from the denominator to turn it into a geometric series!)

Addition and Subtraction of Power Series

When adding or subtracting polynomials, one simply combines like terms, adding or subtracting coefficients of corresponding degree. The same is true for power series. That is,

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n.$$

Exercise 4.3.9. Again in Expanded Form ☕

Write out the above identity for how to add power series in expanded form.

Subtraction is similar, except the coefficients get subtracted instead of added.

Exercise 4.3.10. Subtraction in Sigma Notation and in Expanded Form

As we did above for addition, write out two identities that show how to subtract power series, one in sigma notation and one in expanded form.

Example 4.3.11. Adding Power Series

Suppose we wish to find a power series for the function

$$f(x) = \sqrt{1+x} + \sqrt{1-x}.$$

We can apply the binomial series with $m = 1/2$ to both of the above square roots, and substitute $-x$ for x in the second radical. Carrying this out produces

$$\begin{aligned}
f(x) &= \sqrt{1+x} + \sqrt{1-x} \\
&= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n - \sum_{n=0}^{\infty} \binom{1/2}{n} (-x)^n \\
&= \left(1 + \binom{1/2}{1} x + \binom{1/2}{2} x^2 + \binom{1/2}{3} x^3 + \binom{1/2}{4} x^4 + \binom{1/2}{5} x^5 + \binom{1/2}{6} x^6 + \cdots\right) \\
&\quad + \left(1 - \binom{1/2}{1} x + \binom{1/2}{2} x^2 - \binom{1/2}{3} x^3 + \binom{1/2}{4} x^4 - \binom{1/2}{5} x^5 + \binom{1/2}{6} x^6 + \cdots\right) \\
&= 2 + 2\binom{1/2}{2} x^2 + 2\binom{1/2}{4} x^4 + 2\binom{1/2}{6} x^6 + \cdots \\
&= 2 + 2\frac{(1/2)\cdot(-1/2)}{2\cdot 1} x^2 + 2\frac{(1/2)\cdot(-1/2)\cdot(-3/2)\cdot(-5/2)}{4\cdot 3\cdot 2\cdot 1} x^4 \\
&\quad + 2\frac{(1/2)\cdot(-1/2)\cdot(-3/2)\cdot(-5/2)\cdot(-7/2)\cdot(-9/2)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} x^6 + \cdots \\
&= 2 - \frac{1}{4}x^2 - \frac{5}{64}x^4 - \frac{21}{512}x^6 - \cdots
\end{aligned}$$

The last line lets one see the actual values of the first few coefficients, but for a general sigma form, the third-to-last line is certainly the easiest to draw that from. In particular,

$$f(x) = \sum_{n=0} 2\binom{1/2}{2n} x^{2n}.$$

Exercise 4.3.12. Subtracting Power Series ☕☕☕

Find a power series for the function

$$f(x) = \sqrt{1+x} - \sqrt{1-x}$$

by repeating the method from the above example, but with subtraction instead of addition.

Multiplication of Power Series

Multiplication with power series is substantially more complicated than addition and subtraction. Here, we essentially carry out an "infinite FOIL", multiplying every term of one power series by every term of the other power series, and then combining like terms at the end. Writing this in sigma form, we have

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{n=0}^{k} a_k b_{n-k}\right) x^n,$$

known as the *Cauchy product formula*, which admittedly is correct and also entirely incomprehensible upon first viewing. Let's do a little work to make the above formula more digestable!

Example 4.3.13. Multiplying Together Sine and Cosine

Consider the function

$$f(x) = \sin(x)\cos(x).$$

Let's find its degree five power series centered at zero, by simply multiplying together the two known power series for sine and for cosine. There is one caveat however: as we perform this calculation, we will discard any term that has degree six or more, seeing as we are only interested in the degree five power series. We proceed as follows:

$$
\begin{aligned}
f(x) &= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right)\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\
&= x \cdot \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\
&\quad - \frac{1}{3!}x^3 \cdot \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\
&\quad + \frac{1}{5!}x^5 \cdot \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\
&\quad - \cdots \\
&= \left(x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \cdots\right) \\
&\quad + \left(-\frac{1}{3!}x^3 + \frac{1}{2! \cdot 3!}x^5 - \cdots\right) \\
&\quad + \left(\frac{1}{5!}x^5 - \cdots\right) \\
&\quad - \cdots \\
&= x + \left(-\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{4!} + \frac{1}{2!}\frac{1}{3!} + \frac{1}{5!}\right)x^5 - \cdots \\
&= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \cdots .
\end{aligned}
$$

Exercise 4.3.14. Revisiting the Example ☕☕

- Look at the penultimate step of the above example. Then revisit the Cauchy product formula. Having now seen an example, can you see why the Cauchy product formula makes sense? Describe in words what exactly is going on there with that summation of sums!

- Thanks to trig identities, there is actually an easier solution to the above problem! Find the degree five power series for $f(x) = \sin(x)\cos(x)$ by instead applying the sine double-angle identity to rewrite the product of two trig functions as instead just a single trigonometric function and then using substitution on the resulting expression.

Exercise 4.3.15. Incredibly unnecessary. And incredibly fun. ☕☕☕

In Exercise 4.3.1, we found the power series of the function $f(x) = e^{-2x}$ via substitution (which is certainly the "right" way to find it). Find it again, just out to degree three, by instead taking the function and rewriting as

$$f(x) = e^{-2x} = e^{-x} \cdot e^{-x},$$

rewriting each of the e^{-x} as a power series, and then multiplying them together. Confirm that the answers match!

Division of Power Series

Division of power series is admittedly worse yet. However, it remains as one of the best tools to actually find power series, and in some sense is a technique you already know! If one replicates the method of polynomial long division, but always considers the lowest-degree term to be the leading term instead of the highest-degree term, that is exactly the method of division of power series! Here, there is no particular formula for this method, but instead is just an algorithm that is best understood via example.

Example 4.3.16. Power Series of Tangent

You may have noticed that we have very quietly avoided talking about a power series for the tangent function. This is because there is no particularly nice formula for it! However, we can in fact get as many terms as we like by dividing the power series of sine and cosine, since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ as follows.

In essence, we will perform multiplication of power series but in reverse. If the unknown power series is called

$$\tan(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

then we can solve for these coefficients using the relationship

$$\sin(x) = \tan(x) \cdot \cos(x),$$

or written out as power series,

$$x - \frac{1}{3!}x^3 + \cdots = \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \cdot \left(1 - \frac{1}{2!}x^2 + \cdots\right).$$

We now use the above equation to solve for the coefficients one at a time

- **Solving for a_0.** We ask ourselves, what should a_0 be such that a_0 times 1 (the constant term of cosine) will equal 0 (the constant term of sine)? Thus, we need $a_0 = 0$, and the equation becomes

$$x - \frac{1}{3!}x^3 + \cdots = \left(a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \cdot \left(1 - \frac{1}{2!}x^2 + \cdots\right).$$

- **Solving for a_1.** What should a_1 be such that $a_1 x$ times 1 (the constant term of cosine) will equal x (the linear term of sine)? Thus, we need $a_1 = 1$, and the equation becomes

$$x - \frac{1}{3!}x^3 + \cdots = \left(x + a_2 x^2 + a_3 x^3 + \cdots\right) \cdot \left(1 - \frac{1}{2!}x^2 + \cdots\right).$$

  Let's now get that known term out of the way with some algebra. To accomplish this, we first distribute x times the cosine power series, rewriting the equation as

$$x - \frac{1}{3!}x^3 + \cdots = \left(a_2 x^2 + a_3 x^3 + \cdots\right) \cdot \left(1 - \frac{1}{2!}x^2 + \cdots\right) + x - \frac{1}{2!}x^3 + \cdots,$$

  after which we subtract those terms from both sides, producing

$$\frac{1}{3}x^3 + \cdots = \left(a_2 x^2 + a_3 x^3 + \cdots\right) \cdot \left(1 - \frac{1}{2!}x^2 + \cdots\right).$$

  Notice that the degree 1 term has completely cancelled out of the equation. This is no coincidence, since we purposefully chose a_1 to make the product generate the degree 1 term of sine.

- **Solving for a_2.** What should a_2 be such that $a_2 x^2$ times 1 (the constant term of cosine) will equal $0x^2$ (the degree 2 term of what is left of sine, after having subtracted off those previous terms)? Thus, we need $a_2 = 0$, and the equation becomes

$$\frac{1}{3}x^3 + \cdots = \left(a_3 x^3 + \cdots\right) \cdot \left(1 - \frac{1}{2!}x^2 + \cdots\right).$$

- **Solving for a_3.** What should a_3 be such that $a_3 x^3$ times 1 (the constant term of cosine) will equal $\frac{1}{3}x^3$ (the degree 3 term of what is left of sine, after having subtracted off those previous terms)? Thus, we need $a_3 = \frac{1}{3}$.

We now have found the degree three power series for tangent! In particular,

$$\tan(x) = 0 + 1x + 0x^2 + \frac{1}{3}x^3 + \cdots,$$

or more simply,

$$\tan(x) = x + \frac{1}{3}x^3 + \cdots.$$

Exercise 4.3.17. Checking those terms! ☕

Check the degree three power series that we found for tangent above by taking the degree three power series for cosine and multiplying it by our degree three tangent power series. Does it in fact give you the degree three power series for sine?

The steps in the long division of tangent can be summarized, and generalized, in the following manner:

1. Find the next term of the quotient by asking what we would need to multiply by to get the lowest order terms to cancel.

2. Multiply by this quotient term and subtract.

3. Repeat until the desired number of terms is found.

Notice that these are exactly the steps you do when you perform long division, except that you typically focus on the highest place value digit (when doing integer long division) or the highest degree monomial (when doing polynomial long division), whereas here we focus on the lowest degree term.

Rewriting the tangent example to make it look more like polynomial long division could be formatted like this:

$$
\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \cdots \\
\hline
1 - \frac{1}{2!}x^2 + \cdots & x - \frac{1}{3!}x^3 + \cdots \\
 & -\left(x - \frac{1}{2}x^3 + \cdots\right) \\
\cline{2-2}
 & \frac{1}{3}x^3 + \cdots \\
 & -\left(\frac{1}{3}x^3 + \cdots\right) \\
 & \vdots
\end{array}
$$

Exercise 4.3.18. Practice with Long Division ☕☕☕

Find the degree three power series, centered at zero, for the following functions using power series long division!

- $f(x) = \frac{1}{1-x}$, by dividing 1 by $1 - x$

- $f(x) = \sec(x)$, by dividing 1 by the power series representation of $\cos(x)$

- $f(x) = \frac{x}{1-x-x^2}$, by dividing x by $1 - x - x^2$

Exercise 4.3.19. Limitations of the method ☕☕

Try to find the power series centered at zero for the function $f(x) = \cot(x)$ by dividing the power series for cosine by the power series for sine. What goes wrong? Why was this doomed to fail?

Partial Fraction Decomposition

Though we initially met PDF in the context of finding antiderivatives, it is far from the only use of such a decomposition. Taking a partial fraction decomposition of a rational function also gets it in a form which makes it easier to find its power series!

Example 4.3.20. So much PFD in this PDF

Suppose we wish to find the power series centered at 1 for the function $f(x) = \frac{2x-2}{x^2-2x}$. Here, brute force would be horrible because of all the quotient rules in the derivatives. Instead, we take a partial fraction decomposition, finding that

$$\frac{2x-2}{x^2-2x} = \frac{1}{x} + \frac{1}{x-2},$$

and then rewrite the terms on the right-hand side by adding and subtracting 1 to each x in order to get it expressed in terms of $(x-1)$, since our goal is to find power series centered at 1. Accordingly,

we have

$$f(x) = \frac{1}{1+(x-1)} - \frac{1}{1-(x-1)},$$

which by substitution into the geometric series produces

$$f(x) = \left(1-(x-1)+(x-1)^2-(x-1)^3+\cdots\right) - \left(1+(x-1)+(x-1)^2+(x-1)^3+\cdots\right).$$

Combining like terms then produces

$$f(x) = -2(x-1) - 2(x-1)^3 - 2(x-1)^5 - \cdots = \sum_{n=0}^{\infty} -2(x-1)^{2n+1}.$$

Exercise 4.3.21. Checking the algebra!

Verify all of the glossed over algebraic details in the above calculation. In particular, set up and carefully solve for the unknowns to verify that PFD shown was correct, and make sure that the "add and subtract 1 from all the x's" step does in fact produce what is claimed.

Exercise 4.3.22. Using PFD for Power Series

Finding power series for the function $f(x) = \frac{1}{x^2-x-12}$ centered at 0 by brute force would be brutal because the quotient rules would quickly become terrible lizards. Instead, find its power series by using PFD to split into two separate terms, each of which can then be turned into a power series using a variation on the geometric series. When you're done, find the IOC.

Differentiation and Antidifferentiation

Often we differentiate a known power series term-by-term to find a new series.

Exercise 4.3.23. Practice with Differentiation ☕☕☕

- Find a power series and IOC for $\frac{1}{x^2}$ centered at 5. (**Hint.** Take the power series for the function $1/x$ centered at 5 and differentiate both sides.)

- Find a power series and IOC for $\frac{1}{x^3}$ centered at 5. (**Hint.** Use the answer from the last problem.)

- Take the term-by-term derivative of the power series for e^x centered at 0. Verify that you do in fact get e^x back!

Furthermore, we often antidifferentiate a power series term-by-term to find a new power series. This will create a "$+C$" to solve for by plugging in an x-value, likely the center of the power series, after taking the antiderivative. Notice that this C is really just a_0, and plugging in the center for x is just the first step of the brute force method.

Exercise 4.3.24. Practice with Antidifferentiation ☕☕☕

- Take the term-by-term antiderivative of the power series for e^x centered at 0. Verify that

you do in fact get e^x back!

- Find a power series and IOC for $\ln(1-x)$ centered at zero.

Exercise 4.3.25. Arctan and Arcsine! ☕☕☕

Armed with the idea of antidifferentation, we can find power series for inverse trig functions now!

- Find the power series for $\arctan(x)$ by starting with the power series for $\frac{1}{1+x^2}$, obtained by substituting $-x^2$ into a geometric series, and then integrating.

- Find the power series for $\arcsin(x)$ by starting with the power series for $\frac{1}{\sqrt{1-x^2}}$ (obtained via the substitution of $-x^2$ into the binomial series) and then integrating.

Mixed Practice with New Series from Old

While the above were presented as separate methods, they of course overlap, and often there are many options for the same power series.

Exercise 4.3.26. Four Different Methods for Finding the Same Power Series

Compute the power series for the function

$$f(x) = \frac{1}{(1-x)^2}$$

by following the four different methods outlined below:

1. Finding a power series using differentiation!
   - Write out the power series for $\frac{1}{1-x}$.
   - Differentiate both sides.

2. Finding a power series by multiplying together two known series!
   - Write out the power series for $\frac{1}{1-x}$.
   - Square both sides. Note that this will involve a gigantic infinite FOIL on the right-hand side!

3. Finding a power series using long division!
   - Expand the denominator of $f(x)$, rewriting our function as $f(x) = \frac{1}{1-2x+x^2}$.
   - Perform polynomial long division using the lowest degree term on each step to identify your quotient.

4. Finding a power series via brute force!
   - Write a general unknown series $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$.

- Plug in zero and differentiate to repeatedly solve for the coefficients one at a time (our brute force method).

4.4 Error Bounds

Here we state a theorem (without proof) regarding just how accurate a finite degree polynomial approximation is. It is commonly named after Brook Taylor, who stated it in the 1700s, however Cauchy was the first to actually prove it, roughly 100 years later!

Theorem 4.4.1. Taylor's Error Theorem

Let $f(x)$ be a function, n a natural number, and $P_n(x)$ the degree n power series approximation centered at a real number a. Let M be an upper bound for $\left|f^{(n+1)}(z)\right|$ where z is any number between x and a. Then the error (also called the remainder) in the approximation

$$f(x) \approx P_n(x)$$

is no worse than the quantity

$$\frac{M|x-a|^{n+1}}{(n+1)!}$$

That is,

$$|f(x) - P_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

Note that in the above theorem, the expression $f^{(n+1)}$ represents $n+1$ derivatives applied to the function f. It is not an exponent. Notice also that we can be a bit careless when choosing a value of M. If M is exactly the max value of $f^{(n+1)}$ between x and a, that will give us the tightest error bound. Often, for simplicity's sake, we will intentionally choose a M value that is a bit too large. This still provides a valid error bound, just not one that is as tight as it could have been.

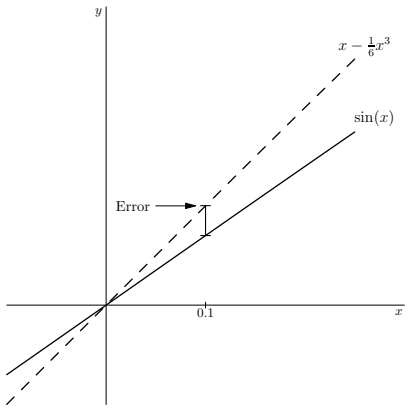
Example 4.4.2. Taylor's Error Theorem Applied to Sine

Suppose we wish to compute $\sin(0.1)$ *by hand*! Here is a feasible approach. Consider the third degree polynomial approximation $P_3(x)$ of sine, centered at zero. We have that for x values near zero,

$$\sin(x) \approx x - \frac{1}{6}x^3.$$

Suppose we wish to compute $\sin(0.1)$ *by hand* using this approximation. We evaluate

$$\begin{aligned}\sin(0.1) &\approx 0.1 - \frac{1}{6}\cdot(0.1)^3 \\ &= 0.1 - (0.16666\ldots)\cdot(0.001) \\ &= 0.1 - 0.00016666\ldots \\ &= 0.099833333\ldots.\end{aligned}$$



Taylor's Error Theorem allows us to analyze how accurate this approximation is. We list all the components to plug into the error formula below:

- The function $f(x)$ is $\sin(x)$, the function that we took the power series approximation of.

- The value of n is 3, the degree of the polynomial approximation. Notice though here since the degree four coefficient in the power series for sine is zero. Thus, $P_3(x) = P_4(x)$, so we can get away with using $n = 4$ which will give a better error bound.
- The center of the power series, a, is zero in this case since we used just powers of x, not powers of $x - a$ for some nonzero a.
- The value of x is 0.1, since that is the input value to the function.
- The upper bound $M = 1$ will suffice, since any derivative of a sine or cosine is again a sine or cosine (plus or minus) and thus has outputs of magnitude less than or equal to one.

Plugging all of this information into our error bound, we get that our error is no worse than

$$\frac{M|x-a|^{n+1}}{(n+1)!} = \frac{1|0.1-0|^5}{(5)!} = (0.000001) \cdot (0.008333\ldots) = 0.00000008333\ldots.$$

This tells us that our approximation of $\sin(0.1)$ is incredibly accurate! The difference between the true value of $\sin(0.1)$ and our approximation $P_4(0.1) = 0.099833333\ldots$ is less than $0.00000008333\ldots$. Another way to say this is that if we were writing out the digits of $\sin(0.1)$ as a decimal, our approximation would have the correct first seven digits past the decimal point (up to rounding).

Exercise 4.4.3. Checking Our Work ☕

Compute $\sin(0.1)$ on a calculator or CAS. Verify that the first seven digits after the decimal are correct, and verify that the difference between the true and approximate values is less than $0.00000008333\ldots$ as claimed.

4.5 What the Cosine Button on Your Calculator Does

It is worth noting that the definitions of common non-polynomial functions (roots, logs, trig functions, etc) are often very useful for intuition, understanding, and proving theoretical results. However, they are often atrocious for practical computation! For example, consider (in year 1 BC, 1 year Before Computers) trying to compute the quantity $\sqrt{4.1}$. What are you going to do, guess and check? Will you play the high-low game?

What is needed is a method for expressing less computationally tractable functions as polynomials (which can be evaluated using good old arithmetic). We describe our method below!

To compute $f(x)$ for a non-polynomial function f :

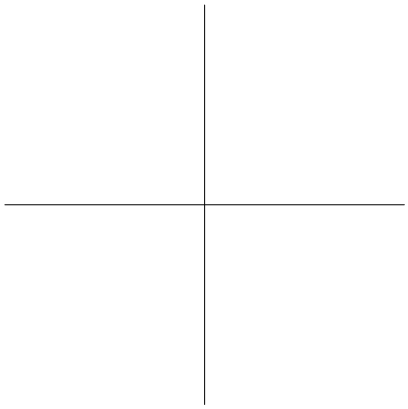
1. Find a power series expansion for f, preferably centered near x.

2. Take a finite degree approximation to this power series (or if the full power series is too difficult to obtain, maybe only compute finitely many terms in the first place).
3. Plug the value for x into your finite degree approximation.
4. Use Taylor's Error Theorem to be sure that your approximation is sufficiently accurate for your purposes.

This is the idea behind calculators and computer programs that can so efficiently compute so many wacky functions. Let's try it by hand together a bit to get a feel for the method.

Exercise 4.5.1. Seeing Taylor's Error Theorem on a Graph

- What is the degree 2 power series for $\cos(x)$ centered at zero? Write the function and plot the graph of both cosine and this parabolic approximation.



- On your graph, label y-coordinates for both functions (cosine and the parabolic approximation) at $x = 1$. What is the gap between the two y-coordinates at $x = 1$? That is, how far is the true value of $\cos(1)$ from the estimated value using the degree two polynomial approximation for cosine? (This gap is what we refer to as the error.)
- Apply Taylor's Error Theorem to this situation. What bound does it give on the error for approximating $\cos(1)$ via the degree two power series centered at zero? How does this relate to your measurement of the error above?
- Suppose we had a calculator that displays ten digits on the screen. What degree power series *would* you need to guarantee ten digits of accuracy in computing $\cos(1)$? Show explicitly

how you used Taylor's Error Theorem to get your result. (**Hint:** Since any derivative of cosine will just again be plus or minus cosine or sine, you can take $M = 1$.)

Exercise 4.5.2. The Error in a Square Root Calculation ☕☕☕

Suppose we wish to compute the number $\sqrt{4.1}$.

- Compute a degree one power series for the function $f(x) = \sqrt{x}$ centered at $x = 4$.

- Plug 4.1 into your degree one power series to get an approximation for $\sqrt{4.1}$. (Note that this is *exactly* what you did in Calc 1 when you looked at tangent lines and linearization as an approximation tool. The only difference here is we're not stuck at degree 1; we can crank up the degree as much as we like to improve accuracy!)

- Compute a degree two power series for the function $f(x) = \sqrt{x}$ centered at $x = 4$.

- Plug 4.1 into your degree two power series to get an approximation for $\sqrt{4.1}$.

- Compute a degree three power series for the function $f(x) = \sqrt{x}$ centered at $x = 4$.

- Plug 4.1 into your degree three power series to get an approximation for $\sqrt{4.1}$.

- Apply Taylor's Error Theorem in all three cases above, the degree one, two, and three cases. How many digits of accuracy do we get in each case?

Exercise 4.5.3. Clear-Cut Logging ☕☕☕

Compute $\ln(0.9)$ *by hand* accurate to three decimal places. Really. Just by hand. Show all work (including all arithmetic!) below. (**Hint:** Use Taylor's Error Theorem to make sure you aren't

working harder than you need and accidentally using too many terms in the power series.)

It is worth noting that for most of the history of mathematics, logarithms were computed in essentially this manner, by hand, and then stored in massive tables which then people would use to look them up. A particularly successful 19th Century French mathematician Gaspard de Prony led a group which compiled a table of logarithms of integers between 1 and 200,000 accurate to 19 decimal places! His name is one of only 72 engraved onto the Eiffel Tower.

Exercise 4.5.4. You're Still Smarter Than the Machines ☕☕☕

Complete the following entirely by hand with no calculator use whatsoever!

- Compute $\sqrt[3]{1001}$ by using a degree two power series approximation for the function $f(x) = \sqrt[3]{x}$ centered at $a = 1000$.

- How many digits of accuracy does Taylor's Error Theorem guarantee in this case?

4.6 Graphing Using Power Series

Second Derivative Test via Power Series

Recall the Second Derivative Test from Calculus I.

Exercise 4.6.1. Go Ahead, Recall It ☕

State the Second Derivative Test.

How does this relate to looking at the degree two power series of a function? We investigate below.

Exercise 4.6.2. Power Series Interpretation of the Second Derivative Test ☕☕

Let c be a critical point of the function $f(x)$.

- If $f''(c) < 0$, what does that tell you about the degree two power series for $f(x)$ centered at c? Explain.

- If $f''(c) > 0$, what does that tell you about the degree two power series for $f(x)$ centered at c? Explain.

- When $f''(c) = 0$, in Calculus I we would say the Second Derivative Test gave no information. With power series, how can you get around this situation and figure out the graph's behavior at that point?

Let us now put this to work, analyzing a complicated function!

Exercise 4.6.3. An Ugly Polynomial ☕☕☕

Graph the function by hand $f(x) = 36x - 30x^2 + \frac{28}{3}x^3 - x^4$ using the following steps:

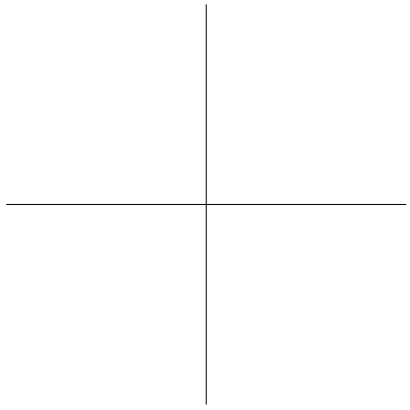
- Determine the end behavior of the function. That is, figure out the limits as x approaches infinity and minus infinity.

- Compute $f(x)$ at the x-values $x = 0, 2, 4, 6$ and plot those points to get started.

- Compute $f'(x)$ and set it equal to zero to find any critical points.

- Find the power series of $f(x)$ centered at each of those critical points. From these series conclude max/min/saddle at each critical point.

- Sketch the graph!



Symmetry via Power Series

One powerful technique in mathematics is the exploitation of symmetry. In the context of graphing, two commonly used types of symmetry are even symmetry and odd symmetry. Recall the definitions of even and odd symmetry below.

Exercise 4.6.4. Definitions of Even and Odd

Complete the definitions.

- We say a function $f(x)$ has *even* symmetry if and only if...
- We say a function $f(x)$ has *odd* symmetry if and only if...

Exercise 4.6.5. Symmetry of Sine and Cosine ☕☕

- Use the power series for cosine to prove that cosine has even symmetry.

- Use the power series for sine to prove that sine has odd symmetry.

As the above exercise demonstrates, the key to a function having even (odd) symmetry is for the power series to only have terms of even (odd) degree! This not only justifies the names of the types of symmetry, but also gives us a quick and easy way to determine the symmetry of a function.

Exercise 4.6.6. Testing Our Known Series for Symmetry ☕☕

Scan through our list of known power series. Classify each function as odd, even, or neither. List any symmetries that you found below!

4.7 Evaluating Limits Using Power Series

Often when faced with an indeterminate form of a limit, it can be resolved by replacing functions with power series. Specifically, since a limit is trying to study a function as x approaches some value a, we use a power series centered at a for the function causing us trouble (or a series centered at whatever value the input is approaching).

Example 4.7.1. Evaluating a Limit Using Power Series

Consider the limit

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right).$$

First, notice this is of the indeterminate form $0\cdot\infty$. This is a situation we would have typically handled via LHR, but now we have different tools! As x approaches infinity, $\frac{1}{x}$ approaches zero, so it makes sense to replace sine by its power series centered at zero. This will allow us to resolve the indeterminate form using ordinary algebra!

$$\begin{aligned}
\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) &= \lim_{x\to\infty} x\left(\frac{1}{x} - \frac{1}{3!}\frac{1}{x^3} + \frac{1}{5!}\frac{1}{x^5} + \cdots\right) \\
&= \lim_{x\to\infty} 1 - \frac{1}{3!}\frac{1}{x^2} + \frac{1}{5!}\frac{1}{x^4} + \cdots \\
&= 1 - \frac{1}{3!}0^2 + \frac{1}{5!}0^4 + \cdots \\
&= 1
\end{aligned}$$

Exercise 4.7.2. Checking Against LHR ☕☕

Verify the previous limit calculation using LHR.

Exercise 4.7.3. Practice with Limits via Power Series ☕☕☕

Evaluate each of the following limits two ways:

1. Using a power series centered at the x value that x is approaching.
2. Via L'Hospital's Rule.

- $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

- $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

- $\lim_{x\to 1} \frac{\ln(x)}{(x-1)^2}$

We at last have the tools to revisit Exercise 3.9.11 and show where the rabbit came from.

Example 4.7.4. Cosine of Reciprocals

Once again, consider the series $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n}\right)\right)$. Our rabbit was the decision to compare to the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. This decision was not actually arbitrary, but in fact motivated by power series! As n approaches ∞, the quantity $\frac{1}{n}$ approaches zero. Thus, it makes sense to replace $\cos\left(\frac{1}{n}\right)$ by its power series centered at zero. Specifically, we have

$$\begin{aligned}1-\cos\left(\frac{1}{n}\right) &= 1-\left(1-\frac{1}{2!}\left(\frac{1}{n}\right)^2+\frac{1}{4!}\left(\frac{1}{n}\right)^4-\frac{1}{6!}\left(\frac{1}{n}\right)^6+\cdots\right)\\ &= \frac{1}{2!}\frac{1}{n^2}-\frac{1}{4!}\frac{1}{n^4}+\frac{1}{6!}\frac{1}{n^6}+\cdots\end{aligned}$$

This power series expansion motivates the choice of $\frac{1}{n^2}$ as comparison function, as it is the domi-

nant term in the expression above. For large n, all other terms are insignificant in comparison. We again demonstrate the summands have the same growth order by taking a limit of their ratios. This time, instead of doing LHR, we use the above power series expansion.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1/n^2}{1-\cos\left(\frac{1}{n}\right)} &= \lim_{n\to\infty} \frac{1/n^2}{\frac{1}{2!}\frac{1}{n^2} - \frac{1}{4!}\frac{1}{n^4} + \frac{1}{6!}\frac{1}{n^6} + \cdots} \\
&= \lim_{n\to\infty} \frac{\left(1/n^2\right)\cdot n^2}{\left(\frac{1}{2!}\frac{1}{n^2} - \frac{1}{4!}\frac{1}{n^4} + \frac{1}{6!}\frac{1}{n^6} + \cdots\right)\cdot n^2} \\
&= \lim_{n\to\infty} \frac{1}{\frac{1}{2!} - \frac{1}{4!}\frac{1}{n^2} + \frac{1}{6!}\frac{1}{n^4} + \cdots} \\
&= \lim_{n\to\infty} \frac{1}{\frac{1}{2} - 0 + 0 - 0 + \cdots} \\
&= \frac{1}{\frac{1}{2}} \\
&= 2.
\end{aligned}$$

Exercise 4.7.5. An Insignificant Calculation ☕

In the above example, we made the claim that for large n, the later terms of

$$\frac{1}{2!}\frac{1}{n^2} - \frac{1}{4!}\frac{1}{n^4} + \frac{1}{6!}\frac{1}{n^6} + \cdots$$

are insignificant compared to the term $\frac{1}{2!}\frac{1}{n^2}$. To check this, fill out the following table of values:

n	$\frac{1}{n^2}$	$\frac{1}{n^4}$	$\frac{1}{n^6}$
10			
100			

Exercise 4.7.6. Now You Too Can Be a Magician ☕☕☕

Determine the convergence or divergence of the following series by using power series to find a suitable comparison function and then applying LCT.

- $\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n}\right)$

- $\sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right)$

- $\sum_{n=1}^{\infty} \arcsin\left(\arctan\left(\frac{2}{n}\right)\right)$

Also, as promised in Section 2.1, we provide a LHR justification using power series!

Exercise 4.7.7. Seeing LHR through a Power Series Lens ☕☕☕☕

Here we analyze the case where $f(x)$ and $g(x)$ both are functions with convergent power series expansions at a real number c. Assume also

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0.$$

We now write out the power series for each function centered at c.

$$f(x) = a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$
$$g(x) = b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots$$

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ and $b_1 \neq 0$, then

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} \frac{a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots}{b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots} \\
&= \lim_{x\to c} \frac{0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots}{0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots} \\
&= \lim_{x\to c} \frac{(x-c)\left(a_1 + a_2(x-c) + a_3(x-c)^2 + \cdots\right)}{(x-c)\left(b_1 + b_2(x-c) + b_3(x-c)^2 + \cdots\right)} \\
&= \lim_{x\to c} \frac{a_1 + a_2(x-c) + a_3(x-c)^2 + \cdots}{b_1 + b_2(x-c) + b_3(x-c)^2 + \cdots} \\
&= \frac{a_1}{b_1} \\
&= \lim_{x\to c} \frac{a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots}{b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \cdots} \\
&= \lim_{x\to c} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

- How can we handle the case where $a_1 = b_1 = 0$?

- How can we handle the case where $b_1 = 0$ but $a_1 \neq 0$?

- What if $c = \infty$ instead of a real number?

- What if the indeterminate limit is of the form $\frac{\infty}{\infty}$ instead of $\frac{0}{0}$?

4.8 The Fibonacci Numbers via Power Series

Recall the Fibonacci numbers are the recursively defined sequence given by

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n. \end{aligned}$$

Exercise 4.8.1. Listing a Few Terms ☕

Use the above recursion to compute the first five Fibonacci numbers. List these below.

We now compare these numbers to the coefficients of a particular power series.

Exercise 4.8.2. Coefficients of a Particular Power Series ☕☕☕

- Use long division to find the first five coefficients of the power series of the following function:

$$f(x) = \frac{x}{1 - x - x^2}.$$

- Whoa. What do you notice about the Fibonacci numbers vs the coefficients in that power series?

We had a recursively defined sequence and a sequence of power series coefficients. We now compare these to an explicitly defined sequence.

Exercise 4.8.3. Really? ☕☕

Define a sequence a_n via the following explicit formula:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

- Compute the first five terms of this sequence by simply plugging in n values and crunching numbers on a calculator or CAS. List your answers below.

- What? Yes, really. Right?

Ok let's figure out what the heck is going on. Let the function $f(x) = F_0 + F_1x + F_2x^2 + F_3x^3 + \cdots$. That is, f is defined to be a function whose power series has the Fibonacci sequence as its coefficients. This is called the *generating function* for the Fibonacci numbers.

Exercise 4.8.4. Studying the Generating Function ☕☕☕

- Find a power series for the function $xf(x)$.

- Find a power series for the function $x^2f(x)$.

- Use the above to find a power series for the function $f(x) - xf(x) - x^2f(x)$.

- Solve the above equation for $f(x)$ to get $f(x) = \frac{x}{1-x-x^2}$.

We now treat the function $f(x) = \frac{x}{1-x-x^2}$ as a "New Series from Old" style exercise. Once we find a formula its coefficients, we will have a formula for the Fibonacci numbers!

Exercise 4.8.5. Finding an Explicit Formula for the Coefficients ☕☕☕

- Factor the polynomial $1 - x - x^2$ via the quadratic formula. In particular, factor into the form $\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right)$ where r_1 and r_2 are the roots.

- Use this factorization to find the partial fraction decomposition of $f(x) = \frac{x}{1-x-x^2}$.

- Use the geometric series formula to find a power series for each term in the PFD, then add them together to get a power series for the Fibonacci generating function $f(x) = \frac{x}{1-x-x^2}$.

- Equate the general degree n coefficient with F_n to obtain the above explicit formula for the

Fibonacci numbers!

The explicit formula for the Fibonacci Numbers is known as *Binet's Formula*. We state it again here, just because it is so nice to look at.

Theorem 4.8.6. Binet's Formula

For all $n \in \mathbb{N}$,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Exercise 4.8.7. Ratio of Consecutive Fibonacci Numbers ☕☕

We now revisit Exercise 3.2.4, armed with Binet's Formula! Use Binet's Formula to compute the limit of the ratio of consecutive Fibonacci numbers. That is, compute an exact value for the following limit:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

Exercise 4.8.8. IOC ☕☕

Find the IOC for the Fibonacci generating function. How does this relate to vertical asymptotes

on the graph of $\frac{x}{1-x-x^2}$?

Now armed with Binet's Formula, we revisit Exercise 3.3.22.

Exercise 4.8.9. Sums of Consecutive Fibonacci Numbers ☕☕☕

Use Binet's Formula to verify that a sum of consecutive Fibonacci numbers is always one less than a Fibonacci number. More specifically, show that

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

by rewriting the left-hand side summand using Binet's Formula and then summing the geometric series that result.

Here we offer another proof of the same summation identity! In this argument, we build yet another generating function for the sums of the Fibonacci numbers rather than for the Fibonacci numbers themselves.

Exercise 4.8.10. Another Take on Sums of Consecutive F_n ☕☕☕

- By expanding and multiplying power series, demonstrate that the following product is valid:

$$\left(\sum_{n=0}^{\infty} F_n x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} F_i\right) x^n.$$

- Explain why the left-hand side above is equal to the function $g(x) = \frac{x}{(1-x-x^2)(1-x)}$.

- Verify the "PFD" $g(x) = \frac{x}{(1-x-x^2)(1-x)} = \frac{1+x}{1-x-x^2} - \frac{1}{1-x}$. (Note that it is technically not PFD precisely as we had defined it since the first quadratic factor is not being separated into linear factors, but it is still a valid decomposition in the spirit of PFD.)

- Use that "PFD" along with our known series for the Fibonacci generating function and the geometric series to show the power series for $g(x)$ has $F_{n+1} + F_n - 1$ as its general degree n coefficient for natural numbers $n > 0$. (**Hint.** Separate the $(1+x)/(1-x-x^2)$ term into the Fibonacci generating function plus that same function divided by x.)

- Conclude that the identity $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ is true, as both the left- and right-hand sides represent the degree n coefficient of $g(x)$.

4.9 Evaluating Infinite Sums Using Power Series

In Section 3.6, we developed many ways to determine if an infinite series converges, but we had almost no methods for determining what value a series converges to. Unless the series was geometric or telescoping, we were stuck. Armed with our list of power series, we have much better tools! Given an infinite series we wish to evaluate, we can now do the following:

- Look for key identifying features of the series that remind us of some known power series.
- See what x-value we can plug into our known power series to obtain the infinite series (or something close enough to it).
- Check that x-value was in the IOC of the power series, to be sure we were using a valid input.

Example 4.9.1. Evaluating an Infinite Series

Consider the infinite series

$$-\frac{3}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\frac{3^5}{6!}+\cdots.$$

We can convince ourselves that it converges using the Ratio Test. But what value does it converge to? We begin by noticing the similarity to the power series for e^x based on the consecutive factorials in the denominator. We then notice the ascending powers of 3 and the alternating signs in the infinite sum; this motivates $x=-3$ as a good choice for input. We write this infinite series down and then play with the equation until we obtain the infinite series above. In particular,

$$e^{-3}=1-\frac{3}{1!}+\frac{3^2}{2!}-\frac{3^3}{3!}+\frac{3^4}{4!}-\frac{3^5}{5!}+\cdots.$$

Notice that the powers of three in our infinite sum are one less than the corresponding factorial, and the signs are off. Dividing both sides by negative three fixes this.

$$-\frac{1}{3}e^{-3}=-\frac{1}{3}+\frac{1}{1!}-\frac{3^1}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\cdots$$

The first two terms can be moved over to left-hand side.

$$-\frac{1}{3}e^{-3}+\frac{1}{3}-\frac{1}{1!}=-\frac{3}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\cdots$$

We combine those terms with arithmetic, and we have our total for the infinite series!

$$-\frac{3}{2!}+\frac{3^2}{3!}-\frac{3^3}{4!}+\frac{3^4}{5!}-\cdots=-\frac{1}{3}e^{-3}-\frac{2}{3}$$

Exercise 4.9.2. Checking with Ratio Test and Some Numerics ☕

- In the above example, it is claimed that the series converges by the Ratio Test. Verify by

applying the Ratio Test and showing all details of the computation below.

- Evaluate $-\frac{1}{3}e^{-3} - \frac{2}{3}$ in a calculator or CAS. Compute a large partial sum of terms from the infinite series and verify that the answer is reasonable. (A quick numeric check like this is very valuable for catching minus sign mistakes or other algebra errors!)

Ok, try a few!

Exercise 4.9.3. Evaluating Infinite Series Using Power Series ☕☕☕

Evaluate each of the following to a number in closed form. Or, if the series does not converge, simply say "diverges" and give a brief explanation why.

- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

- $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots$

- $3 - \frac{3}{2!} + \frac{3}{3!} - \frac{3}{4!} + \frac{3}{5!} - \cdots$

- $3 - \frac{3^2}{2!} + \frac{3^3}{3!} - \frac{3^4}{4!} + \frac{3^5}{5!} - \cdots$

- $3 - \frac{3^2}{2} + \frac{3^3}{3} - \frac{3^4}{4} + \frac{3^5}{5} - \cdots$

- $1 - \frac{1}{2} + \frac{1}{3 \cdot 2} - \frac{1}{4 \cdot 3} + \frac{1}{5 \cdot 4} - \cdots$

- $6 + \frac{6}{4} + \frac{6}{9} + \frac{6}{16} + \frac{6}{25} + \cdots$

- $\binom{40}{0} + \binom{40}{1} + \binom{40}{2} + \binom{40}{3} + \binom{40}{4} + \cdots$

- $\binom{40}{0} - \binom{40}{1} + \binom{40}{2} - \binom{40}{3} + \binom{40}{4} - \cdots$

Notice also that if you simply skip the step of plugging in a number for x, you can often evaluate a power series to a closed form. This will be particularly useful in Chapter 7.

Exercise 4.9.4. Finding Closed Forms for Power Series ☕☕☕

Evaluate each of the following into a closed form.

- $5x - \frac{5}{2!}x^2 + \frac{5}{3!}x^3 - \frac{5}{4!}x^4 + \frac{5}{5!}x^5 - \cdots$

- $-\frac{5}{2!}x^2 + \frac{5}{3!}x^3 - \frac{5}{4!}x^4 + \frac{5}{5!}x^5 - \cdots$

- $1 + 5x - \frac{5^2}{2!}x^2 + \frac{5^3}{3!}x^3 - \frac{5^4}{4!}x^4 + \frac{5^5}{5!}x^5 - \cdots$

- $5 + 5^2x - \frac{5^3}{2!}x^2 + \frac{5^4}{3!}x^3 - \frac{5^5}{4!}x^4 + \frac{5^6}{5!}x^5 - \cdots$

- $5 + 5^2x - \frac{5^3}{2!}x^2 - \frac{5^4}{3!}x^3 + \frac{5^5}{4!}x^4 + \frac{5^6}{5!}x^5 - \frac{5^7}{6!}x^6 - \frac{5^8}{7!}x^7 \cdots$

- $5x + 5^2x^2 - \frac{5^3}{2!}x^3 - \frac{5^4}{3!}x^4 + \frac{5^5}{4!}x^5 + \frac{5^6}{5!}x^6 - \frac{5^7}{6!}x^7 - \frac{5^8}{7!}x^8 \cdots$

Exercise 4.9.5. Converting Back and Forth ☕☕☕

For each of the following series, convert it into a closed form $f(x)$. Afterwards, find a power series centered at 1 for the function you came up with and verify that it matches the original series.

- $\sum_{n=0}^{\infty}(x-1)^n$

- $\sum_{n=0}^{\infty}\binom{1/2}{n}(x-1)^n$

- $\sum_{n=1}^{\infty}\frac{1}{n}(x-1)^n$

- $\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

- $\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(x-1)^n$

4.10 Power Series Reference Sheet

Here we compile our list of known series. Write each below, both listing terms and in sigma notation, indicating the center and the interval of convergence in each case (except Binomial which will not fit in that little IOC block):

Function	Series in Sigma Notation	Series in Expanded Form	IOC
Geometric:			
Exponential:			
Sine:			
Cosine:			
Natural Log:			
Arctangent:			
Binomial:			*Long Story*
Fibonacci:			

4.11 Chapter Summary

In this chapter, we undertook the mission of rewriting the many different types of nonpolynomial functions we have (especially rational, exponential, logarithmic, trigonometric, inverse trigonometric, and radical) as polynomials (usually of infinite degree).

1. **Building the theory of power series!**

   (a) **Given a function, find its power series centered at a:** Given some function $f(x)$, rewrite it as
   $$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots$$
   via one of the following methods:

   - **Brute force method:** Repeatedly differentiate the function, plug in $x = a$, and solve for the coefficients one at a time until a pattern appears or you have enough terms for your purposes.
   - **New series from old:** Using a known power series as a starting point and then manipulating via substitution, algebra, differentiation, antidifferentiation, or other valid forms of trickery.

   (b) **Given a power series, find the interval of convergence:** Use the ratio test to find the interior of the interval. The endpoints then will have to be plugged in one at a time and convergence can be determined using some test other than the ratio test.

   (c) **Use Taylor's Error Bound to determine the accuracy in approximating functions with finite degree power series:** When trying to compute the value of $f(x)$ via a finite power series approximation, we can obtain an upper bound for the error as follows:
   $$\left| f(x) - \underbrace{\left(a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n\right)}_{\text{degree } n \text{ power series for } f(x) \text{ centered at } a} \right| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$
   where M is an upper bound for the absolute value of the $(n+1)^{\text{st}}$ derivative of f between a and x.

2. **Applications of power series!** There are of course many many many more applications of power series, but we focused on a few in particular.

   (a) **Analyzing graphs using power series:** Just as we approximated graphs using tangent lines in Calc I, we can now approximate the shapes of graphs via tangent lines, parabolas, cubics, etc.
   $$f(x) = \underbrace{\underbrace{\underbrace{\underbrace{a_0}_{y \text{ coordinate at } x=a}}_{\text{tangent line at } x=a}}_{\text{approximating parabola at } x=a}}_{\text{approximating cubic at } x=a} + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$
   $$\ddots$$

   (b) **Evaluate indeterminate forms of limits using power series:** We can justify LHR (or bypass it) by instead replacing functions by power series centered at the value the input is approaching, and then evaluate the limit via algebra.

   (c) **Evaluate infinite series using power series:** Given an infinite series, find a closed form (if possible) by identifying a power series it resembles and performing appropriate manipulations (substitutions for x, algebra, etc) to make it match.

4.12 Mixed Practice

Exercise 4.12.1. Reference Sheet ☕

Make sure your power series reference sheet is completely filled out. Once you do, see if you can rewrite it correctly from memory without looking at the original. It is essential to have those formulas internalized in order to be proficient with power series.

Exercise 4.12.2. ☕☕

By hand, (or using your custom built Arduino calculator) compute the cubed root of e accurate to one place past the decimal. (That is, error less that one one-tenth) Use Taylor's error theorem to justify your answer is correct!

Exercise 4.12.3. ☕☕

For each of the following, identify whether the sum converges to a real number or if it diverges. If it converges, find a closed form for the real number it converges to. You may cite results stated in class or in homework.

a.) $1 + \binom{1/2}{1} + \frac{\left(\binom{1/2}{1}\right)^2}{2!} + \frac{\left(\binom{1/2}{1}\right)^3}{3!} + \frac{\left(\binom{1/2}{1}\right)^4}{4!} + \cdots$

b.) $1 + \binom{1/2}{1} + \binom{1/2}{2} + \binom{1/2}{3} + \binom{1/2}{4} + \cdots$

c.) $0.20202020202020\ldots$

d.) $5 + 5^2/1! + 5^3/2! + 5^4/3! + 5^5/4! + \cdots$

Exercise 4.12.4. ☕☕

Consider the following limit:

$$\lim_{x\to 0} \frac{x^2}{\sec(x) - 1}$$

a.) Evaluate the limit by L'Hospital's Rule.

b.) Evaluate the limit by replacing $\sec(x)$ with its power series centered at zero. (HINT: You only need a few terms of this series in order to evaluate the limit!) Verify that your results match.

Exercise 4.12.5. ☕☕

Consider the function

$$f(x) = \sqrt{1 + x^2}$$

a.) Explain why the graph of the function above is equivalent to just the top half of the hyperbola $y^2 - x^2 = 1$.

b.) Show that $f(x)$ has slant asymptotes at $y = \pm x$ by computing the limits:

$$\lim_{x \to \infty} \frac{f(x)}{x}$$

and

$$\lim_{x \to -\infty} \frac{f(x)}{-x}$$

c.) Use the Binomial Theorem to find the degree two power series approximation for $f(x)$ centered at zero. What kind of shape is given by the graph of this degree two power series?

d.) Assemble the information from parts a),b), and c) to sketch the graph of $f(x)$.

Exercise 4.12.6. ☕☕☕

a.) Find a closed form for the function $f(x)$ whose power series is

$$f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{6}x^6 + \frac{1}{8}x^8 - \cdots .$$

b.) Find the Interval of Convergence for the series from part a).

Exercise 4.12.7. Yet More Practice! ☕☕☕

- – Find a degree two power series for the function $\sqrt[3]{x}$ centered at 8.

  – Use your approximation from part a) to estimate the value of $\sqrt[3]{8.05}$.

  – Use Taylor's Error Theorem to give a bound on how bad the error could be in your estimation of the value of $\sqrt[3]{8.05}$. Type the exact value into a calculator or CAS and confirm that you have obtained the desired accuracy.

- Evaluate each of the following infinite series to a closed form. Explain your reasoning.

(a) $\sum_{n=0}^{\infty} \frac{3^n}{n!2^{n+1}}$

(b) $\sum_{n=3}^{\infty} \frac{(-1)^n}{n}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!2^{2n}}$

- Find a rational number that approximates the square root of e accurate to three places past the decimal. (**Hint:** Think of the function e^x evaluated at $x = 1/2$. Then use Taylor's Error

Theorem to guarantee sufficient accuracy.)

- Find power series and Interval of Convergence for the following functions:
  - $f(x) = \frac{1}{x}$ centered at -3
  - $f(x) = \frac{1}{1-x}$ centered at -3
  - $f(x) = \frac{1}{e^x}$ centered at 0
  - $f(x) = \frac{1-x-x^2}{1-x}$ centered at 0

- $f(x) = 2^x$ centered at 1

- $f(x) = x^2 + x + 1$ centered at 5

- Find a degree three power series centered at zero for the function $\frac{1}{4-x^2}$ in five different ways:
  - Brute force.

  - Via geometric series with a substitution for x.

– Via a multiplication of the series for $\frac{1}{2-x}$ with $\frac{1}{2+x}$.

– Via a sum of the series that result in a partial fraction decomposition of $\frac{1}{4-x^2}$.

– Via long division, dividing the numerator 1 by the denominator $4 - x^2$.

Part III

Coming Attractions

Chapter 5

Introduction to Calculus III: Parametric and Polar

We begin by briefly thinking about the word *dimension*.

Exercise 5.0.1. Dimension

One intuitive notion of dimension comes from the idea of how you would assign units to measure it. If an object has length, you would call it one-dimensional. If an object has area, it is called two-dimensional. If it has volume, it is called three-dimensional. State the dimension of each of the following objects:

- $\{x \in \mathbb{R} : x < 2\}$
- $\{(x, y) \in \mathbb{R}^2 : x < 2\}$
- The closed interval $[2, 3]$
- The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
- The disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

In Calculus III, you will redo all of the key concepts of Calculus I and II but in three (or more) dimensions. Often the difficultly of higher-dimensional calculus is notational more than anything! In three or more dimensions, it becomes messier to write down the same concepts. To make this cleaner, we develop better languages for points and curves beyond our standard coordinate system.

5.1 Parametric Curves

Many of the objects we study, like circles or graphs of functions, are one-dimensional objects even though we usually view them as embedded in a two-dimensional plane. Thus, we can represent both x and y

(the two dimensions) in terms of the same parameter t.

Definition 5.1.1. Parametric Curve

Let $x(t)$ and $y(t)$ be functions of t and let $D \subset \mathbb{R}$. The corresponding *parametric curve* is the set of points

$$\{(x(t), y(t)) : t \in D\}.$$

Typically, D is an interval or union of intervals. We can graph most curves by just selecting t values from the domain D and plotting the corresponding points.

Exercise 5.1.2. A Warm-up Parametric Curve ☕☕

Consider the parametric curve

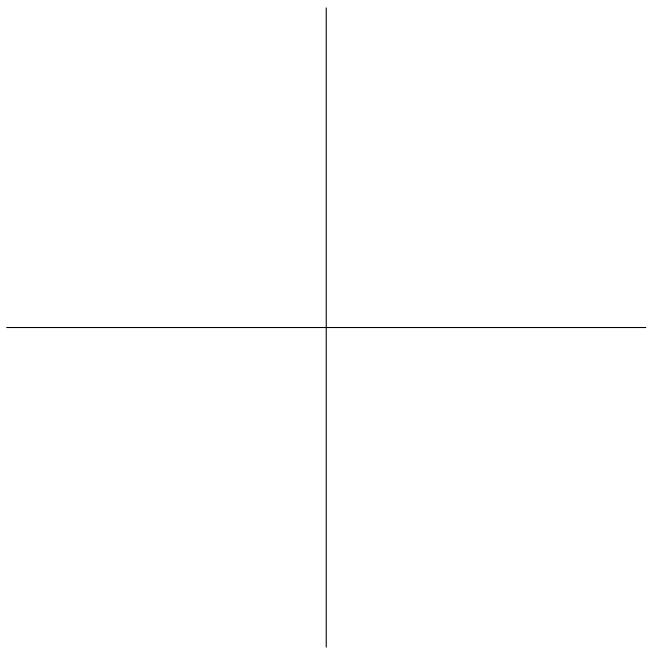
$$\{(2t, 3t + 1) : t \in [-1, 3]\}.$$

That is, $x(t) = 2t$, $y(t) = 3t + 1$, and $-1 \leq t \leq 3$.

- Use the above formulas for $x(t)$ and $y(t)$ and the following t-values selected from $D = [-1, 3]$ to fill out the following table:

t	-1	0	1	2	3
$x(t)$					
$y(t)$					

- Plot those five points on the axes below. What type of shape does it appear to be?



- Solve the equation $x = 2t$ for t. Substitute this expression for t into the equation $y = 3t + 1$. What does this new equation tell you about the parametric curve?

Here is an example of a parametric curve used in Trigonometry (though not called so at the time).

Exercise 5.1.3. The Unit Circle ☕☕

- Explain why the parametric curve

$$C_1 = \{(\cos(t), \sin(t)) : t \in [0, 2\pi]\},$$

is the familiar unit circle from trigonometry.

- Consider the curve

$$C_2 = \{(\sin(t), \cos(t)) : t \in [0, 2\pi]\}.$$

How are the curves C_1 and C_2 similar? How are they different?

5.2 Derivatives of Parametric Curves: Slopes of Tangent Lines

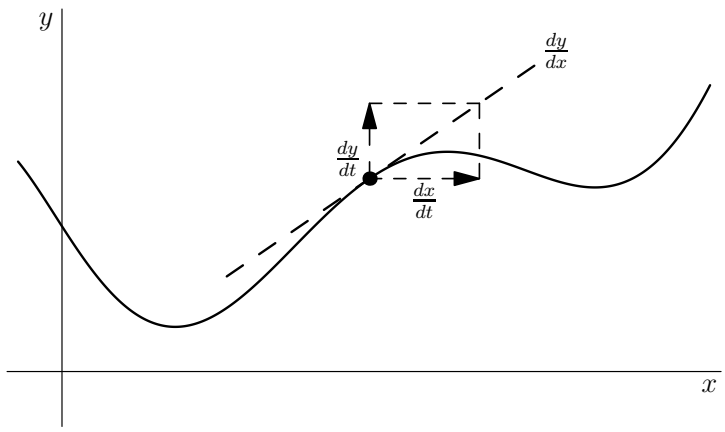
To compute the derivative of a parametric curve, we recall that the slope of a line is the change in y-coordinate divided by the change in x-coordinate. In the context of parametric curves, these can be

computed as rates of change with respect to the parameter t.

Definition 5.2.1. Parametric Derivatives

Let $(x(t), y(t))$ be a parametric curve. Then the slope of the tangent line can be computed as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$



Notice that the above formula is just a slightly rearranged version of the chain rule. In particular, if we consider a portion of the graph of $(x(t), y(t))$ that passes the Vertical Line Test, then we can consider y as a function of x, which x in turn is a function of t. So if we wanted to ask how y changes with respect to t, we would have to take the rate of change of y with respect to x and multiply it by the rate of change of x with respect to t (by the chain rule). Expressing this Chain Rule in symbols instead of words, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Exercise 5.2.2. Understanding the Definition ☕

How would you get from the chain rule application shown above to our definition of parametric derivatives?

Example 5.2.3. Parametric Derivatives on a Parabola

Consider the parametric curve given by

$$\left\{ \left(t^2, t\right) : t \in [0, \infty) \right\}.$$

To find the slope of a tangent line to this parabola, we can use the parametric derivative formula as follows:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}(t)}{\frac{d}{dt}(t^2)} \\ &= \frac{1}{2t}.\end{aligned}$$

Alternately, we could convert the curve to a cartesian equation and differentiate with respect to y. Proceeding, we notice this curve is contained in the graph of $y = \sqrt{x}$, since the formulas $x = t^2$ and $y = t$ satisfy that relationship. Thus, we can differentiate y with the power rule.

$$\begin{aligned}\frac{dy}{dx} &= \left(\sqrt{x}\right)' \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

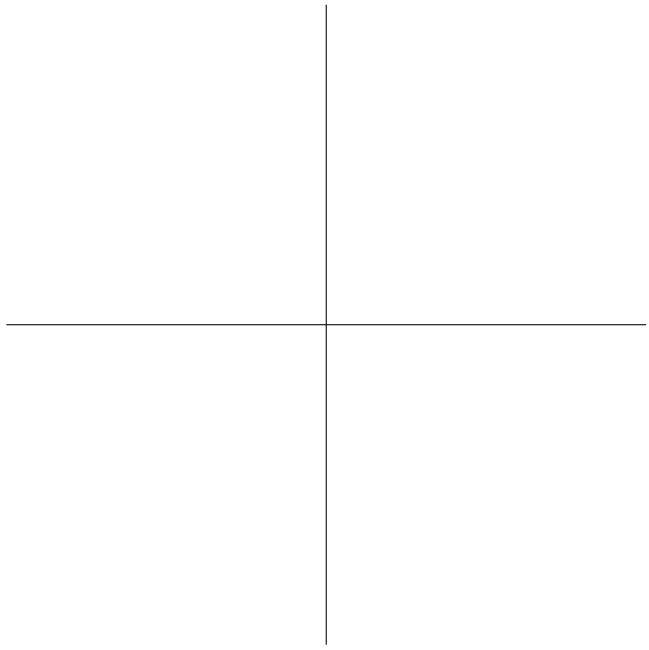
Exercise 5.2.4. Equivalence of the Results

In the above example, we have two distinct expressions for $\frac{dy}{dx}$. Explain why they are in fact equivalent.

Exercise 5.2.5. The Tangent Line to an Ellipse

- Plot the parametric curve given by

$$\{(2\cos(t), \sin(t)) : t \in [0, 2\pi]\}.$$



- Find the point on the graph located at $t = \pi/4$, and find the slope of the tangent line at that point using the parametric derivative formula. Sketch the tangent line on your graph above.

- Verify the above curve is in fact the ellipse given by $\frac{x^2}{4} + y^2 = 1$.

- Use implicit differentiation on the equation $\frac{x^2}{4} + y^2 = 1$ to find $\frac{dy}{dx}$ at that same point and verify your answers match!

Exercise 5.2.6. Finding a Parameterization ☕☕☕

Find a parameterization of the path that consists of two full clockwise laps around the ellipse given by

$$\frac{(x-3)^2}{4} + (y-3)^2 = 1$$

starting from the point (3,2).

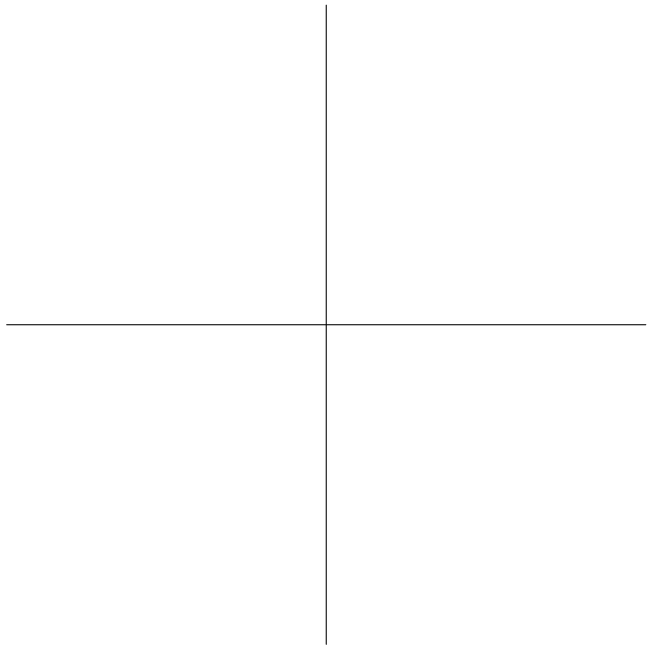
Exercise 5.2.7. A Hyperbola ☕☕

Consider the parametric curve given by the following:

$$\begin{aligned} x(t) &= e^t - e^{-t}, \\ y(t) &= e^t + e^{-t}, \\ t &\in [0, \infty). \end{aligned}$$

- Show that the above curve is contained in the hyperbola $y^2 - x^2 = 4$.

- Graph the parametric curve.



- Find dy/dx using the parametric formula for derivatives. Take the limit as t approaches infinity and interpret on your graph.

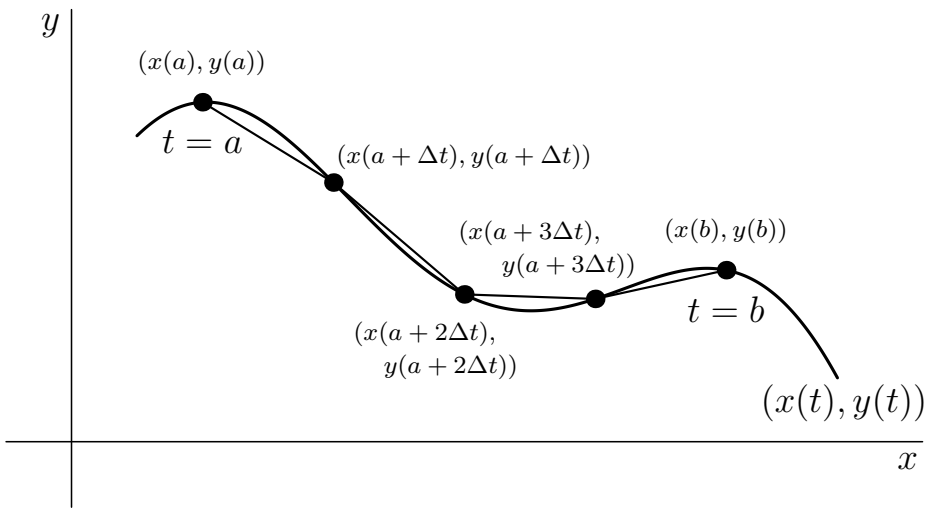
5.3 Integrals of Parametric Curves: Arc Length

The length of a parametric curve is given by the following formula.

Theorem 5.3.1. Parametric Arc Length

Let a parametric curve C be given by $(x(t), y(t))$ for $a \leq t \leq b$. Then the arc length is computed via

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathtt{d}t.$$



The construction here is nearly identical to the construction of the arc length of the graph of a function in Section 2.4. We select points corresponding to t-values along the curve, compute the sum of the lengths of the line segments connecting them, and take the limit as the number of line segments goes to infinity.

Exercise 5.3.2. Fill in the Blanks! Derivation of the Arc Length Formula ☕☕

Let $t_0, t_1, t_2, \ldots, t_n$ be equally spaced points in the interval $[a, b]$. That is, $t_0 = a$, $t_n = b$, and for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta t =$ _____.
With this setup, if we want the length of a line segment connecting points $(x(t_{i+1}), y(t_{i+1}))$ and $(x(t_i), y(t_i))$, we would use the Pythagorean Theorem to obtain

$$\sqrt{\left(\qquad\qquad\qquad\qquad\right)}$$

as the length.

$$
\begin{aligned}
L &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{}\sqrt{(\Delta t)^2} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{}\Delta t \\
&= \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \mathtt{d}t
\end{aligned}
$$

As usual, when first trying out a new tool, it is best to use it in a case where you already know the answer.

Exercise 5.3.3. Checking the Circumference of a Circle ☕☕

Consider the parametrization

$$
\begin{aligned}
x(t) &= r\cos(t) \\
y(t) &= r\sin(t)
\end{aligned}
$$

for $t \in [0, 2\pi]$.

- Explain why this is a parameterization of a circle of radius r.

- Use the parametric arc length formula to compute the length of the curve. Compare it to your known formula for the circumference of a circle. Does the answer make sense?

Exercise 5.3.4. A Familiar Conic in Disguise ☕☕☕

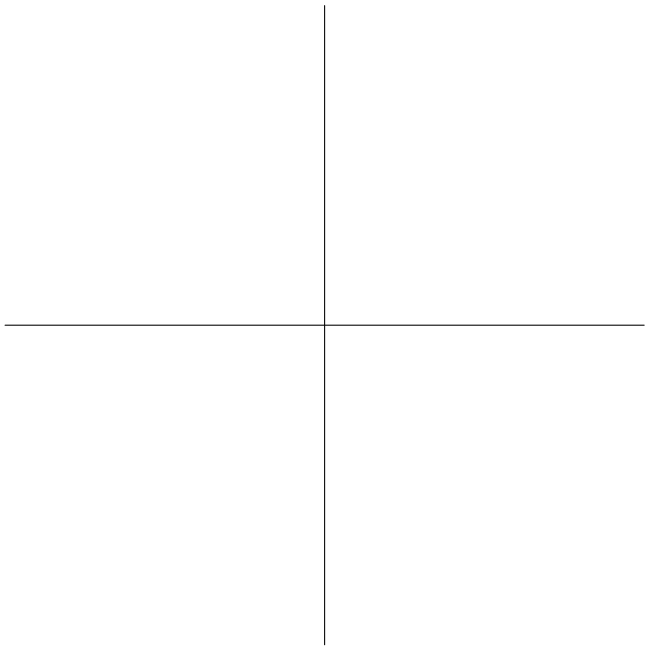
Consider the parametrization

$$x(t) = t - 1$$
$$y(t) = 3t^2$$

for $t \in [-2, 2]$.

- Convert this to a cartesian equation. What kind of shape is it?

- Sketch the curve. Indicate any vertical or horizontal tangent lines and where they occur.



- Use the parametric arc length formula to compute the length of the curve. Does the answer

make sense?

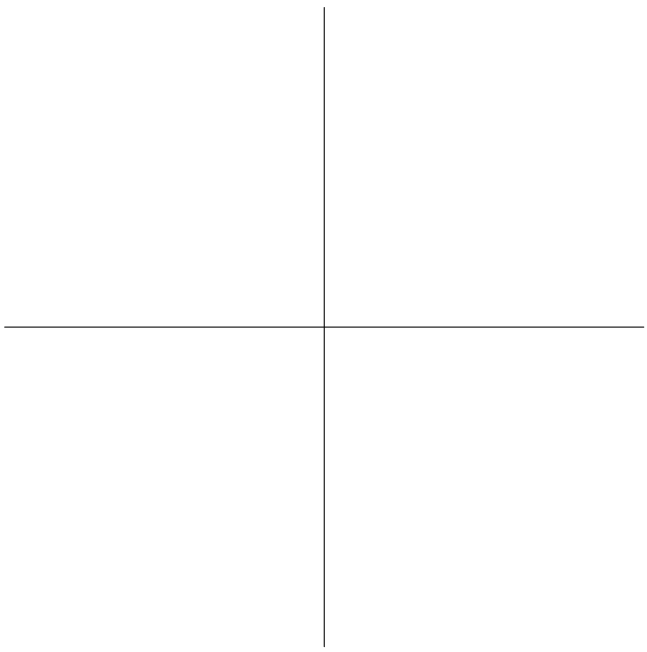
Ok, time to finally play with a curve that is not just a conic.

Exercise 5.3.5. Analyzing a Stranger Curve ☕☕☕

- Sketch the graph of the following parametric curve C:

$$C = \left\{ \left(e^t \cos(t), e^t \sin(t) \right) : 0 \leq t \leq 2\pi \right\}.$$

Include labels of points on the graph at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.



- Where does the above graph have vertical tangent lines? Where does the above graph have horizontal tangent lines? Mark them on your graph.

- What is the length of C?

5.4 Hyperbolic Sine and Cosine

You may have seen in a previous course (or if not, then here they are!) the definitions of the *hyperbolic sine* and *hyperbolic cosine* functions. They are typically defined as follows:

- $\cosh(t) = \frac{e^t + e^{-t}}{2}$
- $\sinh(t) = \frac{e^t - e^{-t}}{2}$

This of course prompts the question: "why do these e things get called sine or cosine?" We answer this question below.

Exercise 5.4.1. Power Series for Hyperbolic Sine and Cosine

- Find a power series for cosh by using what we know about the series for the exponential function. How does the resulting series relate to cosine?

- Find a power series for sinh by using what we know about the series for the exponential

function. How does the resulting series relate to sine?

And of course there are more questions prompted here: "Why do these e things get called hyperbolic? What do these hyperbolic functions have to do with hyperbolas?"

Exercise 5.4.2. Parametric Curve Generated by Hyperbolic Sine and Cosine ☕☕☕

Consider the parametric curve

$$\{(\cosh(t), \sinh(t)) : t \in \mathbb{R}\}.$$

- Verify this parametric curve satisfies the cartesian equation for a hyperbola given by

$$x^2 - y^2 = 1$$

by plugging the exponential definitions for our hyperbolic trig functions in for x and y.

- Again, verify this parametric curve satisfies the cartesian equation for the same hyperbola given by

$$x^2 - y^2 = 1$$

by plugging the power series formulas for our hyperbolic trig functions in for x and y.

5.5 Polar Coordinates

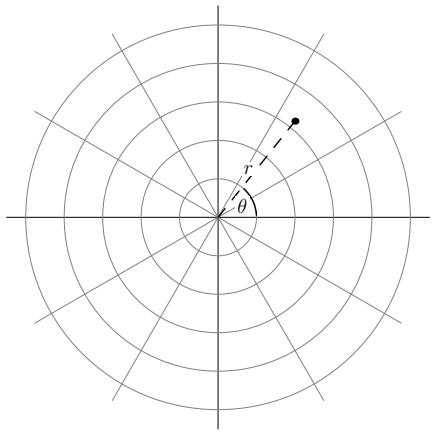
We use the *(horizontal,vertical)* coordinate system so much that it is easy to think that it is somehow inherent to a plane. However, a plane is just a geometric object; a coordinate system is an arbitrary system of labels that we slap on after the fact. Here we explore a different commonly used coordinate system, *polar coordinates*.

Points in Polar Coordinates

Assume we chose an origin in the plane, a direction that we call the positive x-axis, and some point along that ray that marks off unit distance. We now define coordinates for the rest of the plane based on these choices.

Definition 5.5.1. Plotting Points in Polar Coordinates

The point (θ, r) is the point located at an angle θ radians counterclockwise from the positive x-axis, a distance of r units from the origin.



Notice the angles are measured in the same manner as on the unit circle in trigonometry. The difference here is we allow any real number r as radius, rather than only radius one. We do allow r to be a negative number, in which case we travel "backwards" along the ray given by θ.

Example 5.5.2. Polar Coordinates are not Unique!

Be warned that any given point will have many different representations in polar coordinates. For example, consider the cartesian point (1,-1). In polar coordinates, we have many ways to represent this point. We can think of the angle as $\theta = -\pi/4$ and the radius as $r = \sqrt{2}$. We can also think of the angle as $\theta = 7\pi/4$ and the radius as $r = \sqrt{2}$. Yet another valid way to reach that same point is to use angle $\theta = 3\pi/4$ and the radius $r = -\sqrt{2}$. Thus, in polar coordinates we have that

$$\left(-\pi/4, \sqrt{2}\right) = \left(7\pi/4, \sqrt{2}\right) = \left(3\pi/4, -\sqrt{2}\right)$$

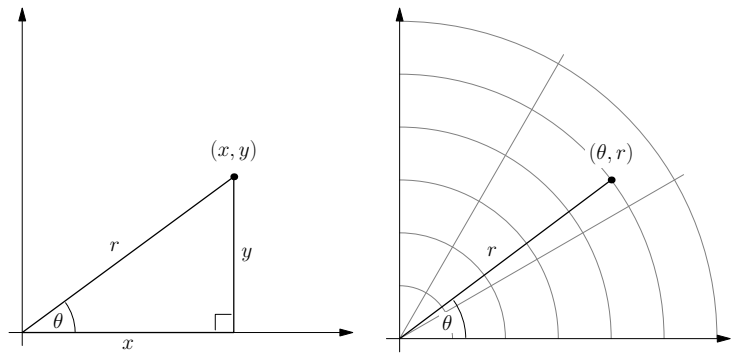
all represent the same point.

We now see how right-triangle trigonometry allows us to convert between polar coordinates and

cartesian coordinates.

Exercise 5.5.3. Converting Between Polar and Cartesian ☕☕☕

- See the diagram below, with a point in QI labeled with both cartesian and polar measurements. For each of the conversion formulas listed below, write a short sentence afterwards explaining how it comes from the diagram.



  - $x^2 + y^2 = r^2$
  - $x = r\cos(\theta)$
  - $y = r\sin(\theta)$
  - $\tan(\theta) = \frac{y}{x}$
  - $\theta = \arctan\left(\frac{y}{x}\right)$

- If the point of interest were in a different quadrant, do the above formulas still hold? Do any of them require adjustment? Explain.

Exercise 5.5.4. Plotting in Polar ☕☕

- Plot the polar point ($5\pi/4$,4). What are its cartesian coordinates?

- Consider the cartesian point (2,0). What are *all* possible ways of writing that point in polar

coordinates?

Exercise 5.5.5. Do Any Points Have the Same Name? ☕☕☕

Do any points happen to have the same label in both polar and cartesian coordinates? Find all points that do, and explain why there are no more!

It is worth noting why "polar coordinates" are called what they are called. Cartesian coordinates look like a grid of horizontal and vertical lines. This is a great approximation of what latitude and longitude lines look like if you are standing at a random point on earth and think of your surroundings as approximated by a plane. But, if you are standing at the north or south pole, the latitude and longitude lines do not in any way look like a grid!

Exercise 5.5.6. Justifying the Name ☕☕

What do the lattitude and longitutde lines look like if you are standing at the north or south pole? Draw a small graph below.

Exercise 5.5.7. The Idea of Coordinate Systems ☕☕☕☕

Create another coordinate system for the plane that is not cartesian and is not polar! Describe your system of labeling all the points!

Graphs of Equations and Functions in Polar Coordinates

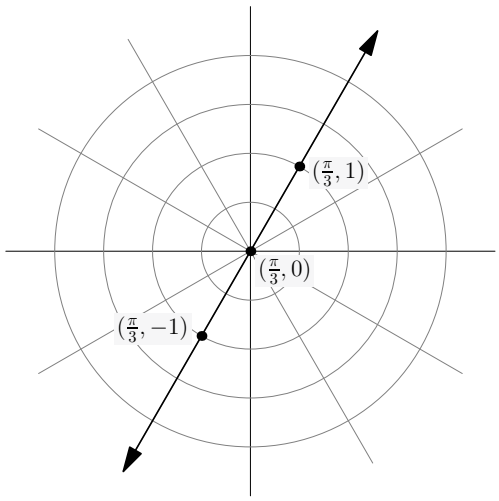
An equation in polar coordinates is an equality between expressions involving r and θ. We often wish to view the solutions visually by plotting all points (θ, r) in the plane that make the equations true (just as one would in cartesian).

Example 5.5.8. Graphing a Polar Equation

Suppose we wish to graph the equation

$$\theta = \pi/3$$

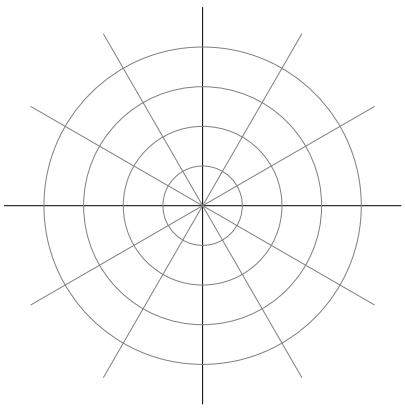
A point satisfies that equation if and only if the angle is $\pi/3$. The radius r is free to be any real number -positive, negative, or zero. For example, points that satisfy the equation include $(\pi/3, 1), (\pi/3, 0)$, and $(\pi/3, -1)$. Thus the graph is a line through the origin at $60°$ to the positive x-axis.



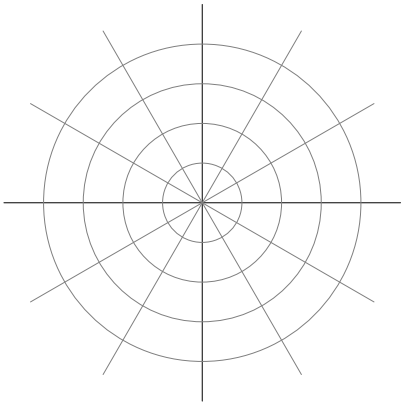
Exercise 5.5.9. Graphing Equations

Graph the following equations.

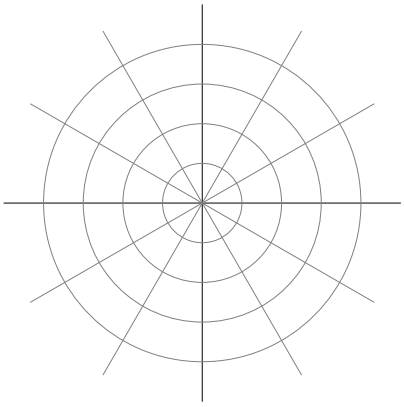
- $r = 2$



- $r = -2$



- $\theta^2 = \pi^2/4$



If the equation can be solved for r, we can consider r as a function of the independent variable θ. To graph a function, we simply make an input-output table of θ values and corresponding $r(\theta)$ values and plot the corresponding points $(\theta, r(\theta))$.

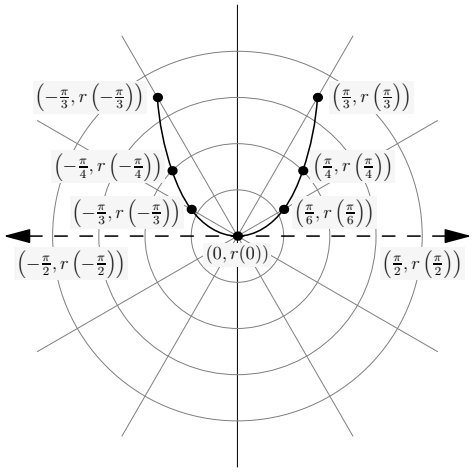
Example 5.5.10. Graphing a Polar Function

Plot the polar function

$$r(\theta) = \tan(\theta)$$

over the domain $-\pi/2 < \theta < \pi/2$. We select input values for θ that are clean unit circle values to plot.

θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$r(\theta)$	DNE	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	DNE



Exercise 5.5.11. Analyzing the Graph ☕☕

Does the graph appear to have any asymptotes? If so, where?

Looking at the graph prompts the question "can we find a Cartesian equation that describes the same set of points"? Here we use the formulas from Exercise 5.5.3 to rewrite all instances of r and θ in terms of x and y. Also, perhaps the cartesian equation can confirm our asymptote suspicions above!

Example 5.5.12. Converting to Cartesian

Let's find a cartesian equation for the graph of $r(\theta) = \tan(\theta)$ from the previous example. Since we do not have a particularly clean conversion formula for r itself but rather for r^2, it can be helpful to either multiply both sides by r or square both sides. In this case, squaring both sides will be

cleaner so we take that path. Proceeding:

$$\begin{aligned} r &= \tan(\theta) \\ r^2 &= (\tan(\theta))^2 \\ x^2 + y^2 &= \left(\frac{y}{x}\right)^2 \\ x^2 + y^2 &= \frac{y^2}{x^2} \\ x^4 + x^2y^2 &= y^2 \\ x^4 + \left(x^2 - 1\right) y^2 &= 0 \\ y^2 &= \frac{x^4}{1 - x^2} \\ y &= \pm\frac{x^2}{\sqrt{1 - x^2}} \end{aligned}$$

Exercise 5.5.13. Analyzing the Graph, Round II ☕☕

Does the cartesian formula tell you anything further about the apparent asymptotes on the graph? More specifically, calculate

$$\lim_{x \to 1^-} y(x)$$

and

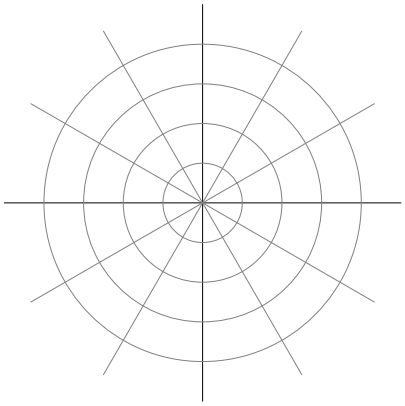
$$\lim_{x \to -1^+} y(x)$$

for the cartesian formula $y(x)$ we found above. What does each limit say about the graph?

The next exercise shows why converting a polar graph to cartesian coordinates can help analyze the geometry of the graph.

Exercise 5.5.14. Graphing a Function and Converting ☕☕☕

- Graph the function $r(\theta) = \sin(\theta)$. Does it look like a circle?



- Is it a circle? If so, what is the center and radius? Convert the equation to cartesian coordinates to confirm!

5.6 Derivatives in Polar Coordinates

Suppose we have the graph of a polar function $r(\theta)$, and we would like to find the slope of the tangent line at a point. We can consider this graph to be a parameterized curve by treating $t = \theta$ as the parameter. Specifically, the parameterization is given by

$$\begin{aligned} x(t) &= r(t)\cos(t) \\ y(t) &= r(t)\sin(t). \end{aligned}$$

Exercise 5.6.1. Deriving the Derivative ☕☕

Use the formula for the derivative of a parametric curve to find the formula for the derivative of a polar graph.

Exercise 5.6.2. Using the Formula

Use the polar derivative formula above to find the slope of the graph of $r(\theta) = \sec(\theta)$. What does this let you conclude about that graph?

5.7 Area in Polar Coordinates

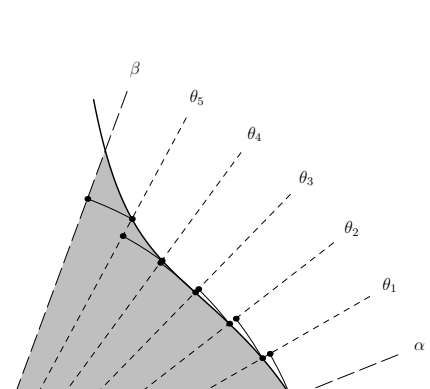
To compute area in polar coordinates, we essentially repeat the process of taking a Riemann sum. Rather than using rectangles however, we use sectors of circles.

Exercise 5.7.1. Area of a Single Sector

- What is the area of an entire circle with radius r? Draw the circle.
- Within your circle, draw a sector of that circle with angle θ. What proportion of the area of the entire circle does that sector occupy?
- Explain why the area of that sector is $A = \frac{1}{2}r^2\theta$.

We now repeat the process of taking a Riemann sum using sectors of circles. In particular, say we wish to find the area under the graph of $r(\theta)$ between two rays specified by angles $\theta = \alpha$ and $\theta = \beta$.

Let $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ be equally spaced angles from α to β. That is, $\theta_0 = \alpha$, $\theta_n = b$, and for each



$i \in \{0, 1, 2, \ldots, n-1\}$, $\Delta\theta = \theta_{i+1} - \theta_i = \frac{\beta-\alpha}{n}$.

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2} r^2(\theta_i)\,\Delta\theta \\ &= \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} r^2(\theta_i)\,\Delta\theta \\ &= \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta \end{aligned}$$

Thus, we have the formula for polar area!

Theorem 5.7.2. Polar Area

The area under a polar curve $r(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta$$

Sometimes a helpful way to remember the above formula is to write it as

$$A = \int_{\theta=\alpha}^{\theta=\beta} \pi r^2(\theta) \frac{\mathrm{d}\theta}{2\pi}.$$

This way you can think of the integrand as the area of a circle being multiplied by what ratio of 2π radians the change in θ is occupying. Canceling the two π's and pulling the $\frac{1}{2}$ outside of the integral lands you back at the Polar Area formula.

Exercise 5.7.3. Looking for Patterns ☕☕☕

Fill out the table! Carry out the instructions listed below for each given value of n.

- Plot the graph of $r(\theta)$.
- Find the area inside just one "petal".
- What patterns in n do you see? What can you say about the percent of the unit circle that lies inside rather than outside the graph?

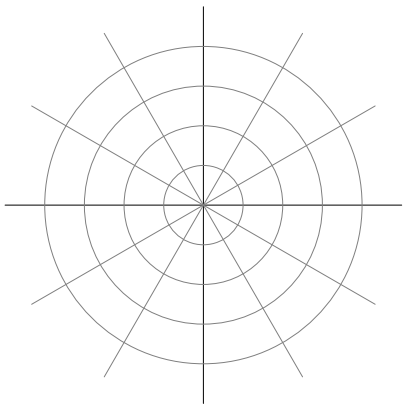
n	Graph of $r(\theta) = \sin(n\theta)$	Area of One Petal
2		
3		
4		
5		
n		

n	Graph of $r(\theta) = \cos(n\theta)$	Area of One Petal
2		
3		

4		
5		
n		

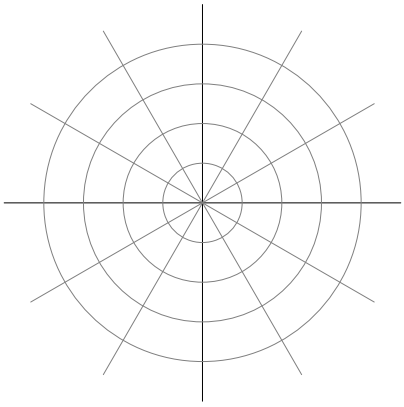
Exercise 5.7.4. Area Bounded by Two Polar Curves ☕☕☕

Plot both $r_1(\theta) = \frac{1}{2}\sec(\theta)$ and $r_2(\theta) = \cos(\theta)$ on the same set of axes. Find the area of the region that is to the right of r_1 but inside r_2.



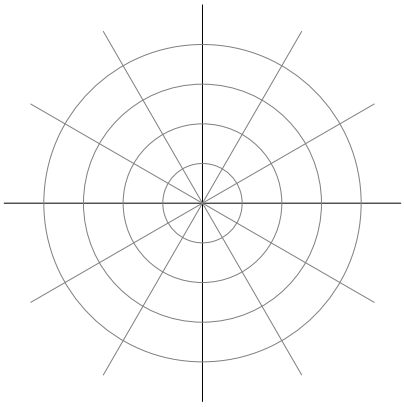
Exercise 5.7.5. Mixed Practice with Polar Curves ☕☕☕

- – Sketch the graph of $r(\theta) = 2\cos(2\theta)$.



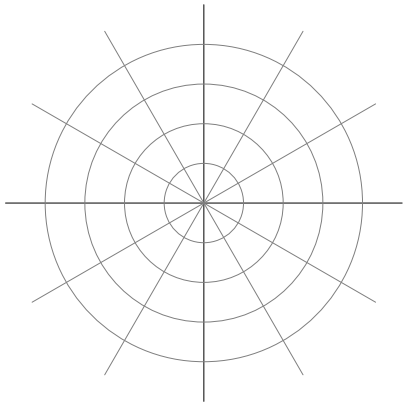
- Convert the above curve to cartesian. That is, find a polynomial equation in x and y whose solution set describes the same set of points. (**Hint:** Begin by applying the cosine double-angle identity!)

- Sketch the graph of $r(\theta) = \frac{1}{2} + \cos(\theta)$.



- Find the area enclosed by the inner loop of the graph.

- – Plot the graphs of both $r_1(\theta) = 1 + \cos(\theta)$ and $r_2(\theta) = 1 - \cos(\theta)$ on the same axes.



  – Shade the region contained inside both curves. Find its area.

5.8 Microphone Design

A microphone is a device that picks up sound (variations in air pressure) and produces an electrical signal. For any microphone, sound engineers want what is called the *polar pattern*, a graph indicating all locations from which sound is picked up with equal intensity. Microphones that are physically designed differently will have different polar patterns.

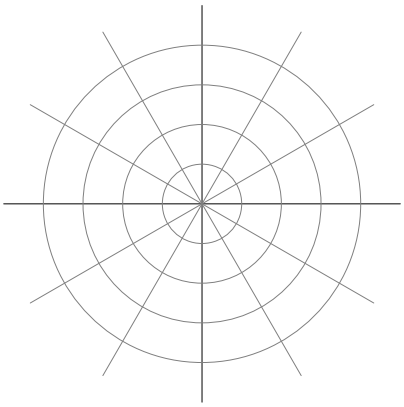
The key element to a microphone is some mechanical device that the waves of air pressure can compress. There are two basic types of devices:

Diaphragm

A spherical diaphragm responds equally to changes in air pressure from any side. Thus given a sound of a particular volume, the response in the microphone sensitivity is proportional to the distance from the diaphragm. A microphone with such a diaphragm is called an *omnidirectional microphone* and is represented by the polar pattern $r(\theta) = 1$, since the microphone has equal sensitivity to all points on a circle.

Exercise 5.8.1. Diaphragm Microphones ☕

Plot the polar pattern for a diaphragm microphone function below.

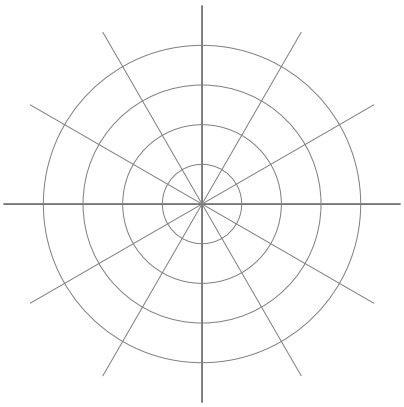


Ribbon

The other main type of device is a ribbon that floats in a magnetic field. Since it is a horizontal ribbon, it picks up changes in air pressure proportion to the sine of the angle to the source. (Imagine for example in physics a force pushing on a wall at an angle... the force that goes into the wall is not equal to the magnitude of the whole force but rather the magnitude times sine of the angle.) A microphone equipped with such a device is called a *ribbon microphone* or a *figure eight microphone* and has polar pattern given by $r(\theta) = |\sin(\theta)|$. Here we are taking absolute values because we are just denoting sensitivity, not the wave itself.

Exercise 5.8.2. Ribbon Microphones ☕☕

Plot the polar pattern for a ribbon microphone function below.



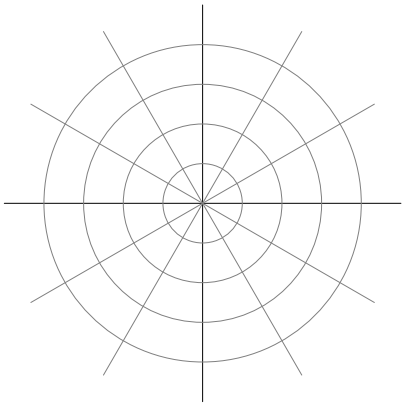
Cardioid

There are many situations where one of the above microphones is perfect for the purpose at hand. However, when a band is playing live music on a stage, the above two microphones do not work. The basic setup is the following: if a singer sings into the microphone, the main speakers are pointed towards the audience and not towards the singer. Thus, it is necessary to have monitors (smaller speakers pointing the opposite direction) so that the singer can hear herself. However, if the microphone picks up the sound coming out of the monitor, it's going to be again reproducing the same sound it just heard. The waves combine amplitude again and again, and this leads to that horrible high-pitched screeching noise known as feedback.

The solution to this is to design a microphone that picks up sound from one side but not from the other. The ingenious way engineers figured out how to do this was to simply make a microphone with *both* a ribbon and a diaphragm inside! The waves produced add to each other to make a single signal. Thus the sine of the ribbon will combine with the diaphragm's signal on one side, but cancel it out on the other! The polar pattern is given by the function $r(\theta) = 1 + \sin(\theta)$ (adding the waves together). Such a mic is called a *cardioid microphone* and is the standard mic for onstage live sound. The Shure 57 and Shure 58 are cardioid microphones and have been the standard mic used onstage for about 40 years now!

Exercise 5.8.3. Cardiod Microphones

- Plot the polar pattern corresponding to the above described cardioid microphone.



- Find the area of the region where sounds are at least as sensitive as they are on the boundary of that cardioid. (That is, find the area enclosed by the above polar curve.)

5.9 Chapter Summary

Here we introduced two new languages for describing curves in the plane, parametric and polar.

1. **Parametric:** A **parametric curve** is the set of points $(x(t), y(t))$ for some specified domain of t values.

   (a) **Graphing:** Pick a helpful spread of t values and plot the resulting points $(x(t), y(t))$ to get some idea of the shape. Often **converting to cartesian** by eliminating t and finding a direct relationship between x and y can be helpful.

   (b) **Derivatives:** The **slope of the tangent line to a parametric curve** can be found by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}$.

   (c) **Integrals:** The **length of a parametric curve** can be found by integrating the distance formula. If the parameter domain is the closed interval $D = [a, b]$, then the length is

$$\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\mathrm{d}t.$$

2. **Polar:** The system of **polar coordinates** describes the plane as (θ, r) where θ is the counterclockwise angle from the positive x axis and r is the signed distance from the origin.

   (a) **Graphing:** Given a polar function $r(\theta)$, pick a helpful spread of θ values and plot the resulting points $(\theta, r(\theta))$ to get some idea of the shape. Often **converting to cartesian** can be helpful. To convert, use the relationships

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r^2 &= x^2 + y^2 \end{aligned}$$

   or any other helpful relationship that follows from the triangle with angle θ, adjacent side x, opposite side y, and hypothenuse r.

   (b) **Derivatives:** To find the **slope of the tangent line to a polar graph**, convert to parametric by letting $t = \theta$. Specifically, set

$$\begin{aligned} x(t) &= r(t)\cos(t) \\ y(t) &= r(t)\sin(t) \end{aligned}$$

   and then use the formula for a parametric derivative.

   (c) **Integrals:** To find **area under a polar graph**, perform a Riemann sum with sectors of circles (rather than rectangles as we did initially). The π cancels to give us our polar area formula as seen below.

$$\begin{aligned} A &= \int_{\theta=\alpha}^{\theta=\beta} \underbrace{\pi r^2(\theta)}_{\text{Area of a circle}} \underbrace{\frac{\mathrm{d}\theta}{2\pi}}_{\text{proportion of full circle's radians}} \\ &= \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2(\theta)\,\mathrm{d}\theta \end{aligned}$$

5.10 Mixed Practice

Exercise 5.10.1. ☕☕

a.) Graph the polar function $r(\theta) = 4\sec(\theta)$ by picking a spread of θ values and making an input-output table.

b.) Convert to cartesian coordinates to show that the graph is in fact just a line!

c.) Sketch the polar region whose area corresponds to the following integral:

$$A = \int_{\theta=0}^{\theta=\pi/4} \frac{1}{2}\left(4\sec(\theta)\right)^2 \, \mathtt{d}\theta$$

d.) What shape is the region sketched in c)? Find the area by basic geometry.

e.) Find the area by computing the integral. Verify your answers match.

Exercise 5.10.2. ☕☕

Consider the parameterization

$$\{(4t-1, 6t) : t \in [0,2]\}$$

That is, $x(t) = 4t - 1$ and $y(t) = 6t$ as t roams from 0 to 2.

a.) Describe the shape of this parametric curve.

b.) What is the slope of the tangent line to the parametric curve at $t = 1$?

c.) Use the parametric arc length formula to compute the length of the parameterized curve from part a).

Exercise 5.10.3. ☕☕☕

State the power series definitions for hyperbolic sine and cosine:

- $\cosh(t) =$
- $\sinh(t) =$

a.) Use the power series for hyperbolic sine to compute its derivative.

b.) Use the power series for hyperbolic cosine to compute its derivative.

c.) Consider the parametric curve

$$\{(\cosh(t), \sinh(t)) : t \in \mathbb{R}\}$$

Verify out to degree six that this parametric curve also satisfies the cartesian equation for the same hyperbola given by:

$$x^2 - y^2 = 1$$

by plugging the power series formulas for our hyperbolic trig functions in for x and y.

d.) Find the slope of the tangent line at $t = 0$ to the parametric curve using your derivatives computed above.

e.) Graph the parametric curve along with the tangent line you found in the previous part.

Exercise 5.10.4. ☕☕☕

a.) Graph the polar function $r(\theta) = \sin(2\theta)$.

b.) Find the area enclosed by one loop of that function.

Chapter 6

Introduction to Complex Numbers

The extension from the real numbers to the complex numbers has far-reaching affects. In this chapter, we give a brief introduction to complex numbers and then show how they interact with almost every topic in the course!

The complex numbers arise out of the fact that the simple little equation $x^2 + 1 = 0$ has no solution over the reals. Thus, we create the number i to represent a root of that polynomial. That is, $i^2 + 1 = 0$.

Definition 6.0.1. Complex Numbers

The set of *complex numbers* is the set of all numbers that can be written in the form $a + bi$ for real numbers a and b.

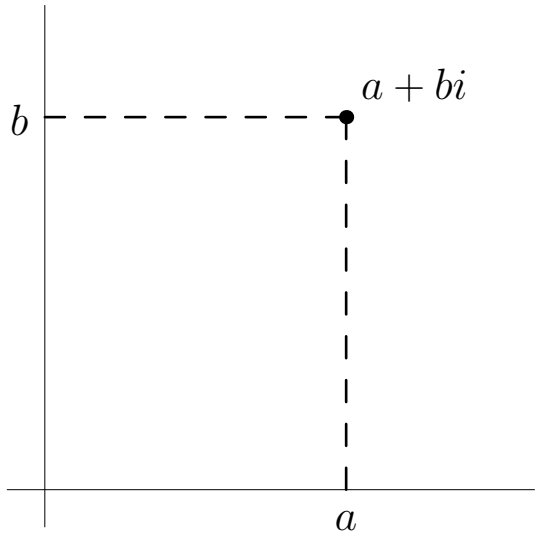
We perform arithmetic in the complex numbers using the usual rules of arithmetic and algebra along with the extra identity $i^2 = -1$.

Exercise 6.0.2. Containment of the Reals

- Is 3 a complex number? Can you write 3 in the form $a + bi$ for real numbers a and b?

- Does the set of complex numbers contain all real numbers?

We can visualize complex numbers in the complex plane, where a (the *real part*) is the horizontal component and b (the *imaginary part*) is the vertical.


$a + bi$
b
a

6.1 Euler's Identity and Consequences

Look again at the power series for the exponential function, sine, and cosine:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots \\ \cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \end{aligned}$$

You have to wonder if there is some way to add together sine and cosine to get the exponential function! Sure the signs are off, but otherwise things seem so right. Sine has all the odd factorial denominators, cosine has all the even factorial denominators, and the exponential function has all of them! It turns out that i is exactly the constant we need to fix those minus signs!

Exercise 6.1.1. Proof of Euler's Identity

- Write out a power series for $e^{i\theta}$.

- Write out a power series for $\cos(\theta) + i\sin(\theta)$.

- Verify the two are equal!

The fact that there is any relationship whatsoever between sine, cosine, and e is very surprising when you think of how differently those quantities are defined! We again state this incredible theorem, Euler's Identity!

Theorem 6.1.2. Euler's Identity

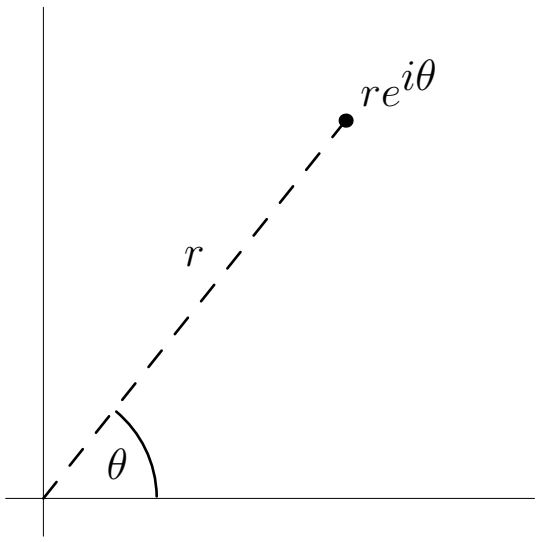
For any real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

If we multiply both sides by a real number r, we then obtain

$$re^{i\theta} = r\cos(\theta) + ir\sin(\theta).$$

We notice that the horizontal component, $r\cos(\theta)$, is in fact the conversion for x into polar coordinates. Likewise, $r\sin(\theta)$ is the conversion for y into polar coordinates. This means that the complex number $re^{i\theta}$ is in fact the point located at angle θ and radius r in the complex plane.



Complex Roots

One interesting fact about the complex numbers is that the number of nth roots of *every* real number is exactly n. So every number has two square roots, three cubed roots, and so on. We use $re^{i\theta}$ form to find these complex roots.

Example 6.1.3. The Cubed Roots of Two

To find all cubed roots of two, we solve the equation

$$z^3 = 2.$$

We begin by putting both z and 2 in complex polar form. We write $z = re^{i\theta}$ and $2 = 2e^{i0}$. We plug these into the equation, expand the powers.

$$\begin{aligned} z^3 &= 2 \\ \left(re^{i\theta}\right)^3 &= 2e^{i0} \\ r^3 e^{i3\theta} &= 2e^{i0} \end{aligned}$$

We now equate the radius and the angles as two separate equations.

- **Radius:** Since r is a real number, we obtain $r^3 = 2$, which implies $r = \sqrt[3]{2}$.
- **Angle:** The angles need to be equivalent but not necessarily equal. If they differ by a multiple of 2π, that is fine! Thus, we have $3\theta = 0 + 2\pi k$ for any integer k. Dividing both sides by 3, we have

  $$\theta = \frac{0 + 2\pi k}{3} = \ldots, \frac{-4\pi}{3}, \frac{-2\pi}{3}, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \ldots.$$

  However, if we use more values of θ beyond just $0, \frac{2\pi}{3}, \frac{4\pi}{3}$, the solutions will repeat since cosine and sine have period 2π. Thus, we use just those three angles.

Putting together our r and θ values, we have the following three roots:

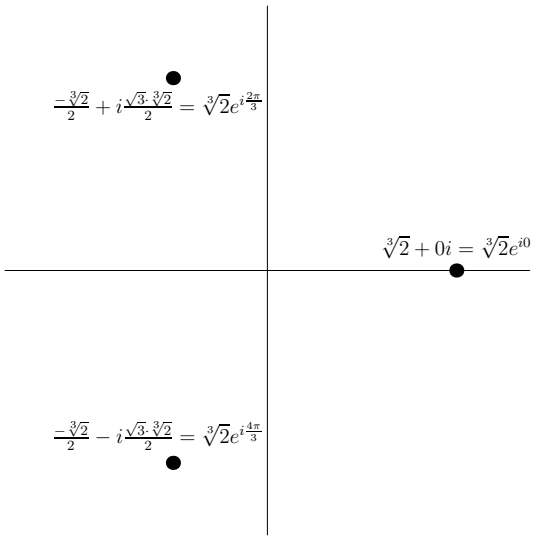
$$z = \sqrt[3]{2}e^{i0}, \sqrt[3]{2}e^{i\frac{2\pi}{3}}, \sqrt[3]{2}e^{i\frac{4\pi}{3}}.$$

Thus, we have our roots in complex polar form. We use Euler's Identity to turn these back into complex cartesian form as follows:

$$z = \sqrt[3]{2}\cos(0) + i\sqrt[3]{2}\sin(0), \sqrt[3]{2}\cos(\frac{2\pi}{3}) + i\sqrt[3]{2}\sin(\frac{2\pi}{3}), \sqrt[3]{2}\cos(\frac{4\pi}{3}) + i\sqrt[3]{2}\sin(\frac{4\pi}{3}).$$

At last, we use the unit circle to evaluate these and plot in the complex plane.

$$z = \sqrt[3]{2}, -\frac{\sqrt[3]{2}}{2} + i\frac{\sqrt[3]{2}\sqrt{3}}{2}, -\frac{\sqrt[3]{2}}{2} - i\frac{\sqrt[3]{2}\sqrt{3}}{2}$$



Exercise 6.1.4. Checking Once Again ☕☕

Cube each of the answers from the previous problem. Verify in each case you get 2!

It turns out to be of particular importance to find roots of 1. Define the nth *roots of unity* to be the solutions to the equation

$$z^n = 1.$$

Lets play around and see if we can find some neat properties!

Exercise 6.1.5. Roots of Unity ☕☕☕☕

- Find all square roots of unity. Write your answers in both cartesian and polar complex form, and plot them in the complex plane. (the case where $n = 2$)

- Find all third roots of unity.

- Find all fourth roots of unity.

- Find all fifth roots of unity.

- Find all sixth roots of unity. Let $z^6 = 1$ so that

$$(re^{i\theta})^6 = e^{0\theta}.$$

Then $\theta = \frac{1}{3}\pi k$ so that $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$.
Plugging these into the complex polar equations

$$\cos(0) + i\sin(0) = 1$$
$$\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$\cos(\pi) + i\sin(\pi) = -1$$

$$\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

- Fill out the following table:

n	Σ_n	Π_n
2		
3		
4		
5		
6		

  where Σ_n represents the sum of all n^{th} roots of unity and Π_n represents the product of all n^{th} roots of unity. (**Hint:** It's easier to add in cartesian, and easier to multiply in polar.)

- Based on your above data gathered, conjecture a formula for both Σ_n and Π_n. Prove your conjecture is correct. (**Hint:** Consider the roots of the polynomial $z^n - 1$ and how that polynomial would factor based on those roots. Then consider the degree zero and degree $n - 1$ coefficients.)

Using the same techniques we can answer the following question, "what is the square root of i?" Keep in mind there are technically two square roots of i, the two solutions to the equation $z^2 = i$.

Exercise 6.1.6. Square Roots of i ☕☕

- Find the square roots of i. Write your answers in complex cartesian form.

- Square your answers back out (in complex cartesian form) and verify that you do in fact get i when you square them.

Exercise 6.1.7. Cubed Roots of i ☕☕☕

- Find all cubed roots of i. That is, find all complex numbers z such that $z^3 = i$. Write your answers in $a + bi$ form.

- Take the cube of each of your roots to verify that you do in fact get i as the third power.

Proving Trig Identities

Remember how there are 47,000 useful but impossible to remember trigonometric identities? No? Well, that shows how hard they are to remember. Believe it or not, most of them can be constructed very quickly and easily from Euler's Identity!

Example 6.1.8. The Sine and Cosine Double-Angle Identities

To construct the sine and cosine double-angle formulas, we can manipulate the expression $e^{2\theta}$. We proceed with the following chain of equality:

$$\begin{aligned}
\cos(2\theta) + i\sin(2\theta) &= e^{i\cdot 2\theta} \\
&= \left(e^{i\theta}\right)^2 \\
&= (\cos(\theta) + i\sin(\theta))^2 \\
&= \cos^2(\theta) + 2\cos(\theta) i \sin(\theta) + i^2 \sin^2(\theta) \\
&= \left(\cos^2(\theta) - \sin^2(\theta)\right) + i\left(2\sin(\theta)\cos(\theta)\right).
\end{aligned}$$

We now equate real parts to obtain the cosine double-angle identity

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

Similarly, we equate imaginary parts to obtain the sine double-angle identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Exercise 6.1.9. Annotate! ☕

Write a short justification alongside each line of computation above.

Exercise 6.1.10. Angle-Sum Identities

- Expand the expression $e^{i(A+B)}$ into real and imaginary parts using Euler's Identity.

- Expand the expression $e^{iA}e^{iB}$ into real and imaginary parts using Euler's Identity twice, once per factor. Multiply out the resulting terms into an expression of the format

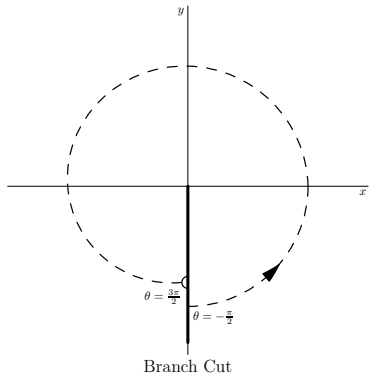
$$f(A, B) + ig(A, B)$$

where f is the function corresponding the real part of that expression, and g corresponds to the imaginary part.

- Equate real and imaginary parts to produce the angle sum identities for cos and sin, respectively! (**Hint:** We're using the fact that two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.)

Natural Logarithm of a Complex Number

We now show how to compute the natural logarithm of a complex number. As usual, polar form will be critical.



- Given a complex number z, we first write z in polar form $z = re^{i\theta}$, where r is a positive real number and $\theta \in [-\pi/2, 3\pi/2)$. This choice of interval for θ is often called a *branch cut* and is essentially a domain restriction for the exponential function (since it fails to be one-to-one over the complex numbers).
- Split apart using the property of logarithms and cancel the log with the exponential as follows:

$$\ln(z) = \ln\left(re^{i\theta}\right) = \ln(r) + \ln\left(e^{i\theta}\right) = \ln(r) + i\theta.$$

Example 6.1.11. Natural Log of i

Here we compute the mysterious quantity $\ln(i)$. We begin by rewriting i as

$$i = 1 \cdot e^{i\pi/2}.$$

Notice that we chose the angle $\theta = \pi/2$ to be within the branch cut specified above. From here, we split using log properties as follows:

$$\begin{aligned}\ln(i) &= \ln\left(1 \cdot e^{i\pi/2}\right)\\ &= \ln(1) + \ln\left(e^{i\pi/2}\right)\\ &= 0 + i\frac{\pi}{2}.\end{aligned}$$

Thus, $\ln(i) = \frac{\pi}{2}i$.

Note that in principle there is no reason we had to pick our angle θ in that particular interval. One can construct a perfectly well-defined logarithm from choosing a different domain for θ. This is similar to the construction of the inverse trig functions, where one must restrict the domain in some manner, so we tend to just choose a default interval to restrict to and stick with it.

Exercise 6.1.12. Complex Logarithms ☕☕

Try the above method to compute each of the following logarithms. Write each in the standard complex cartesian form $a + bi$.

- $\ln(2)$
- $\ln(-2)$

- $\ln(1+i)$

- $\ln(3-4i)$

Complex Exponentials

Recall our trick for dealing with strange bases:

$$w_1{}^{w_2} = e^{\ln(w_1{}^{w_2})} = e^{w_2 \ln(w_1)}.$$

This provides the advantage of moving us back to the familiar base e from the unfamiliar base w_1. This will make a complex exponential base manageable!

Example 6.1.13. Computation of Hammurabi

Here we perform two exponentials; we have an i to an i, and a 2 to the 2th. Using the above trick and the value of $\ln(i)$ computed in Example 6.1.11, we find

$$\begin{aligned} i^i 2^2 &= 4i^i \\ &= 4e^{\ln\left(i^i\right)} \\ &= 4e^{i\ln(i)} \\ &= 4e^{i\frac{\pi}{2}i} \\ &= 4e^{-\frac{\pi}{2}}. \end{aligned}$$

Notice there was no need to decompose further using Euler's Identity here; the end result of i raised to the i power is in fact a real number!

Exercise 6.1.14. Complex Exponentials ☕☕

- Use the above trick to compute $(1+i)^i$.

- Use the above trick to compute i^{1+i}.

- Use the above trick to compute $(1+i)^{1+i}$.

6.2 Even and Odd Parts of Functions

In your college algebra or precalculus course, you likely encountered the notions of *even functions* and *odd functions.* Let us begin by recalling these definitions (or, look them up with a brief internet search if you hadn't seen them).

Exercise 6.2.1. Even and Odd Functions ☕

- Complete the definition below.

  A function f is an *even function* if and only if...

- Complete the definition below.

  A function f is an *odd function* if and only if...

Exercise 6.2.2. Even and Odd Symmetry ☕

- What kind of symmetry does the graph of an even function have?

- What kind of symmetry does the graph of an odd function have?

Now let us think about some famous functions with respect to these properties.

Exercise 6.2.3. Examples of Even and Odd Functions ☕☕

Consider the list of functions given below.

- $f(x) = \sin(x)$
- $f(x) = \cos(x)$
- $f(x) = e^x$
- $f(x) = \arctan(x)$
- $f(x) = \frac{1}{1-x}$
- $f(x) = \frac{1}{1-x^2}$
- $f(x) = \frac{x}{1-x^2}$
- $f(x) = \cosh(x)$
- $f(x) = \sinh(x)$

For each function on the above list, decide if it is even, odd, or neither.

Exercise 6.2.4. Power Series of Even and Odd Functions

Consider the same list of functions from the previous exercise. For each function on that list, write out its power series. Do you see any correspondence between the power series and the state of being even, odd, or neither?

So what does all of this have to do with complex numbers? Why is it in this chapter? Got lost on the way home from the bar?

Theorem 6.2.5. Even and Odd Parts

Any function can be decomposed into a sum of an even function and an odd function. In particular, given a function $f(x)$, define

$$f_o(x) = \frac{f(x) - f(-x)}{2}$$

and

$$f_e(x) = \frac{f(x) + f(-x)}{2}.$$

Then, $f_o(x)$ is odd, $f_e(x)$ is even, and

$$f(x) = f_e(x) + f_o(x).$$

Exercise 6.2.6. Verifying the odd (and even) claims

Verify the above theorem for yourself. In particular, use the formulas given for f_o and f_e to prove the following:

- f_o is odd
- f_e is even
- $f_o + f_e = f$

Ok, so really, what does all of this have to do with complex numbers?

Exercise 6.2.7. Odd and Even Parts of Some Basic Functions

- Consider the polynomial $f(x) = 3 + 2x - x^2$. Find f_o and f_e for it!
- Consider the cubed root function $f(x) = \sqrt[3]{x}$. Find f_o and f_e for it!

Exercise 6.2.8. Odd and Even Parts of a Power Series

If a function has a power series representation centered at zero, there is an extremely nice, and much simpler interpretation of the even and odd parts. In particular, calculate $f_e(x)$ and $f_o(x)$ for the function

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots .$$

Use your calculation above to describe the even and odd parts of a function $f(x)$ in terms of its power series, in words.

Wait, but what does all of this have to do with complex numbers?

Exercise 6.2.9. Even and Odd Parts of an Exponential Function ☕☕

Find the even and odd parts of the function $f(x) = e^x$, both as a power series, and by recognizing those power series as a famous function.

But what about Mirabel?

Exercise 6.2.10. Even and Odd Parts of a slightly tweaked exponential Function ☕☕

Find the even and odd parts of the function $f(x) = e^{ix}$, both as a power series, and by recognizing those power series as a famous function.

The consequence of the past few exercises is profound: we now have a beautiful, unifying perspective on the trig functions and hyperbolic trig functions. Moreover, the proof of Euler's Identity can be seen as a decomposition of e^{ix} into its even and odd parts. In particular, we have proven the following:

- Hyperbolic cosine and sine are respectively the even and odd parts of the function $f(x) = e^x$. That is,
$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$
and
$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots .$$
- Ordinary cosine and sine (times i) are respectively the even and odd parts of the function $f(x) = e^{ix}$. That is,
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
and
$$i\sin(x) = \frac{e^{ix} - e^{-ix}}{2} = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots ,$$
or if we divide both sides by i, then we have
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

It should be noted that, although complex numbers are involved, it is often easier to work with sine and cosine via their exponential formulas of

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

In particular, they can often reduce tricky triggy calculations to basic algebra!

Exercise 6.2.11. Proving trig identities ☕☕

Use the exponential formulas for sine and cosine above to prove the double angle formula for sine, that is, that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ by simply plugging into the exponential formulas on both the right and left hand sides and making sure that they're equal.

Exercise 6.2.12. Revisiting Products of Sines and Cosines ☕☕☕

(a) Recall the technique of integration from Section 1.3. Use it to calculate the integral $\int \sin^2(x)\cos^2(x)dx$.

(b) Calculate that integral again but this time by subbing the exponential formulas in for sine and cosine, and doing algebra on those exponential functions to expand everything out. Show that eventually one ends up with the equivalent integral

$$-\frac{1}{16}\int \left(e^{i4x} - 2 + e^{-i4x}\right) dx,$$

which can then be integrated term-by-term. (Note that even though there is the complex number i appearing in this integral, it is ok to perform the antiderivative as if it were a real constant. In a complex analysis class you will define integration over the complex numbers in much more generality!) However, you answer will then be in terms of complex exponentials. Use Euler's identity to turn these back into sines and cosines, and verify your answer is in fact equivalent to what we got in part (a).

6.3 Inverse Euler's Identity

Since Euler's Identity links exponential functions to trigonometric functions, there then just *has* to be a link between logarithms and inverse trigonometric functions, right? They're the inverse functions corresponding to the functions that appear in Euler's identity, so it *has* to be, right? This is Kirk speaking, by the way, not Spock.

The technique we use to construct this link is quite surprising: partial fraction decomposition over the complex numbers! Note that if we are using complex numbers, there are no more irreducible quadratics! This gives us an interesting alternate way to perform PFD, since all polynomials will fully factor into linear factors.

Exercise 6.3.1. PFD over the Complex Numbers ☕☕☕

- Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the real numbers.

- Find an antiderivative of $\frac{4-2x^2}{x^3+4x}$ via a partial fraction decomposition over the complex numbers. That is, when you factor the denominator, don't stop at $x \cdot (x^2+4)$, but instead, keep factoring to get all factors to be of the form $(x-z)$ where z is a complex number. Then

proceed with PFD like normal from there.

- Verify your answers are compatible.

At last, we construct our sought-after "inverse Euler's Identity".

Exercise 6.3.2. Relationship between Inverse Tangent and Natural Log ☕☕☕

Recall that by trigonometric substitution, we have

$$\int \frac{1}{x^2+1}\,\mathtt{d}x = \arctan(x) + C.$$

Compute the same antiderivative but using a PFD over the complex numbers. In particular, carry out the following steps:

- Factor x^2+1 over the complex numbers.

- Find A and B in the decomposition

$$\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i}.$$

- Find the antiderivative, simply integrating terms of the form $\frac{c_1}{x+c_2}$ as $c_1 \ln(x + c_2)$ for complex constants c_1 and c_2, just as you would for real constants. (Note that in a complex analysis class, you will define integration over the complex numbers in much more generality!)

- Conclude that
$$\arctan(x) = \frac{1}{2i} \ln\left(\frac{x-i}{x+i}\right) + C$$
for some constant C.

- Solve for C by letting $x \to \infty$ on both sides to get a relationship between inverse tangent and the natural logarithm! (It is valid for positive x.)

6.4 Chapter Summary

The set of **complex numbers** is the set of all numbers expressible as $a + bi$ for real numbers a and b. In the **complex plane**, we plot a as the horizontal and b as the vertical. Allowing complex numbers in our calculus adventures relates many seemingly unrelated objects!

1. **Euler's Identity and consequences:** By setting $x = i\theta$ in our power series for the exponential function, we obtain **Euler's Identity**. This is the relationship

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

which is usually then multiplied by r to obtain

$$re^{i\theta} = \underbrace{r\cos(\theta)}_{x \text{ in polar coords}} + i \underbrace{r\sin(\theta)}_{y \text{ in polar coords}}.$$

This means that in the complex plane we have $re^{i\theta}$ as the point at angle θ and radius r. This relationship has many consequences, including the following:

   (a) **Proving trig identities:** Properties of exponentials can be turned into trig identities using Euler's Identity.

   (b) **Calculating roots of complex numbers:** To find the nth roots of a complex number $a+bi$, notice that this is the same as solving the equation $z^n = a + bi$. Rewrite everything in polar form, distribute the n power, and equate radius and angles to find the roots.

   (c) **Calculating logarithms of complex numbers:** To compute $\ln(a + bi)$, write $a+bi$ in polar form with an angle chosen in the branch cut $-\pi/2 \leq \theta < 3\pi/2$. From there, use properties of logs to simplify the expression.

   (d) **Calculating exponentials with a complex base:** Rewrite as "e to the ln" of the expression and then use the method for complex logarithms described above.

2. **Even and Odd Parts of Functions:** Any function $f(x)$ can be decomposed into a sum of an even part, $f_e(x)$, and an odd part, $f_o(x)$, given by the formulas $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$. In particular, if one does this for the functions $f(x) = e^x$ and $f(x) = e^{ix}$, it generates the hyperbolic and regular sine (off by a factor i) and cosine functions. That is,

   - $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$
   - $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$
   - $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$
   - $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$

3. **Revisiting PFD with complex numbers:** With complex numbers, there is no need for the irreducible quadratic case in a PFD. Instead, we can completely factor the denominator of any rational function into a product of powers of linear factors. We can use this to prove what we call "inverse Euler's Identity":

$$\arctan(x) = \frac{1}{2i}\ln\left(\frac{x-i}{x+i}\right).$$

6.5 Mixed Practice

Exercise 6.5.1. ☕☕

(a) Find all square roots of $-i$. Write your answers in complex cartesian form.

(b) Take your answers and square them to verify their square is $-i$ as you claim above.

Exercise 6.5.2. ☕☕☕

Compute the following complex numbers in standard $a + bi$ form for $a, b \in \mathbb{R}$. List all values if there are multiple answers.

(a) i^{2015}

(b) $i^{(2i)}$

(c) $1 + i\pi - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{\pi^4}{4!} + \frac{i\pi^5}{5!} - \frac{\pi^6}{6!} - \frac{i\pi^7}{7!} + \frac{\pi^8}{8!} + \cdots$

(d) $\sqrt[3]{i}$

(e) $\ln\left(4\sqrt{3} + i\right)$

Exercise 6.5.3. ☕☕

(a) State and prove Euler's Identity using power series.

(b) Multiply both sides of Euler's Identity by r. Explain how this formula relates to our conversion between polar and cartesian coordinates.

(c) Find the sine triple-angle identity using Euler's Identity.

Exercise 6.5.4. ☕☕☕

Consider the integral

$$\int \cos(x)\cos(2x)dx.$$

(Note: these kinds of integrals will be extremely important when you study Fourier series!)

(a) Integrate by replacing $\cos(2x)$ by the formula given by the double-angle identity $\cos^2(x) - \sin^2(x)$, expanding the product, and applying the technique from Section 1.3 to each resulting term.

(b) Integrate by turning all of the cosines into exponentials over the complex numbers, multiplying out the resulting expression, and integrating each exponential separately. Then turn your answer back into sines and cosines using Euler's Identity.

(c) Your answers may appear very different! Verify they are in fact compatible.

Exercise 6.5.5. ☕☕☕

Consider the integral

$$\int \frac{1}{x^3 + x} dx.$$

(a) Integrate by performing a partial fraction decomposition over the real numbers.

(b) Integrate by performing a partial fraction decomposition over the complex numbers.

(c) Your answers may appear very different! Verify they are in fact compatible.

Chapter 7

Introduction to Differential Equations

In this course, we got really good at two things: finding antiderivatives and using power series. It is no accident that the study of differential equations relies primarily on those two techniques! Here we show just two methods for solving differential equations: separation of variables, based on antidifferentiation, and power series solutions, based on power series (really!).

7.1 What is a Differential Equation?

Definition 7.1.1. Differential Equation

A *differential equation* (DE) is an equation involving a variable (say y) that stands for some unknown function, and also involving one or more derivatives of y. The *solution* to a differential equation is the set of all functions y that make the equation true.

We begin with a nice bridge troll riddle. We ask "What functions are equal to their own derivative?".

Example 7.1.2. Functions Equal to Their Own Derivative

To state this question in the language of differential equations, we say that we wish to solve the DE

$$y' = y.$$

Exercise 7.1.3. Guess and Check ☕

Can you think of any functions that are equal to their own derivative? Do you think you have all

of them, or are some likely still out there?

As you can see, guess and check is not a good method for solving even the simplest of differential equations. We now take a more structured approach.

7.2 Separable Equations

Definition 7.2.1. Separable

Let x be the independent variable and let y represent an unknown function of x. A differential equation is *separable* if and only if it can be written in the form

$$\frac{dy}{dx} = F(x)G(y)$$

for some functions F and G.

Our method for solving a separable differential equation is as follows:

1. Write right-hand side of the differential equation in factored form, one function of x times one function of y.
2. Separate variables by multiplying both sides by $\frac{1}{G(y)}\,\mathtt{d}x$.
3. Antidifferentiate both sides.
4. Solve for y, if possible. (If not, we at least have an implicit solution.)

We try out this method on the previous example.

Example 7.2.2. Separation of Variables

Notice the differential equation

$$y' = y$$

is separable because it can be rewritten as

$$\frac{\mathtt{d}y}{\mathtt{d}x} = (1)(y).$$

That is, our factored form uses the functions $F(x) = 1$ and $G(y) = y$. We now perform separation

of variables and antidifferentiate both sides.

$$\begin{aligned}
\frac{1}{y}\,\mathtt{d}y &= 1\,\mathtt{d}x \\
\int \frac{1}{y}\,\mathtt{d}y &= \int 1\,\mathtt{d}x \\
\ln|y| &= x + C \\
|y| &= e^{x+C} \\
y &= \pm e^{C} e^{x} \\
y &= Ce^{x}
\end{aligned}$$

Notice that on the last line for simplicity, we clean up the constant $\pm e^C$ by just calling it C.

Exercise 7.2.3. Analyzing the Example ☕

- Why were we able to just put a $+C$ on one side when we integrated? What would have happened if we put it on both sides?

- When we renamed $\pm e^C$ as C, we technically introduced a new solution. The expression $\pm e^C$ is incapable of being equal to zero, but C can be. Verify that the $C = 0$ solution is valid to include as a solution to the differential equation.

Exercise 7.2.4. More Complicated DEs ☕☕☕

- Solve the following differential equation via separation of variables:

$$\frac{dy}{dx} = xy + x$$

- Solve the following Initial Value Problem via separation of variables:

$$\frac{dy}{dx} = e^{y-x}\sec(y)(1+x^2)$$

Note that you will not be able to obtain an explicit formula for y in terms of x but rather an implicit solution. Use the initial condition $y(0) = 0$ to solve for C.

7.3 Power Series Solutions

Power series provide a very effective method for solving differential equations. The steps are simple:

- Set the unknown function y equal to an unknown power series.
- Plug the power series in for all occurrences of y. Expand and combine like terms.
- Equate coefficients one degree at a time (much like we do when solving for unknowns in a PFD).
- Solve for the coefficients a_0, a_1, $a_2, \ldots$ one at a time in terms of a_0.
- Plug those coefficients back into the power series expansion for y to obtain a power series solution.
- Try to recognize that power series as a known function or variant thereof.

The process is thus very mechanical, but sometimes working through the details becomes a bit messy. We repeat the previous example with this new method.

Example 7.3.1. Revisiting Our First DE

Here we solve

$$y' = y$$

using power series. First, let y be written as a generic unknown power series as follows:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Plug this expression into the differential equation and expand.

$$\begin{aligned}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)' &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\end{aligned}$$

We now equate coefficients one degree at a time, and solve for every coefficient in terms of a_0.

$$\begin{array}{lrcl}
\text{Degree 0:} & a_1 = a_0 & \Longrightarrow & a_1 = a_0 \\
\text{Degree 1:} & 2a_2 = a_1 & \Longrightarrow & a_2 = \frac{1}{2} a_0 \\
\text{Degree 2:} & 3a_3 = a_2 & \Longrightarrow & a_3 = \frac{1}{3!} a_0 \\
\text{Degree 3:} & 4a_4 = a_3 & \Longrightarrow & a_4 = \frac{1}{4!} a_0 \\
\vdots & \vdots & & \vdots \\
\text{Degree } n-1: & n a_n = a_{n-1} & \Longrightarrow & a_n = \frac{1}{n!} a_0
\end{array}$$

We can now plug all coefficients back into our expression for y and simplify until we obtain a closed form for y.

$$\begin{aligned}y(x) &= a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \frac{1}{4!} a_0 x^4 + \cdots \\ &= a_0 \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots\right) \\ &= a_0 e^x\end{aligned}$$

Notice we have obtained the same solution as via separation of variables! Clearly, the power series solution was way more work. The reason it is so valuable though is that there are many DEs which are not separable but for which the power series method works just fine.

Exercise 7.3.2. Comparing the Methods ☕☕☕

- – Show the differential equation $\frac{dy}{dx} = yx$ is separable and use this to separate variables and solve the differential equation.

  – Solve the same differential equation via power series. Confirm you get the same answer.

- – Explain why the differential equation $\frac{dy}{dx} = yx + x + 1$ is not separable.

- Solve the same differential equation via power series.

- Check your answer is correct by plugging it back into the original DE.

- Consider the DE given by

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \\ y'' &= -y. \end{aligned}$$

Solve this DE via power series (use the initial conditions to solve for a_0 and a_1).

7.4 Modeling with Differential Equations

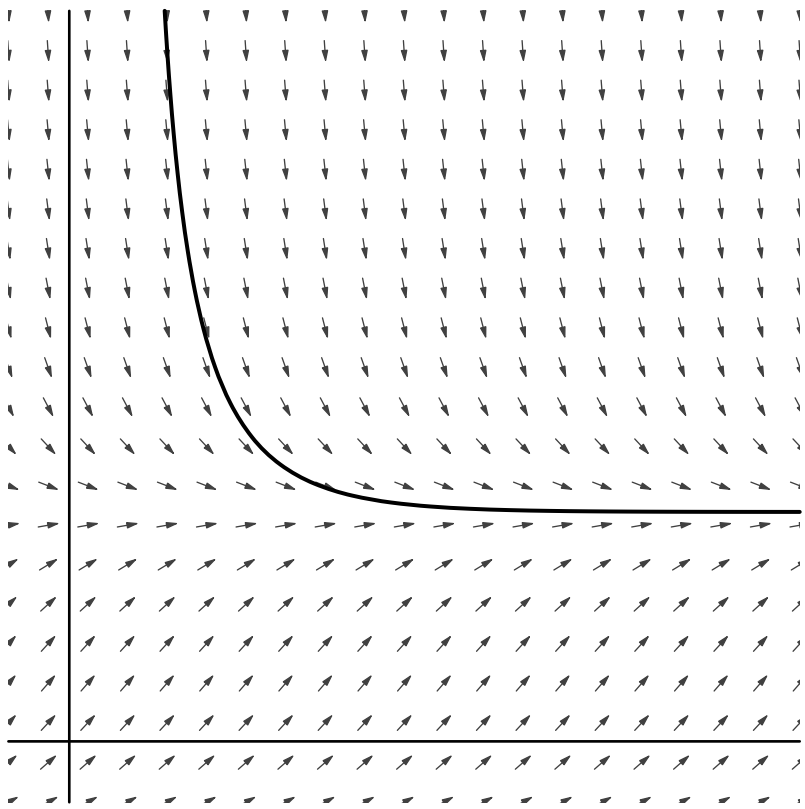
Differential equations are used extensively in applied mathematics and the sciences to describe models, which are then solved using mathematics to find explicit formulas for the quantities of interest.

Example 7.4.1. Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature of a small object in a room is proportional to the difference between room temperature and the temperature of the object. If A is the constant that represents the ambient temperature (room temperature), $T(t)$ represents the temperature of the room at time t, and k is the constant of proportionality, then this situation can be modeled by

$$\frac{dT}{dt} = k(T - A).$$

Here we introduce the idea of a *slope field*, a grid of small dashes that indicate the slope $\frac{\mathrm{d}T}{\mathrm{d}t}$ at every point (t, T) in the plane. Here we draw a slope field that governs solution curves to this model and show one sample solution curve.



Exercise 7.4.2. Newton's Law of Cooling

- Label the above diagram. What variables do the axes correspond to? Can you find where the horizontal line $T = A$ is located?

- In this model, would it make sense that the proportionality constant k is positive or negative? Why?

- Solve the differential equation by separation of variables.

- Solve the differential equation by power series.

Note that those solutions give explicit formulas for the solution curves above, which is significantly more useful than just thinking of it intuitively as "following the arrows".

Exercise 7.4.3. Malthusian Population Model

A simple intuitive population model can be stated as follows:

If there are more individuals in a population, there will be more babies produced.

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is proportional to the size of the population.

- Let $P(t)$ be the size of the population at time t. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series.

- Solve your differential equation using separation of variables and confirm that your answers

match.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Under what real life conditions might this model be realistic? Under what conditions might this model be unrealistic?

As you probably noticed, the above model is slightly ridiculous for large time values since it would claim that eventually any species would fill up the entire visible universe with bodies. So let's adjust it to fix that unrealistic assumption. Here's an upgrade:

Exercise 7.4.4. Logistic Population Model ☕☕☕

If there are more individuals in a population, there will be more babies produced, but then it slows down as it approaches some sort of maximum possible population (a limit perhaps based on food supply, available habitat, etc).

Here is a slightly more technical restatement of the same idea:

The rate of growth of a population is jointly proportional to both the size of the population and the distance from some maximum possible population.

- Let $P(t)$ be the size of the population at time t and let M for maximum be a constant that the population cannot exceed. Rewrite the above growth principal in the language of differential equations.

- Solve your differential equation using power series out to a degree two approximation (this will be much too difficult to solve the whole thing using power series!).

- Solve your differential equation using separation of variables and confirm that your answers

match out to the degree two approximation.

- Use your formula to find the limit of $P(t)$ as t approaches infinity.

- Suppose you started with population $P(0) = 2M$. What would your model predict would happen to the population?

7.5 Chapter Summary

A **differential equation** is an equation involving an unknown function $y(x)$ and one or more of its derivatives. The goal is to solve for an infinite family of functions $y(x)$ that satisfy the equation, or to find just a single function that solves the equation if a suitable **initial condition** is provided. There are many methods for solving DEs, but we focused on two in particular.

1. **Separation of variables:** To solve via separation of variables, we follow the following steps:

   - Write right-hand side of the differential equation in factored form, producing a DE in the form $\frac{dy}{dx} = F(x)G(y)$ (if possible).
   - Separate variables by multiplying both sides by $\frac{1}{G(y)}\,\mathtt{d}x$.
   - Antidifferentiate both sides.
   - Solve for y, if possible. (If not, we at least have an implicit solution.)

   This method is usually quite straightforward to carry out. The drawback is that most Differential Equations are not separable, meaning that it is impossible to write it as $\frac{dy}{dx} = F(x)G(y)$. Thus, the method usually fails as soon as it gets started.

2. **Power series solutions:** This method is far more robust as it does not depend on the DE being separable. It applies to more DEs, but be warned it is typically far messier! The steps are as follows:

   - Set the unknown function y equal to an unknown power series:

   $$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$

   - Plug the power series into the DE for all occurrences of y. Expand and combine like terms.
   - Equate coefficients one degree at a time. This will create an infinite system of equations of the following form:

   $$\begin{aligned} \text{Left-hand side degree zero coefficient} &= \text{Right-hand side degree zero coefficient} \\ \text{Left-hand side degree one coefficient} &= \text{Right-hand side degree one coefficient} \\ \text{Left-hand side degree two coefficient} &= \text{Right-hand side degree two coefficient} \\ \text{Left-hand side degree three coefficient} &= \text{Right-hand side degree three coefficient} \\ &\vdots \end{aligned}$$

   - Solve for the coefficients a_0, a_1, $a_2, \ldots$ one at a time in terms of a_0. (If you have an initial condition, a_0 will just be a number. Otherwise leave everything in terms of a_0.)
   - Plug those coefficients back into the power series expansion for y to obtain a power series solution.
   - Try to recognize that power series as a known function or variant thereof.

   One can visualize solutions to DEs via a **slope field**, a grid of arrows that shows the value of $\frac{\mathtt{d}y}{\mathtt{d}x}$ at each point as the slope of the arrow. The solutions to the differential equation are functions that essentially follow the directions given by the arrows, starting at some initial condition.

7.6 Mixed Practice

Exercise 7.6.1. ☕

Find all functions that equal their own second derivative. That is to say, use power series to solve the following differential equation:

$$y'' = y$$

Exercise 7.6.2. ☕☕☕

a.) Find the set of all solutions to the following differential equation using power series. Do not leave your answer as a power series but rather turn it back into a closed explicit formula using familiar functions.

$$\frac{dy}{dx} = y - x - 2$$

b.) Plug your answer back into the DE to verify it is correct.

Exercise 7.6.3. ☕☕☕

a.) Explain why one cannot use separation of variables to solve the differential equation

$$\frac{dy}{dx} = 2y + x$$

b.) Solve the above differential equation using power series. Recognize your answer as a known function!

Chapter 8

Selected Answers and Hints

Exercise 0.3.1. The first is a function, the second is a quantity, and the third is a statement.

Exercise 0.3.6. The first is a quantity, the second is a function, and the third through fifth are garbage. (Note that for comparing sets, there is in fact an entirely different relation called *subset*, written $\subseteq$, which means that the elements of one set are contained in the other set. So what was written in the problem was garbage, but one could say $\mathbb{N} \subseteq \mathbb{Z}$.) The sixth is a statement (that happens to be false), and the last two are statements (that happen to be true). Note that the last one may be a bit counterintuitive, saying that 3 is a complex number. However, the set of complex numbers contains the natural numbers (where 3 certainly lives), so it is valid to say 3 is a complex number. It is sometimes helpful to think of it as $3 + 0i$ to see that it does fit the standard form of a complex number.

Exercise 0.3.7. •In words, it says "For all persons x, there exists a person y, such that x is the mother of y." This is accurate but maybe sounds a bit rigid in English, where we would likely say something more like "Every person had a mother." Thus, this sentence is true, since every human came from some mother. (At least at the time of writing this, monkeys have been cloned, but not humans!) •In words, it says "There exists a person x, such that for all persons y, x is the mother of y. A more natural way to say this is "All people have the same mother!" which is clearly false, since the sentence implies that the one special person x is the mother of every person who has ever existed! •In words, the sentence says "Every real number has another real number bigger than it." This is certainly true. For example, one could satisfy the statement above by choosing $y = x + 1$, which is greater than x no matter what x is. •This statement says that there exists one real number that is bigger than all real numbers! This is false. One might think of infinity as a symbol which plays this role, but infinity is not an element of the set of real numbers.

Exercise 0.4.1. All statements are true except the last one. You cannot say that the polynomial $p(x) = 0$ has any degree at all, because we require the leading coefficient a_n to be nonzero, and here no such coefficient exists! Thus, the degree of the polynomial $p(x) = 0$ is undefined. It is also incorrect to say it has no roots; every number r satisfies $p(r) = 0$, so it actually has infinitely many roots.

Exercise 0.4.6. The roots are $x = \frac{-2\pm\sqrt{2^2-4\cdot 2\cdot 2}}{2\cdot 2} = \frac{-2\pm\sqrt{-12}}{4} = \frac{-2\pm 2\sqrt{3}i}{4}$ which simplifies to the roots $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Thus, it factors as $p(x) = 2\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$.

Exercise 0.4.9. •Here nothing needs to be done: the polynomial already has no linear term. If one wanted to really force it, it would be simply $p(x) = 1\,(x-0)^2 - 1$. •$p(x) = (x - 1/2)^2 + 3/4$ •$p(x) = 3(x-1)^2 - 2$

Exercise 0.4.10. •The multiplications all work out just fine; remember that $i^2 = -1$. •The poly-

nomial $p(x) = 4x^4 - 9$ can be viewed as a difference of two squares: $p(x) = \left(2x^2\right)^2 - 3^2$. Choosing $A = 2x^2$ and $B = 3$ produces a factorization using the difference of two squares formula: $p(x) = \left(2x^2 - 3\right)\left(2x^2 + 3\right)$. Notice the first factor is again a difference of two squares, as it can be written as $\left(\left(\sqrt{2}x\right)^2 - \left(\sqrt{3}\right)^2\right) = \left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x + \sqrt{3}\right)$. If we are factoring only using real numbers, we would then be done, and leave the factorization as $p(x) = \left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x + \sqrt{3}\right)\left(2x^2 + 3\right)$. However, if we allow complex numbers, we can factor further by recognizing the final factor as a sum of two squares (or by using the quadratic formula, but this is excessive). This produces the factorization $p(x) = \left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x + \sqrt{3}\right)\left(\sqrt{2}x - \sqrt{3}i\right)\left(\sqrt{2}x + \sqrt{3}i\right)$. •The polynomial factors over the real numbers as $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$ and over the complex numbers as $x^6 - 1 = (x-1)(x+1)\left(x - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)$.

Exercise 0.4.15. The polynomial factors as $(x+1)^2(2x-3)$.

Exercise 0.4.16. •Simply plug it in and calculate $p(25/6)$. It will be zero. •Long division produces $p(x) = (6x - 25)(x^2 + x + 2)$.

Exercise 0.4.18. First factor the common $4x^2$ out of the degree three and degree two terms. Then think of the $x-9$ as $1\cdot(x-9)$. The two groupings will have the common $(x-9)$ which can then be factored out.

Exercise 0.4.19. Hint: The first one is actually the trickiest. Rewrite the polynomial $x^4-x^3+2x^2-x+1$ as $x^4 - x^3 + x^2 + x^2 - x + 1$, and proceed to factor x^2 out of the first three terms.

Exercise 0.4.21. •We notice the $1, 5, 10, 10, 5, 1$ in the coefficients. This leads us to think about the row $n = 5$ of Pascal's Triangle, and tells us that the polynomial will factor in the form $(A + B)^5$. It only remains to decide what the A and B are. A little strategic guess and check will show that $A = -x^2$ and $B = 1$ works. Thus, the polynomial factors as $-x^{10} + 5x^8 - 10x^6 + 10x^4 - 5x^2 + 1 = \left(-x^2 + 1\right)^5$. Notice this actually factors further by reversing $-x^2 + 1$ to be $1 - x^2$ and then applying the Difference of Two Squares. Thus, we have $-x^{10} + 5x^8 - 10x^6 + 10x^4 - 5x^2 + 1 = ((1-x)(1+x))^5 = (1-x)^5(1+x)^5$. •The coefficients lead us to row $n = 2$, since we see the $1, 2, 1$. Thus, $x^6 + 2x^3 + 1 = (x^3 + 1)^2$. We then factor the inside further using Sum of Two Cubes: $x^6+2x^3+1 = (x+1)^2(x^2-x+1)^2$. If we are factoring over the real numbers, we are done, or over the complex numbers one could continue by applying the quadratic formula to the degree 2 factor.

Exercise 0.5.3. As a hint, reference Example 0.5.2! One needs only to swap the roles of input and output, but also to be mindful of the domain/range restrictions. Arccosine outputs values between 0 and π; thus $\arccos\left(-\frac{1}{2}\right) = 2\pi/3$ works just fine. However, we cannot say that $\arcsin\left(\frac{\sqrt{3}}{2}\right) = 2\pi/3$ as well, because it is not in the proper interval. Arcsine outputs values between $-\pi/2$ and $\pi/2$. So, we need an angle $\theta \in [-\pi/2, \pi/2]$ that also satisfies $\sin(\theta) = \frac{\sqrt{3}}{2}$. A visual inspection of the unit circle reveals $\theta = \pi/3$ will do just fine. Thus, $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \pi/3$.

Exercise 1.0.1. Saying that F is an antiderivative of f is equivalent to saying the derivative of F is f. That is, $F'(x) = f(x)$. The Fundamental Theorem of Calculus states that after antidifferentiating the integrand, one can plug the bounds into the antiderivative and take their difference in order to calculate the integral. Because $F'(x) = f(x)$ by the definition of an antiderivative, a good way to check that your antiderivative F is correct is to take its derivative F'. You should get the original function, f.

Exercise 1.1.1. $\int f'(g(x)) \cdot g'(x)\,\mathtt{d}x = \int \left(f\left(g(x)\right)\right)'\,\mathtt{d}x = f\left(g(x)\right) + C$

Exercise 1.1.6. Use the substitutions $u = x^2 + x + 8, \ln(x)$, and $-x^2$. In the last case, the $\mathtt{d}u$ term has nothing to cancel the x with!

Exercise 1.2.4. By factoring out the quantity $(x+1)^{3/2}$, both answers can be brought into the form $(x+1)^{3/2}\left(\frac{2}{5}x - \frac{4}{15}\right) + C$.

Exercise 1.2.6. Use the substitution $u = 1 - x^2$.

Exercise 1.2.7. The antiderivative is $x\ln(x) - x + C$.

Exercise 1.2.12. The antiderivative is $\frac{1}{2}\left(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\right) + C$.

Exercise 1.2.13. Use the substitution $u = \sqrt{x}$ to transform the first integral into $\int 2u\cos(u)\,\mathtt{d}u$.

Exercise 1.2.14. Choosing $u = \ln(x)$ will make the logarithm disappear upon differentiation. The opposite choice will not clean up the log.

Exercise 1.3.2. The antiderivative is $\frac{1}{3}\sin^3(x) + C$.

Exercise 1.3.4. Since seven is odd, when we pulled out one factor of sine, we ended up with the sixth power of sine remaining. Since six is even, we were able to express it as a power of a perfect square of sine, which in turn let us rewrite as cosines using the Pythagorean identity.

Exercise 1.3.5. The first antiderivative is $-\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$. For the second, rewrite as $(1-\sin^2(x))^4\cos(x)$ and proceed by letting $u = \sin(x)$.

Exercise 1.3.6. Often when trying to show that two antiderivatives are compatible, it is easiest to verify that their difference is a constant.

Exercise 1.3.6. The substitution $u = \sin(x)$ is much cleaner since the other will involve having to expand a binomial to the fifth power. The antiderivative is $\frac{1}{12}\sin^{12}(x) - \frac{1}{14}\sin^{14}(x) + C$.

Exercise 1.3.7. The exponent on sine is zero, which is indeed even. Thus both exponents are even in this case.

Exercise 1.3.9. When all like terms are combined and the one-eighth is distributed, the result is $\frac{5}{16}x + \frac{1}{4}\sin(2x) - \frac{1}{48}\sin^3(2x) + \frac{3}{64}\sin(4x) + C$.

Exercise 1.3.10. The antiderivative to $\cos^6(x)$ came out to

$$\frac{5}{16}x + \frac{1}{4}\sin(2x) - \frac{1}{48}\sin^3(2x) + \frac{3}{64}\sin(4x) + C$$

Before we differentiate, first bash everything back down to an "x" in the argument using double angle identities. This produces

$$\frac{5}{16}x + \frac{1}{2}\sin(x)\cos(x) - \frac{1}{6}\sin^3(x)\cos^3(x) + \frac{3}{16}\sin(x)\cos^3(x) - \frac{3}{16}\sin^3(x)\cos(x) + C$$

Factor out a sine and use the Pythagorean Identity to get everything else in terms of cosine. This produces

$$\frac{5}{16}x + \sin(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) + C$$

Then we differentiate and obtain

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \sin^2(x)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

to which we apply the Pythagorean Identity $\sin^2(x) = 1 - \cos^2(x)$ to produce

$$\frac{5}{16} + \cos(x)\left(\frac{5}{16}\cos(x) + \frac{5}{24}\cos^3(x) + \frac{1}{6}\cos^5(x)\right) - \left(1 - \cos^2(x)\right)\left(\frac{5}{16} + \frac{5}{8}\cos^2(x) + \frac{5}{6}\cos^4(x)\right)$$

This will simplify to $\cos^6(x)$ once you expand and combine like terms.

Exercise 1.3.11. For the first, apply the identity $\sin^2(3x) = \frac{1-\cos(6x)}{2}$ and proceed. For the second, notice that $\sin^4(x)$ can be rewritten as $\left(\sin^2(x)\right)^2$, after which the half-angle identity can be applied.

Exercise 1.4.6. First apply all the product and chain rules to reach the expression

$$\frac{3}{\sqrt{1-\frac{x^2}{4}}} + 4\sqrt{1-\frac{x^2}{4}} + \frac{-x^2}{\sqrt{1-\frac{x^2}{4}}} + \sqrt{1-\frac{x^2}{4}}\left(1 - \frac{3}{2}x^2\right) + \frac{-x}{4\sqrt{1-\frac{x^2}{4}}}\left(x - \frac{x^3}{2}\right)$$

Put all terms over the common denominator $\sqrt{4-x^2}$ and combine like terms in the numerator. Notice the numerator becomes $\left(4-x^2\right)^2$ and then reduce for the win!

Exercise 1.4.7. The antiderivative is $2^{18}\left(\frac{\left(1-x^2/16\right)^{9/2}}{9} - \frac{\left(1-x^2/16\right)^{7/2}}{7}\right) + C$

Exercise 1.4.10. Exercise 1.2.12 will be helpful! The antiderivative is $\frac{x\sqrt{x^2-4}}{2} - 2\ln\left|x + \sqrt{x^2-4}\right| + C$.

Exercise 1.4.14. The antiderivative is $-\frac{1}{5}\frac{2x+1}{x^2+x-1} + \frac{4\sqrt{5}}{25}\ln\left(\frac{2x+1+\sqrt{5}}{2\sqrt{x^2+x-1}}\right) + C$. Note that one can expand using properties of logarithms and then rename C as $C - \frac{4\sqrt{5}}{25}\ln(2)$ since it is anyhow just an arbitrary constant. Thus, we can slightly clean up the answer to become $-\frac{1}{5}\frac{2x+1}{x^2+x-1} + \frac{4\sqrt{5}}{25}\ln\left(2x+1+\sqrt{5}\right) - \frac{2\sqrt{5}}{25}\ln\left(x^2+x-1\right) + C$.

Exercise 1.5.7. Using properties of logarithms, both answers should be able to be put in the form $\ln\left|\sqrt{\frac{x-1}{x+1}}\right| + C$

Exercise 1.5.13. •The function $\frac{1}{x^2-9x+20}$ has $\ln\left|\frac{x-5}{x-4}\right| + C$ as its antiderivative. •The factorization $x^4 - 9 = \left(x^2+3\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$ will produce the following setup:

$$\frac{1}{x^4-9} = \frac{Ax+B}{x^2+3} + \frac{C}{x-\sqrt{3}} + \frac{D}{x+\sqrt{3}}$$

in which you can then solve for the coefficients and antidifferentiate. •The function $\frac{x^4}{x^2+1}$ has an irreducible quadratic for a denominator. However, the degree of the numerator is not smaller than the degree of the denominator. Thus, polynomial long division is the only step of PFD that is required in this case. •The antiderivative of $\frac{2}{x^5+2x3+x}$ is

$$2\ln|x| - \ln\left|x^2+1\right| + \frac{1}{x^2+1}$$

•The PFD will produce

$$\frac{x-2}{x^3+x^2+3x-5} = \frac{-\frac{1}{8}}{x-1} + \frac{\frac{1}{8}x+\frac{11}{8}}{x^2+2x+5}$$

While the first term is easy to integrate, the second is quite tricky! To hack through it, split it as follows:

$$\frac{\frac{1}{8}x+\frac{11}{8}}{x^2+2x+5} = \frac{\frac{1}{8}x+\frac{1}{8}}{x^2+2x+5} + \frac{\frac{10}{8}}{x^2+2x+5}$$

The first fraction can then be integrated via u-sub, while the second can be done via trig sub after completing the square on the denominator.

Exercise 1.5.14. For $\frac{1}{x^4-9x^2}$, keep in mind that x^2 is not an irreducible quadratic factor but rather a repeated linear factor. The PFD and integration will produce

$$\frac{1}{9x} + \frac{1}{54}\ln\left|\frac{x-3}{x+3}\right| + C$$

Exercise 1.7.1. $\ln(1+x) + C$

Exercise 1.7.2. $x + -2\sqrt{x} + 2\ln|\sqrt{x}+1| + C$

Exercise 1.7.3. $\ln\left|\frac{2+\sqrt{3}}{\sqrt{2}+1}\right|$

Exercise 1.7.4. $\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{x} + C$

Exercise 1.7.5. •$\int 2x+1\,\mathtt{d}x = x^2+x+C$ Here no u-sub is required. •We apply the substitution $u = 2x+1$, which implies $\mathtt{d}u/\,\mathtt{d}x = 2$ and $\mathtt{d}x = \mathtt{d}u/2$. Thus, $\int \frac{1}{2x+1}\,\mathtt{d}x = \int \frac{1}{u}\frac{\mathtt{d}u}{2} = \frac{1}{2}\int \frac{1}{u}\,\mathtt{d}u = \frac{1}{2}\ln(u)+C = \frac{1}{2}\ln(2x+1)+C$. •The occurrences of e^x inside parentheses motivate the choice $u = e^x$. Thus, $\mathtt{d}u/\,\mathtt{d}x = e^x$ as well and $\mathtt{d}x = \mathtt{d}u/e^x$. The integral becomes $\int e^x\sec(e^x)\tan(e^x)\,\mathtt{d}x = \int e^x\sec(u)\tan(u)\frac{\mathtt{d}u}{e^x} = \int \sec(u)\tan(u)\,\mathtt{d}u = \sec(u)+C = \sec(e^x)+C$. •Though the initial form might motivate the choice of $u = e^x$, upon trying this one finds the integral becomes worse rather than better. Instead, rewrite as a negative exponent and use $u = -x$ so $\mathtt{d}x = -\,\mathtt{d}u$. Thus, $\int \frac{1}{e^x}\,\mathtt{d}x = \int e^{-x}\,\mathtt{d}x = \int e^u(-1)\,\mathtt{d}u = -\int e^u\,\mathtt{d}u = -e^u + C = -e^{-x}+C$. •One can try the inner function $u = 2x$, but that ends up not cleaning things up enough. A more productive choice is $u = \ln(2x)$. That choice produces $\frac{\mathtt{d}u}{\mathtt{d}x} = \frac{1}{2x}\cdot 2 = \frac{1}{x}$. Solving for $\mathtt{d}x$ produces $\mathtt{d}x = x\cdot\mathtt{d}u$. We now substitute into the integral, creating $\int \frac{\cos(\ln(2x))}{x}\,\mathtt{d}x = \int \frac{\cos(u)}{x}x\cdot\mathtt{d}u = \int \cos(u)\cdot\mathtt{d}u = \sin(u)+C = \sin(\ln(2x))+C$. •One can try $u = 2^x$, but it gets tangled up. Instead, try simplifying the integrand using properties of exponents to become 2^{2x}, at which point $u = 2x$ becomes a natural substitution. Taking the derivative produces $\mathtt{d}u/\,\mathtt{d}x = 2$ so $\mathtt{d}x = \mathtt{d}u/2$. Thus, $\int (2^x)^2\,\mathtt{d}x = \int 2^{2x}\,\mathtt{d}x = \int 2^u\,\mathtt{d}u/2 = \frac{1}{2}\int 2^u\,\mathtt{d}u = \frac{1}{2}\frac{1}{\ln(2)}2^u + C = \frac{1}{2\cdot\ln(2)}2^{2x}+C$. •This seems close enough to arctangent if you just glance at it! Let's complete the square on the denominator to get it there. In particular, $2+2x+x^2 = 1+1+2x+x^2 = 1+(1+x)^2$, which motivates the u-sub $u = 1+x$ with change of differential $\mathtt{d}u/\,\mathtt{d}x = 1$ which implies $\mathtt{d}u = \mathtt{d}x$. Therefore, $\int \frac{1}{2+2x+x^2}\,\mathtt{d}x = \int \frac{1}{1+(1+x)^2}\,\mathtt{d}x = \int \frac{1}{1+u^2}\,\mathtt{d}u = \arctan(u)+C = \arctan(1+x)+C$. •This appears so similar to the previous, yet it works out quite differently! The denominator is a perfect square: $(1+x)^2$. Thus, we simply apply u-sub with $u = 1+x$, which has the squeaky clean consequence that $\mathtt{d}u/\,\mathtt{d}x = 1$ so $\mathtt{d}u = \mathtt{d}x$. This produces the integral $\int \frac{1}{1+2x+x^2}\,\mathtt{d}x = \int \frac{1}{(1+x)^2}\,\mathtt{d}x = \int \frac{1}{u^2}\,\mathtt{d}u = \int u^{-2}\,\mathtt{d}u = u^{-1}/(-1)+C = -\frac{1}{1+x}+C$.

Exercise 1.7.6. $-\frac{\cos^{18}x}{18} + \frac{\cos^{16}x}{8} - \frac{\cos^{14}x}{14} + C$

Exercise 1.7.7. The antiderivative is $-\frac{1}{2}(\csc(x)\cot(x) + \ln|\csc(x)+\cot(x)|) + C$

Exercise 1.7.8. $\frac{1}{8}\ln\left|\frac{x-4}{x+4}\right| + C$

Exercise 1.7.9. •$\frac{x^3}{x^3-3x^2+4} = 1 + \frac{-\frac{1}{9}}{x+1} + \frac{\frac{28}{9}}{x-2} + \frac{\frac{8}{3}}{(x-2)^2}$
•$\int 1 + \frac{-\frac{1}{9}}{x+1} + \frac{\frac{28}{9}}{x-2} + \frac{\frac{8}{3}}{(x-2)^2}\,\mathtt{d}x = x - \frac{1}{9}\ln|x+1| + \frac{28}{9}\ln|x-2| - \frac{8}{3}\frac{1}{(x-2)} + C$

Exercise 1.7.10. $\frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$

Exercise 2.1.4. The limits are 0, $1/\pi$, and -1.

Exercise 2.1.7. The limits are 1, -2, and e.

Exercise 2.1.8. The results are 1, 0, and 1.

Exercise 2.1.11. Their ratio converges to 3 (both numerically in the table, and analytically as evaluated by LHR). Since this is a nonzero constant, the two functions have the same growth order.

Exercise 2.1.12. In the first and third, the ratio between f and g seems to grow without bound, so f has larger growth order. In the second, the ratio of f to g seems to always be right around 2. Thus, they have the same growth order.

Exercise 2.1.23. The integrals evaluate to $2\sqrt{2}, \infty, \infty$, and ∞.

Exercise 2.1.27. •The area under xe^{-x^2} from zero to ∞ is $\frac{1}{2}$. •Splitting into two integrals at $x = 0$ produces one of area one-half and one of area negative one-half, so the total integral is zero. •After applying IBP with $u = x$ and $\mathtt{d}v = xe^{-x^2}\,\mathtt{d}x$, one obtains $\frac{\sqrt{\pi}}{2}$ as the area under the curve. •The area under $\frac{1}{x\ln(x)}$ from 2 to ∞ is infinite. •The area under $\frac{1}{x(\ln(x))^2}$ from 2 to ∞ is $\frac{1}{\ln(2)}$. •An improper integral is defined using a limit, and here the limit does not exist, as the area keeps going up and down by the same amount forever.

Exercise 2.2.8. •The curves $y = x^3 + x^2 - x - 1$ and $y = x^3 - x^2 - x + 1$ intersect on the x axis at -1 and 1 and have area 8/3 between them. •The area inside the unit circle but above the line $y = 1/2$ is $\pi/3 - \sqrt{3}/4$. •Notice graphically that the curves intersect at $x = \pm\pi/4$. The area between curves is $\pi/4 - \ln(2)$.

Exercise 2.2.12. The volume estimate with a single cylinder is 2π. To get the heights of the six cylindrical shells, you'll need to use the fact that $x^2 + y^2 = 1$ for every point on the boundary of the circle. With six shells, the volume estimate is $\pi \cdot \frac{\sqrt{35}+12\sqrt{2}+15\sqrt{3}+14\sqrt{5}+9\sqrt{11}}{108} \approx 1.018\pi$. The first is an overestimate, whereas the second is an underestimate.

Exercise 2.2.13. The function $g(x) = \sqrt{1-x^2}$ represents just the QI y-coordinate. It needs to be doubled to represent the height of the shell since the each shell extends the same vertical distance into QIII. Once the integral is evaluated, it will return the exact volume $\frac{4}{3}\pi$.

Exercise 2.3.4. Notice that if you turn the pyramid sideways, you can get the 2D side view to be almost exactly the same as we had for the cone! The volume is $V = \frac{1}{3}r^2h$.

Exercise 2.3.5. The volume is $V = \frac{1}{6}abc$.

Exercise 2.3.7. The circular cross section has equation $x^2 + y^2 = 1$. If you solve for the y coordinate, you'll have a function for the radius of a circular cross section at position x. This formula can be integrated to produce the volume $V = \frac{4}{3}\pi r^3$.

Exercise 2.3.11. The parabolic bowl has volume $\pi/2$ and occupies exactly fifty percent of the cylinder it sits in!

Exercise 2.4.4. The exact arc length is $\frac{2\sqrt{5}+\ln\left|2+\sqrt{5}\right|}{4}$.

Exercise 2.4.8. The length of the graph of the natural logarithm from (1,0) to (e,1) is

$$\sqrt{e^2+1}+\frac{1}{2}\ln\left|\frac{\sqrt{e^2+1}-1}{\sqrt{e^2+1}+1}\right|-\sqrt{2}+\frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

which is roughly 2.003497. Also, notice that the natural exponential function is just the inverse of the natural logarithm; think about what this means regarding arc length!

Exercise 2.4.9. The two-frusta approximation is

$$\frac{\pi}{8}\left(\sqrt{5}+3\sqrt{13}\right)\approx 5.126$$

The exact value of the surface area is

$$\frac{\pi}{6}\left(5\sqrt{5}-1\right)\approx 5.3304$$

which is just slightly larger, as one would expect.

Exercise 2.5.5. The sine gumdrop is just a translation $\pi/2$ units to the right of the cosine gumdrop. So, we would expect the center of mass to have the same y-coordinate but have an x-coordinate that is $\pi/2$ units larger. Indeed, when computed with the moment integrals, we get $(\bar{x},\bar{y})=(\pi/2,\pi/8)$.

Exercise 2.5.7. The diagonals have the equations

$$y=\frac{b}{a}x \text{ and } y=\frac{b-2c}{a}x+c$$

with intersection point $(a/2,b/2)$, which is also the center of mass of the region.

Exercise 2.5.9. The coordinates of the vertices are (0,0), (0,c), and (a,b). Two of the medians are

$$y=\frac{b+c}{a}x \text{ and } y=\frac{2b-c}{2a}x+\frac{c}{2}$$

and their intersection point (and center of mass of the triangle) is $\left(\frac{a}{3},\frac{b+c}{3}\right)$.

Exercise 2.5.10. The center of mass is $\left(0,\frac{4r}{3\pi}\right)$.

Exercise 2.6.10. The force function is $F(y)=1800-4y$.

Exercise 2.6.11. 225,000 foot-pounds of work

Exercise 2.6.12. 1470 Joules

Exercise 2.6.15. The distance function $D(y)$ is still $4-y$, but the area function will now be $A(y)=4\cdot 2\sqrt{4-y^2}$. The total work is 1,576,330 Newton-meters, or just over 1.5 million Joules. This is roughly three times the work involved in draining the corresponding spherical tank, which makes sense because one can properly fit the spherical tank inside the cylindrical, so the answer should be substantially larger for the cylinder.

Exercise 2.6.17. The equation produces $c=-6\sqrt{2}$. Thus, they should switch after Rho has dug roughly 8.485 feet down, and Arg can take the rest.

Exercise 2.6.18. • No effect, same switching point. • Switch at a depth of $6\sqrt[3]{4}$ or roughly 9.5

feet. • Rho should dig the first 3 feet, Arg digs the next 6, and then Rho digs the last 3.

Exercise 2.6.21. The total force exerted is 156800π N, or just under a half million Newtons.

Exercise 2.6.22. •627000 Newtons • 117600π Newtons

Exercise 2.8.1. a. $e^x >> x^2$ b. $e^x >> x^3$ c. $e^x >> x^4$ d. The above calculations demonstrate that if you compare $p(x)$ to e^x you will have n iterations of LHR resulting in $\lim_{x\to\infty} \frac{p(x)}{e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = 0$ Thus e^x has larger growth order than any polynomial.

Exercise 2.8.2. $V = \sqrt{3}$

Exercise 2.8.3. $V = 3\sqrt{3}\pi + \frac{8\pi^2}{3} \approx 42.643$

Exercise 2.8.4. The area is infinite!

Exercise 2.8.5. $V = \frac{4}{3}\pi r^3$

Exercise 2.8.6. $V = \int_{-r}^{r} \pi(r^2 - x^2)dx = \frac{4}{3}\pi r^3$

Exercise 2.8.7. $\bar{x} = \frac{3}{2}, \bar{y} = \frac{18}{5}$

Exercise 2.8.8. a.) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$
b.) $\lim_{x\to 0^+} \frac{\ln x}{\sqrt{x}} = -\infty$
d.) $\int_0^1 \frac{\ln x}{\sqrt{x}} dx = -4$

Exercise 2.8.9. 4

Exercise 2.8.10. 1. Cylindrical shells takes line segments and revolves them about an axis parallel to those segments in order to find the volume of the solid created by that revolution. Cross sections cuts a 3D object into 2D parallel slices and does not require revolution.
2. You cannot use shells because it is impossible to obtain a tetrahedron via revolution since it has no rotational symmetry. So we use cross sections.
3. The base and height at location x are both given by the function $1 - x$. Thus the volume is $\int_0^1 \frac{1}{2}(1-x)^2 dx = \frac{1}{6}$

Exercise 2.8.11. $\left(\frac{2}{3}, \frac{1}{3}\right)$

Exercise 2.8.12. Roughly 11.9 million Joules are required.

Exercise 3.1.11. •$\frac{1}{n+1}$ •$n+1$ •$(n+2)(n+1)$ •$(2n+2)(2n+1)$

Exercise 3.1.18. Think about what happens if the common difference d is zero and if the common ratio r is 1.

Exercise 3.1.21. The common ratio r is what we multiply by to get from term to term. Listing out the terms $a_0, a_0r, a_0r^2, a_0r^3, \cdots$ shows that a_0r^n is the explicit formula.

Exercise 3.2.5. In the context of computing a limit to infinity, it is fine to replace $n!$ by $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Setting up limits of ratios and testing growth order with LHR and good old algebra will then verify that the order goes $n^2, e^n, n!, n^n$.

Exercise 3.3.4. The totals are 12, -6, and 15.

Exercise 3.3.5. It is easiest to just expand the sums on both sides and see what the terms look like. For example, in the first case the left-hand side is $(ca_j + ca_{j+1} + \cdots + ca_k)$, whereas the right-hand side is $c\,(a_j + a_{j+1} + \cdots + a_k)$. These two expressions are equal, because we can factor the c out of the left-hand side to produce the right-hand side. For the last two summations, think about our discussion of fencepost problems above!

Exercise 3.3.9. In Gauss's formula, the first term a_0 and the common difference d are both 1. The number of terms is N. Plugging these into the Arithmetic Series Formula will produce $N(N+1)/2$.

Exercise 3.3.11. The totals are 500500, 1501500, 214214, and 245.

Exercise 3.3.14. The common ratio $r = 10$. The first term is 1. The number of terms is 6. Putting this all together in the Geometric Series Formula produces $1 \cdot \frac{1-10^6}{1-10} = \frac{-99999}{-9} = 11111$.

Exercise 3.3.15. A finite sum of consecutive powers of two, starting at one, is equal to one less than the next power of two.

Exercise 3.3.18. Sure! If you further factor $A^2 - B^2$ via difference of two squares and further factor $A^3 + A^2B + AB^2 + B^3$ via grouping, you will end up with the same factorizations.

Exercise 3.3.21. \$5 billion $\cdot \frac{1-0.8^{13}}{1-0.8} \approx$ \$23.6 billion

Exercise 3.3.22. Try to notice how the summations relate to the very next Fibonacci number, the first one not being summed.

Exercise 3.4.3. In this series, there are $N+1$ terms, the first of which is 1 and the last of which is $2N+1$. Plugging this information into the Arithmetic Series Formula will produce the desired result.

Exercise 3.4.4. The partial sum A_N represents the total number of push-ups you've done so far in your push-up routine, up to and including day N.

Exercise 3.4.8. The corresponding partial sums are as follows: $\bullet A_N = 10\left(1 - 1/2^{N+1}\right)$ $\bullet A_N = 3/4\left(1 - 1/9^{N+1}\right)$ $\bullet A_N = (10-N)\,(N+1)\,/2$ $\bullet A_N = (N+1)^3$ $\bullet A_N = 1, 0, 1, 0, 1, 0, \ldots = \left(1 - (-1)^{N+1}\right)/2$ $\bullet A_N = 1$

Exercise 3.5.2. Think about an integral of the form $\int_{x=0}^{x=\infty} f(x)\,\mathrm{d}x$. How does one handle that infinity in the bounds?

Exercise 3.5.4. The sequence is $a_n = \frac{1}{2^{n+1}}$. Since this is a geometric sequence, the finite geometric series formula can be applied to then find the sequence of partial sums A_N.

Exercise 3.5.5. It ends up one-third of a meter forward from where it started.

Exercise 3.5.9. Yes, the series is geometric with initial term $\frac{3^5}{2^{11}}$ and common ratio 3/4. The infinite series totals to $\frac{3^5}{2^9}$.

Exercise 3.5.10. Think about what the value of r would be for that series. What restrictions did we have on r in the statement of the infinite geometric series formula?

Exercise 3.5.11. $1 + 2\frac{5}{8} + 2\left(\frac{5}{8}\right)^2 + 2\left(\frac{5}{8}\right)^3 + \cdots = 1 + 2\frac{5/8}{1-5/8} = 1 + 2\frac{5/8}{3/8} = 13/3 = 4.\overline{3}$ meters.

Exercise 3.5.12. The partial sums are $A_N = 2(N+1)$. The infinite series is the limit of A_N as N goes to infinity, which here is clearly again infinity. Thus, the infinite series diverges.

Exercise 3.5.13. The partial sums are

$$A_N = \frac{N+1}{N+3}$$

for an infinite sum of 1.

Exercise 3.5.14. The infinite series $\Sigma_{n=0}^{\infty} a_n$ are •Divergent •Divergent •3 •$\frac{2}{3}$ •Divergent •$\frac{9}{4}$ •Divergent •1

Exercise 3.5.21. It is absolutely convergent, since the series of corresponding positive terms is $0.1 + 0.02 + 0.004 + 0.0008 + 0.00016 + \cdots$ which converges to one-eighth.

Exercise 3.7.5. Since the sequence $1/n$ approaches zero as n goes to infinity, the No Hope Test tells you nothing.

Exercise 3.7.6. The first two summands have limits of $\sqrt{2}$ and 1, respectively. Since these limits are nonzero, the series has no hope of converging and thus diverges. The third summand does approach zero as n goes to infinity, so it gives no information.

Exercise 3.7.8. • Why are you still in your chair?

Exercise 3.7.10. Taking term-by-term absolute values produces the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. Since it totals to a finite value, the original series converges absolutely.

Exercise 3.7.12. The first two and last converge by AST. It does not apply to the third.

Exercise 3.7.15. After three reversals, the bug is within $\frac{1}{32}$ of its final location. The bug would have to reverse course eight times to be guaranteed by the Alternating Series Error Bound to be within one one-thousandth of its final location.

Exercise 3.8.8. For $p > 1$ or $p < 1$, one can repeat the corresponding calculations from Example 3.8.3. If $p = 1$, the series diverges, because the corresponding integral has the natural log as its antiderivative, which goes to infinity as the input gets arbitrarily large.

Exercise 3.9.3. The first one-half would come from the first term itself. But since the total is one, it means the terms $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ must themselves total to be the other one-half. Thus, we can try to group terms to form batches that total to one-half, but the second batch uses up all infinitely many remaining terms!

Exercise 3.9.7. These series are convergent by DCT against $\frac{1}{n^2}$, divergent by DCT against $\frac{1}{n}$, and convergent by DCT against $\frac{1}{n^2}$.

Exercise 3.9.10. The first converges by comparison to $\Sigma\frac{1}{n^2}$. The second diverges by comparison to $\Sigma\frac{1}{n^{-1/2}}$.

Exercise 3.9.13. Use the comparison function $\frac{1}{n}$ to show the series diverges.

Exercise 3.9.16. Try p-series for two different values of p, one value less than 1 and one value greater than 1.

Exercise 3.9.18. •Converges, ratio 0. •No info, ratio 1. •Converges, ratio 0. •Diverges, ratio 2.

Exercise 3.9.23. The ratio test gives a ratio of 1, and thus no information is obtained. This is why the root test is occasionally stronger than the ratio test!

Exercise 3.9.24. The Root Test provides answers of diverge, diverge, converge, converge, converge, and no info, respectively.

Exercise 3.10.1. •Divergent by NHT or Integral Test. •Divergent by Integral Test or LCT against $\frac{1}{n}$. •Convergent by AST, but only conditionally since the absolute value is the previous summand whose series diverged. •Absolutely convergent since taking term-by-term absolute value produces a convergent series (which can be shown convergent via LCT with $\frac{1}{n^3}$).

Exercise 3.10.2. The black region is one-third of the total square and thus must total to one-third. The infinite series for the black square areas is $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{128}+\cdots$. NHT and AST give no information here, but all the rest of the tests work to determine convergence! Use $\frac{1}{n^2}$ as a comparison function for LCT.

Exercise 3.11.7. It is absolutely convergent, since the series of corresponding positive terms is $0.1+0.02+0.004+0.0008+0.00016+\cdots$ which converges to one-eighth.

Exercise 3.13.1. a. For $\epsilon=0.1, N=5$, $\epsilon=0.01, N=50$, $\epsilon=0.001, N=500$ b. $\frac{1}{2\epsilon}=N$ c. Let $\epsilon>0$ also, let $N=\frac{1}{2n}$ and let $n\in\mathbb{N}$ Assume $n>N$ We wish to show that under these circumstances, the distance from $a_n=\frac{1}{2n}$ to $L=0$ will be less than ϵ. $|a_n-L|=\left|\frac{1}{2n}-0\right|=\frac{1}{2n}<\frac{1}{2N}$ since $N<n$ by our assumptions $\frac{1}{2N}=\frac{1}{2\frac{1}{2\epsilon}}=\frac{1}{\epsilon}=\epsilon$ Thus the terms will be within ϵ or 0 past the index $\frac{1}{2\epsilon}$, no matter how small ϵ is chosen. Therefore, $\lim\limits_{n\to\infty}\frac{1}{2n}=0$

Exercise 3.13.2. $3,375,000$

Exercise 3.13.3. a.) 6 Slices, b.) 4.5 Slices, c.)$\left(\frac{3}{4}\right)^{20}8$ Slices, d.) None because as time goes to ∞ the number of slices left goes to 0, since $\left(\frac{3}{4}\right)^n 8$ represents the number of slices left after n minutes and $\lim\limits_{n\to\infty}\left(\frac{3}{4}\right)^n 8=0$ slices.

Exercise 3.13.4. a.) Converges Absolutely by the Ratio Test.
b.) Diverges by the No Hope Test.
c.) Diverges by the No Hope Test.
d.) Converges by the Limit Comparison Test and the p-test.
e.) Diverges by the No Hope Test
f.) Converges Absolutely by the Ratio Test.
g.)Converges Absolutely by the Integral Test or the Ratio Test.

Exercise 3.13.5. Notice a_n is a geometric sequence with initial term $a_0=2$ and the ratio $r=\frac{1}{2}$. We can then use the infinite geometric series formula $\Sigma_{n=0}^{\infty}a_n=\frac{a}{1-r}$ so we have $\Sigma_{n=0}^{\infty}2\cdot\left(\frac{1}{2}\right)^n=2\cdot\frac{1}{1-\frac{1}{2}}=2\frac{1}{\frac{1}{2}}=2\cdot 2=4$. Thus, the sequence of partial sums converges to 4.

Exercise 3.13.6. a.) $\lim\limits_{n\to\infty}a_n=L\iff$ For all $\epsilon>0$ there exists N such that for all $n>N, |a_n-L|<\epsilon$
b.) $\frac{1}{3}$

c.) Let $\epsilon > 0$ then choose $N = \frac{1}{9n} - \frac{1}{3}$ Then choose $n \in \mathbb{N}$ with $n > N$. We now show that any a_n for such n is no more than ϵ away from $\frac{1}{3}$.
$\left|\frac{n}{3n+1} - \frac{1}{3}\right| = \left|\frac{3n}{3(3n+1)} - \frac{3n+1}{3(3n+1)}\right| = \left|\frac{-1}{3(3n+1)}\right| = \frac{1}{3(3n+1)} < \frac{1}{3(3N+1)}$
note here we made the denominator smaller by introducing N for n
$\frac{1}{3(3N+1)} = \frac{1}{3\left(3\left(\frac{1}{9\epsilon}-\frac{1}{3}\right)+1\right)} = \frac{1}{3\left(\left(\frac{1}{3\epsilon}-1\right)+1\right)} = \frac{1}{3\left(\frac{1}{3\epsilon}\right)} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$

Exercise 3.13.7. a.) $\lim_{n\to\infty} (-1)^n \frac{1}{n!} = 0$ So the no hope test gives no information since it requires the limit to be not equal to zero.
b.)It converges absolutely by the Alternating Series Test.
c.)It converges absolutely by the Ratio Test

Exercise 3.13.8. a.)$a_0 = 0, a_1 = 1, a_2 = 1 + 3 \cdot 2^2 - 3 \cdot 2 + 1 = 8, a_3 = 8 + 3 \cdot 3^2 - 3 \cdot 3 + 1 = 27, a_4 = 1 + 3 \cdot 4^2 - 3 \cdot 4 + 1 = 64$
b.)$a_n = n^3$
c.)It diverges by the No Hope Test

Exercise 3.13.9. •The summation $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$ has no common ratio r and thus is not a geometric series. For example, the first three terms are $1, 1/4$, and $1/9$. Thus, the first two ratios between consecutive terms are $1/4$ and $4/9$, which are not equal. •The given geometric series has common ratio $r = -1/3$. After taking the absolute value of each term, it becomes the series $18 + 6 + 2 + \frac{2}{3} + \frac{2}{9} + \cdots$ which still converges as it now has common ratio $r = 1/3$. •It is not possible to build a conditionally convergent geometric series. If we are given a convergent geometric series, then the common ratio r satisfies $|r| < 1$. Taking the absolute value of each term in the series might flip the sign on r, but it will not change the magnitude. Thus, any convergent geometric series must converge absolutely.

Exercise 3.13.10. •Since $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$, the No Hope Test gives no information. •The Integral Test does not apply since the terms are not positive and decreasing. In this case, it is actually even worse than that, as the function $\left(-\frac{1}{2}\right)^x$ is undefined for all half-integer values of x. •The summand is not of the form $1/n^p$, so the very narrow p-Test does not apply.

Exercise 4.1.4. Written in sigma notation, the power series is $\sin(x) = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!} x^{2n+1}$.

Exercise 4.1.5. Written in sigma notation, the power series is $e^x = \Sigma_{n=0}^{\infty} \frac{1}{n!} x^n$.

Exercise 4.1.8. The power series is $\frac{1}{1-x} = \Sigma_{n=0}^{\infty} x^n$. It is a geometric series with initial term $a = 1$ and common ratio $r = x$.

Exercise 4.1.9. When we try to plug in $x = 0$ to find a_0, we get $\ln(0)$ which is not a real number.

Exercise 4.1.12. The power series centered at one for the natural log is $\ln(x) = \Sigma_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$.

Exercise 4.3.3. If we substitute $x - 1$ for x in the power series for sine, we get $\sin(x-1) = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}(x-1)^{2n+1}$. Likewise, substituting $2x$ for x in the power series for sine produces $\sin(2x) = \Sigma_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}(2x)^{2n+1} = \Sigma_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}$.

Exercise 4.3.8. The power series $\frac{1}{x} = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}}(x-5)^n$ has IOC (0,10). It turns out these two examples generalize; for rational functions, the IOC will always just be the interval that goes from the center of the series outwards until it bumps into the nearest vertical asymptote!

Exercise 4.3.12. The answer will have very similar form but involve only odd degree terms instead of

even!

Exercise 4.3.22. The power series $\frac{1}{x^2-x-12} = \Sigma_{n=0}^{\infty}\left(\frac{-1}{21\cdot(-3)^n} - \frac{1}{28\cdot 4^n}\right)x^n$ has IOC (-3,3).

Exercise 4.3.24. Antidifferentiate the geometric series to sneak up on $\ln(1-x)$.

Exercise 4.3.26. Each method should lead to

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

Exercise 4.5.2. If $n = 1$, we have the following degree one power series centered at $a = 4$:

$$f(x) = \sqrt{x} \approx 2 + \frac{1}{4}(x-4)$$

Since $n = 1$, we need the second derivative. We compute $|f''(x)| = \frac{1}{4x^{3/2}}$, which on the interval [4,4.1] has its maximum at $x = 4$. Thus, $M = 1/32$, which provides an error bound of

$$\frac{\frac{1}{32}\cdot|4-4.1|^2}{2!} = \frac{1}{6400}$$

Thus the error is definitely less than one thousandth, but not necessarily less than one ten thousandth. So for the approximation

$$\sqrt{4.1} \approx 2 + \frac{1}{4}(4.1-4) = 2.025$$

we can guarantee three digits past the decimal are correct but not necessarily the fourth. That is, 2.025 is certainly the correct decimal expansion for $\sqrt{4.1}$ rounded to the thousandths place. However, the digit 0 we implicitly have in the ten-thousandths place may or may not be correct. This process can be repeated for the other n values of two and three.

Exercise 4.6.5. Substitute $-x$ into the power series for cosine and simplify to demonstrate that $\cos(-x) = \cos(x)$, and similarly for sine.

Exercise 4.7.6. Choices of comparison functions and conclusions are as follows: $\bullet\frac{1}{n}$, diverges $\bullet\frac{1}{n^2}$, converges $\bullet\frac{1}{n}$, diverges.

Exercise 4.7.7. If the expression $\frac{f(x)}{g(x)}$ is indeterminate of the form $\frac{\infty}{\infty}$, we can trade it out for the expression $\frac{1/g(x)}{1/f(x)}$, which brings us back to the $\frac{0}{0}$ case.

Exercise 4.8.7. The ratio between consecutive terms is $\frac{1+\sqrt{5}}{2}$, the Golden Ratio.

Exercise 4.8.8. The IOC is $\left(\frac{1-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right)$. This is the interval that proceeds symmetrically left and right from the origin as far as it can until it runs into the nearest vertical asymptote of $\frac{x}{1-x-x^2}$.

Exercise 4.9.5. $\bullet\frac{1}{2-x}$ $\bullet\sqrt{x}$ $\bullet-\ln(2-x)$ $\bullet\frac{1}{x}$ $\bullet\frac{e^{x-1}-1}{x-1}$

Exercise 4.12.2. 1.4

Exercise 4.12.3. a.) $\sqrt{e}$, b.) $\sqrt{2}$, c.) $\frac{20}{99}$, d.) $5e^5$

Exercise 4.12.4. a. 2 b. $\lim_{x\to 0}\frac{x^2}{1+\frac{1}{2}x^2} = 2$

Exercise 4.12.5. a.) $y^2 - x^2 = 1 \to y^2 = \pm\sqrt{1+x^2}$, b.) $\lim_{x\to\infty} \frac{f(x)}{x} = 1$ and $\lim_{x\to\infty} \frac{f(x)}{-x} = -1$ c.) $1+1/2x^2$ a Parabola with vertex at (0,1) and squished by 1/2.

Exercise 4.12.6. a. Notice that the function $f'(x)$ has a substantially cleaner power series than $f(x)$. Find a closed form for $f'(x)$ instead, and then integrate both sides and solve for C. This process should produce $f(x) = 1 - \frac{1}{2}\ln(1+x^2)$ b. $[-1, 1]$

Exercise 5.1.2. This parametric curve is the line $y = \frac{3}{2}x + 1$.

Exercise 5.1.3. The two curves are the same points in the plane. Both start at the point (1,0) at time $t = 0$, but C_1 then proceeds counter-clockwise while C_2 proceeds clockwise.

Exercise 5.3.4. The arc length is

$$\frac{6\sqrt{146} + \ln\left(\sqrt{73} + 6\sqrt{2}\right)}{6} \approx 12.55.$$

Also, to handle the absolute value, just find the arc length on the interval [0,2] where you can ignore the absolute value and then apply symmetry.

Exercise 5.3.5. The arc length is $\sqrt{2}\left(e^{2\pi} - 1\right)$.

Exercise 5.5.14. Yes, it is in fact a circle with cartesian center $(0, 1/2)$ and radius 1/2. This can be verified by demonstrating the polar equation converts to the cartesian equation

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

Exercise 5.6.2. The derivative is always undefined (in this case corresponding to infinity); thus, the graph is a straight vertical line!

Exercise 5.7.4. The area between the curves is $\frac{\pi}{8}$.

Exercise 5.7.5. The area inside the inner loop of $r(\theta) = \frac{1}{2} + \cos(\theta)$ is $\frac{\pi}{4} - \frac{3\sqrt{3}}{8}$.

Exercise 5.10.1. a. $r(0) = 4$, $r(\pi/6) = 8/\sqrt{3}$, $r(\pi/4) = 8/\sqrt{2} = 4\sqrt{2}$, $r(\pi/3) = 8$ b. $r = 4\sec\theta \Longrightarrow r\cos\theta = 4 \Longrightarrow x = 4$ is a vertical line. c. d. It is an isosceles triangle with hypotenuse $4\sqrt{2}$ and sides 4 $A = 8$ e. $A = 8$ They are the same.

Exercise 5.10.2. a.) It is a line segment that lies on the line $\frac{x-1}{4} = t = \frac{y}{6} \leftrightarrow y = \frac{3}{2}x + \frac{3}{2}$ between $(-1, 0)$ and $(7, 12)$
b.) $\frac{3}{2}$
c.) $4\sqrt{13}$

Exercise 5.10.3. $\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots$
$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$
a. $\frac{d\ \sinh(t)}{dt} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \cosh(t)$
b. $\frac{d\ \cosh(t)}{dt} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots = \sinh(t)$
c. $\cosh^2 t - \sinh^2 t = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right)^2 - \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)^2 = 1$

d. $\frac{1}{0}$ which is a vertical line

Exercise 5.10.4. b.) $\frac{\pi}{8}$

Exercise 6.1.12. $\bullet\ln(2) = \ln(2) + 0i$ $\bullet\ln(-2) = \ln(2) + \pi i$ $\bullet\ln(1+i) = \ln\left(\sqrt{2}\right) + i\frac{\pi}{4}$ $\bullet\ln(3-4i) = \ln(5) + i\arctan\left(-\frac{4}{3}\right)$

Exercise 6.1.14. The number $(1+i)^{1+i}$ can be written in complex cartesian form as

$$\left(e^{\ln\left(\sqrt{2}\right)-\frac{\pi}{4}}\cos\left(\ln\left(\sqrt{2}\right)+\frac{\pi}{4}\right)\right) + i\left(e^{\ln\left(\sqrt{2}\right)-\frac{\pi}{4}}\sin\left(\ln\left(\sqrt{2}\right)+\frac{\pi}{4}\right)\right).$$

Exercise 6.2.1. Even means that $f(x) = f(-x)$ for all x in the domain. Odd means that $f(x) = -f(-x)$ for all x in the domain.

Exercise 6.2.2. Even means that a reflection across the y-axis leaves the graph unchanged. Odd means a reflection about the y-axis followed by a reflection through the x-axis leaves the graph unchanged.

Exercise 6.2.3. The functions are odd, even, neither, odd, neither, even, odd, even, and odd, in that order.

Exercise 6.2.4. Even functions only have even degree terms in their power series, i.e., the odd degree coefficients are all zero for an even function. The same is true for an odd function, mutatis mutandis.

Exercise 6.2.7. In the first example, $f_o(x) = 2x$ and $f_e(x) = 3 - x^2$. For the second, the even part is zero and the odd is just $f(x)$, since the given function was already odd.

Exercise 6.2.8. If a function $f(x)$ has a power series representation centered at zero, then the odd part $f_o(x)$ is just the sum of all the odd degree terms in the power series, and the even part $f_e(x)$ is just the sum of all the even terms.

Exercise 6.3.1. The PFD over the complex numbers is

$$\frac{4-2x^2}{x^3+4x} = \frac{1}{x} - \frac{\frac{3}{2}}{x+2i} - \frac{\frac{3}{2}}{x-2i}.$$

Exercise 6.3.2. Use the fact that arctangent has a horizontal asymptote at $\pi/2$ as well as the fact that $\ln(-1) = i\pi$ to conclude that $C = 0$.

Exercise 6.5.1. $z = e^{i(-\pi/4)}, e^{i(3\pi/4)} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

Exercise 6.5.2. a.) $-i$
b.) $e^{-\pi}$
c.) -1
d.) $\frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} + \frac{i}{2}, -i$
e.) $\ln 7 + i\arctan\sqrt{3}/12$

Exercise 6.5.3. b. $r\cos(\theta) + ri\sin(\theta) = re^{i\theta}$ This shows that r and θ represent a radius and an angle.

Exercise 7.2.4. Any solution to $\frac{dy}{dx} = xy + x$ can be written as $y = Ce^{\frac{x^2}{2}} - 1$ for some real number C. The second DE with initial condition has the solution

$$\frac{1}{2}e^{-y}\left(\sin(y) - \cos(y)\right) = -e^{-x}\left(3 + 2x + x^2\right) + \frac{5}{2}.$$

Exercise 7.6.1. Linear combinations of hyperbolic sine and hyperbolic cosine functions are the only functions that equal their own second derivatives.

Exercise 7.6.2. a. $y = 3 + x + (a_0 - 3)e^x$ b. If $y = 3 + x + (a_0 - 3)e^x$ then $\frac{dy}{dx} = 1 + (a_0 - 3)e^x$ but $y - x - 2 = 3 + x + (a_0 - 3)e^x - x - 2 = 1 + (a_0 - 3)e^x$ So they match.

Exercise 7.6.3. a.) The right-hand side $2y + x$ does not factor into a function of y times a function of x so there can be no separation of variables.
b.) $y = -\frac{1}{4} - \frac{1}{2}x + Ce^{2x}$

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