Calculus I

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FRONT RANGE COMMUNITY COLLEGE

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Contents

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We hope you have fun with this resource and find it helpful! It is a beautiful subject, and we tried to honor that with a beautiful text. The text is still in its adolescence, and we welcome any and all feedback you could give us. Thank you in advance for any comments, complaints, suggestions, and questions.

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Chapter 0

Overview

0.1 A Three-Part Course

The topics of Calculus I fall into three parts that each have an appropriate place in the story of the calculus sequence.

- **Part I: Limits and Continuity.** The idea of the **limit** is the conceptual framework we will use to build both derivatives and integrals. Intuitively, it is a value that a function becomes arbitrarily close to, whether or not it ever reaches it. We will analyze limits graphically, analytically, and algebraically. We will use limits to define **continuity**, which intuitively just means "no holes or gaps" but ends up being quite tricky to define carefully!
- **Part II: Derivatives.** The rise-over-run slope computation calculates the average rate of change of a quantity. The **derivative** is an extension of this idea, computing the **instantaneous rate of change** of the quantity. We essentially still just perform a rise-over-run calculation, but take the limit as the "run" goes to zero. This is one of the most widely used concepts in mathematics, physics, economics, computer science, and engineering.
- **Part III: Integrals.** Up until this point, the only shape you can really find the area of is a rectangle (or shapes easily constructed from it, like triangles). Even the area of a circle formula has secretly been accepted on faith and not justified. Here we construct the **Riemann Integral**, a general framework for finding the area under any continuous function. It is calculated by taking **sums of areas of rectangles** and then taking the limit as the width of these rectangles goes to zero.

0.2 How to Use This Book

This book is meant to facilitate *Active Learning* for students, instructors, and learning assistants. Active Learning is the process by which the student participates directly in the learning process by reading, writing, and interacting with peers. This contrasts the traditional model where the student passively listens to lecture while taking notes. This book is designed as a self-guided step-by-step exploration of the concepts. The text incorporates theory and examples together in order to lead the student to discovering new results while still being able to relate back to familiar topics of mathematics. Much of this text can be done independently by the student for class preparation. During class sessions, the instructor and/or learning assistants may find it advantageous to encourage group work while being available to assist students, and work with students on a one-on-one or small group basis. This is highly desirable as extensive research has shown that active learning improves student success and retention. (For example, see `www.pnas.org/content/111/23/8410` for Scott Freeman's metaanalysis of 225 studies supporting this claim.)

What is Different about this Book

If you leaf through the text, you'll quickly notice two major structural differences from many traditional calculus books:

1. The exercises are very intermingled with the readings. Gone is the traditional separation into "section" versus "exercises". (The exception here is the Mixed Practice section at the end of each chapter, placed for the convenience of the student looking for more practice problems.)

2. Whitespace was included for the student to write and work through exercises. Parts of pages have indeed been intentionally left blank.

A consequence of this structure is that the readings and exercise are closely linked. It is intended for the student to do the readings and exercises sequentially. If the student jumps into an exercise without having read what preceded it, confusion is likely to follow.

Ok... Why?

The goal of this structure is to help the student simulate the process by which a mathematician reads mathematics. When a mathematician reads a paper or book, he/she always has a pen in hand and is constantly working out little examples alongside and scribbling incomprehensible notes in bad handwriting. It takes a *long* time and a lot of experience to know how to come up with good questions to ask oneself or to know what examples to work out in order to to help oneself absorb the subject. Hence, the exercises sprinkled throughout the readings are meant to mimic the margin scribbles or side work a mathematician engages in during the act of reading mathematics.

The Legend of Coffee

A potential hazard of this self-guided approach is that while most examples are meant to be simple exercises to help with absorption of the topics, there are some examples that students may find quite difficult. To prevent students from spinning their wheels in frustration, we have labeled the difficulty of all exercises using coffee cups as follows:

Coffee Cup Legend

Symbol	Number of Cups	Description of Difficulty
☕	A One-Cup Problem	Easy warm-up suitable for class prep.
☕☕	A Two-Cup Problem	Slightly harder, solid groupwork exercise.
☕☕☕	A Three-Cup Problem	Substantial problem requiring significant effort.
☕☕☕☕	A Four-Cup Problem	Difficult problem requiring effort and creativity!

0.3 Prerequisites in the Language of Mathematics

Types of Expressions

In Precalculus, there is a wide range of how much focus there is on the language of mathematics itself as opposed to calculation. To get everyone on the same page, here is a short list of some language, symbols, and ideas that we will use in this text.

Quantities, Functions, and Statements

Any valid collection of symbols in mathematics is called an *expression*. Every expression has a **type** which tells you what kind of expression that is. The most common types of expressions we will use in this course are the following:

- **Quantity.** Anything that represents a numerical value is a *quantity*. The following are some examples of expressions of type quantity:
  (i) 28
  (ii) $-\pi$
  (iii) The real root of the polynomial $x^3 - x + 10$.
- **Function.** A well-defined rule for mapping inputs to outputs is called a *function*. (See your precalculus text for a much more detailed and precise definition.) Here are some examples of expressions whose type would usually be interpreted as function:
  (i) cosine
  (ii) $f(x) = x^2$
  (iii) $f''(x)$
- **Statement.** Any expression that could be considered true or false is called a *statement*. You do not need to be certain which it is, just that it is possible to be true or false. Here are some examples of statements:
  (i) The number 28 is larger than the number 6.
  (ii) The number 28 is smaller than the number 6.
  (iii) The number 28 is smaller than my favorite number.

Note that all three of the above are perfectly good statements, even though the second and third may sound a bit odd! The first statement is true, the second statement is false, and the third statement is impossible to determine because you do not know my favorite number. But, it is a perfectly valid statement since it is either true or false.

If you have a background in computer programming, the above discussion of types should feel somewhat familiar; many programming languages require that one declares a data type when declaring a variable. The first type, quantity, is usually represented by something like `int` or `float` depending on what you want to use it for. The second type, function, usually corresponds to declaring a method or a subroutine. The third type, an expression which is true or false, is often called a `boolean`.

Also, be aware that our most common notation for functions, in which we write something like "$f(x) =$ formula" can easily be mistaken for a statement, since you could interpret the equals sign to be asking whether or not those two expressions are equal as opposed to creating an assignment of input to output. In programming languages, they often distinguish the different contexts by using a single equals to mean assignment and a double equals sign to mean a statement in which you are testing the equality of two expressions. It is extremely common in mathematics to use the same symbol for both meanings; we stick with this convention and will rely on context to interpret which is meant when.

Exercise 0.3.1. Types of Objects ☕

Each of the following objects is either a quantity, a function, or a statement. Identify which is which!

- $\cos(x)$
- $\cos(\pi)$
- $\cos(\pi) = 0$

Be aware that we often identify a quantity with the corresponding constant function. That is, 3 is a quantity, but it often is useful to think of it as the constant function $f(x) = 3$.

Sets and Elements

There is another very important (and somewhat more complicated) expression type we will frequently use in this course: the type *set*. A *set* is just a collection of objects. Amazingly, this simple idea is often used as the foundation of all of modern mathematics! Here is some notation.

- If an object x is in a set A, we say x is an *element of* A and write $x \in A$.
- If an object x is not in a set A, we say x is *not an element of* A and write $x \notin A$.

Any particular object is either an element of a given set or it is not. We do not allow for an object to be partially contained in a set, nor do we allow for an object to appear multiple times in the same set. Often we use curly braces around a comma-separated list to indicate what the elements are.

Example 0.3.2. A Prime Example

Suppose P is the set of all prime numbers. We write

$$P = \{2, 3, 5, 7, 11, 13, 17, \ldots\}.$$

For example, $2 \in P$ and $65,537 \in P$, but $4 \notin P$.

Invalid Expressions

Be aware that it is easy to write down expressions which do not have a valid type. In fact, most collections of symbols have no meaning in the language of mathematics, much as if you typed a random string of letters, it would be very unlikely to spell a valid word in the English language. We call these expressions *garbage* (and can be thought of as the equivalent of a compiler error in programming).

Example 0.3.3. Garbage

The expression

$$2 \in 3$$

might look like a statement. However, it is not. The relation $\in$ expects a set on the right, however we handed it a quantity. Thus, we did not assemble our types of objects into a valid expression. Thus the above expression is neither true nor false, but simply garbage. Nobody likes garbage.

Some Famous Sets of Numbers

The following are fundamental sets of numbers used throughout mathematics.

- **Natural Numbers:** The set $\mathbb{N}$ of natural numbers is the set of all positive whole numbers, along with zero. That is,
$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$
Note that in many other sources, zero is not included in the natural numbers. Those who authored such sources are bad people, and you should tell them you are very disappointed in them when you see them.
- **Integers:** The set of integers $\mathbb{Z}$ is the set of all whole numbers, whether they are positive, negative, or zero. That is,
$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$
- **Rational Numbers:** The set of rational numbers $\mathbb{Q}$ is the set of all numbers expressible as a fraction whose numerator and denominator are both integers.
- **Real Numbers:** The set of real numbers $\mathbb{R}$ is the set of all numbers expressible as a decimal.
- **Complex Numbers:** The set $\mathbb{C}$ of complex numbers is the set of all numbers formed as a real number (called the real part) plus a real number times i (called the imaginary part), where i is a symbol such that $i^2 = -1$.

Set-Builder Notation

The most common notation used to construct sets is *set-builder notation*, in which one specifies a name for the elements being considered and then some property $P(x)$ that is the membership test for an object x to be an element of the set. Specifically,

$$A = \{x \in B : P(x)\}$$

means that an object x chosen from B is an element of the set A if and only if the claim $P(x)$ is true about x. Sometimes the "$\in B''$ gets dropped if it is clear from context what set the elements are being chosen from. The set-builder notation above gets read as "the set of all x in B such that $P(x)$". One can think of this as running through all elements of B and throwing away any that do not meet the condition described by P.

Example 0.3.4. Interval Notation

Interval notation can be expressed in set-builder notation as follows.

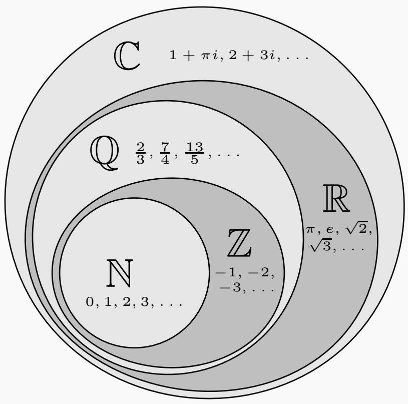
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

- $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Example 0.3.5. Rational, Real, and Complex in Set-Builder Notation

Set-builder notation is often used to express the sets of rational, real, and complex numbers as follows:

- $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$
- $\mathbb{R} = \{0.a_0a_1a_2a_3a_4 \ldots \times 10^n : n \in \mathbb{N}, a_i \in \{0,1,2,3,4,5,6,7,8,9\}$ where $i \in \mathbb{N}\}$ Note this is essentially scientific notation; the concatenation of the a_i's represents the digits in a base-ten decimal expansion.
- $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$



Exercise 0.3.6. Sorting Recyclables and Taking Out the Trash ☕

Identify each of the following expressions as a **quantity**, **function**, **statement**, **set**, or **garbage**.

- The number of atoms in the universe.
- The process by which a US citizen is assigned a social security number.
- $+++$
- $\mathbb{N} < \mathbb{Z}$
- $\mathbb{N} \in \mathbb{Z}$
- $\pi \in \mathbb{Q}$
- $i \in \mathbb{C}$
- $3 \in \mathbb{C}$

Quantifiers

There are two symbols from logic that we will occasionally use.

- **Universal Quantifier:** The symbol $\forall$ is a shorthand for the phrase "for all", representing the A from All, but it tripped on a comma and landed on its head. For example, the expression

$$\forall x \in \mathbb{R}, x^2 \geq 0$$

is just a shorter way to say the sentence

Every real number has a square that is greater than or equal to zero.

- **Existential Quantifier:** The symbol $\exists$ is a shorthand for the phrase "there exists", representing the E from Exists, but it too met a comma. One typically includes the phrase "such that" when reading an existential quantifier to make it sound more natural. For example, the expression

$$\exists x \in \mathbb{R}, x^3 + x + 1 = 0.$$

is just a shorter way to say the sentence

There exists a real number x such that $x^3 + x + 1 = 0$.

A single quantifier is not usually that complicated to deal with. However, when a statement contains two or more quantifiers, it can quickly become difficult to extract exactly what it is saying! The following exercise demonstrates how slightly altering the order of quantifiers can drastically change the meaning of a sentence.

Exercise 0.3.7. Order Matters ☕☕

Let P be the set of all humans who have ever existed. Write each of the following statements out in words. Then, decide if it is true or false.

- $\forall x \in P, \exists y \in P$, x is the mother of y.
- $\exists x \in P, \forall y \in P$, x is the mother of y.
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, $x < y$.
- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}$, $x < y$.

0.4 Prerequisites from Algebra

Ever since Leonhard Euler's incredibly influential work in the 1700s, mathematics has largely become the study of functions. Today's algebra and trigonometry curricula reflect that! Here are the most important functions from those courses and a few very important things to know about them.

Polynomials

For our purposes in this course, a *polynomial* is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where n is a natural number and the numbers $a_0, a_1, \ldots, a_n$ are complex numbers (called *coefficients*) with $a_n \neq 0$. The coefficient a_n is called the *leading coefficient* and n is the *degree* of the polynomial. The number a_0 is called the *constant term* of the polynomial.

The above form of polynomials is often called *expanded form* or *standard form.* There is another form for polynomials called *factored form* in which a polynomial is written as the product of other smaller degree polynomials.

Polynomials are best understood through their *roots.* Any complex number r for which $p(r) = 0$ is called a *root* of the polynomial.

Exercise 0.4.1. Language of Polynomials ☕

Identify each of the following statements as true or false.

- The function $p(x) = x^{1/2}$ is not a polynomial because $1/2$ is not a natural number.
- The polynomial $p(x) = x - 5$ has degree 1 and just a single root, namely 5.
- The polynomial $p(x) = 5$ has leading coefficient of 5 and degree zero.
- The polynomial $p(x) = 5$ has no roots.
- The polynomial $p(x) = x^2 - 9$ has exactly two roots, namely 3 and -3.
- The polynomial $p(x) = 0$ has degree zero and thus it has no roots.

Theorems about Polynomials

Here we state without proof several useful theorems regarding polynomials.

Theorem 0.4.2. Factor Theorem

The complex number r is a root of the polynomial $p(x)$ if and only if $(x-r)$ is a factor of $p(x)$. That is to say, $p(r)=0$ if and only if $p(x)=(x-r)q(x)$ for some polynomial $q(x)$.

Note that the above theorem does not say that a polynomial $p(x)$ with root r has to be divisible by $(x-r)$ only once. Perhaps it is divisible by some higher power of $(x-r)$, like $(x-r)^2$. This leads to the idea of *multiplicity*: the multiplicity of a root r in a polynomial $p(x)$ is the highest power of $(x-r)$ that divides $p(x)$.

Theorem 0.4.3. Fundamental Theorem of Algebra

Every degree n polynomial has exactly n roots in the complex numbers when counted with multiplicity.

Example 0.4.4. Counting Roots with Multiplicity

Consider the polynomial

$$p(x)=(x-1)^3(x+2).$$

If we multiply everything out, we get

$$p(x)=-2+5x-3x^2-x^3+x^4$$

which has degree 4. Thus, the Fundamental Theorem of Algebra promises four roots. When we count with multiplicity, we see that is the case: the list of roots is

$$1,1,1,-2.$$

Said another way, the polynomial has a root $r=1$ with multiplicity 3 and a root $r=-2$ with multiplicity 1.

Exercise 0.4.5. Checking Algebra ☕

Multiply out the product $p(x)=(x-1)^3(x+2)$ and verify that the expanded form of $p(x)$ shown above is correct.

Because of the Fundamental Theorem of Algebra, when we say to factor a polynomial, we typically mean to factor it into degree 1 factors with complex roots (since it is always possible to do so). Sometimes, however, we simply factor over the real numbers, in which case one may end up with degree 2 factors that have no real roots (for example something like x^2+1 whose only roots are i and $-i$). It is not the slightest bit obvious why, but it turns out that any polynomial of degree 3 or more with real coefficients will factor into a product of smaller degree polynomials with real coefficients.

Notice that taking a polynomial from factored form to expanded form is not that difficult; one simply multiplies it out. However, going the other direction, from expanded form to factored form, is far more difficult. The next subsection is dedicated to this complicated task!

Polynomial Factoring Techniques

The most difficult part of working with polynomials is usually finding roots (or factoring, which is equivalent thanks to the Factor Theorem). The next few results give some methods towards that goal. Note that none of those are guaranteed to work in general; these results are merely a collection of special cases in which something works out nicely.

Quadratic Formula/Completing the Square

The famous *quadratic formula* gives an explicit formula for the roots of a degree 2 polynomial in terms of the coefficients. Specifically, the degree 2 polynomial

$$p(x) = ax^2 + bx + c$$

has roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and thus has factorization

$$p(x) = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

Notice that we need the leading coefficient a out in front in addition to the factors promised by the factor theorem; otherwise if you multiplied out the right-hand side, the leading term would be simply x^2 rather than ax^2.

Exercise 0.4.6. Trying out the Quadratic Formula ☕

Find the roots of the polynomial $p(x) = 2x^2 + 2x + 2$ using the quadratic formula, and then use that to write it in factored form.

It is worth noting that there are cubic and quartic formulas (i.e., similar formulas for degree 3 and 4 polynomials, respectively) but they are far messier and thus typically not memorized, but rather used as theoretical tools or looked up when needed. There provably cannot exist a general formula for the roots when the degree is greater than or equal to five.

It is sensible to ask where the quadratic formula comes from! There is a process known as *completing the square* that can be used to prove it. Specifically, completing the square is just rewriting a quadratic in another form:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Exercise 0.4.7. Check It

- Expand and simplify the right-hand side given above, namely

$$a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a},$$

and verify that it does equal the left-hand side as claimed.

- Set $a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$ equal to zero and solve for x. Verify that it produces the quadratic formula.

One can apply the technique of completing the square by simply memorizing the above formula. However, in practice nobody does. It is usually implemented in a sequence of four steps:

1. Factor out the leading coefficient a from the x^2 and x terms.
2. Add and subtract the square of half of the remaining linear coefficient. That is, add and subtract the quantity $\left(\frac{b}{2a}\right)^2$.
3. Notice that $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2$ is now a perfect square trinomial, and factor it as $\left(x+\frac{b}{2a}\right)^2$.
4. Distribute the leading coefficient to the $-(b/2a)^2$ that is left over from step 2, and combine like terms.

Example 0.4.8. Revisiting Exercise 0.4.6

Suppose we once again consider the polynomial $p(x)=2x^2+2x+2$. Let us follow the four steps given above to complete the square on it.

1. We factor out 2, the leading coefficient:

$$p(x)=2\left(x^2+x\right)+2.$$

2. The remaining coefficient on x is just 1. We take half of that $(1/2)$ and then square that quantity to get $(1/2)^2=1/4$. This is the quantity we add and subtract, so it becomes

$$p(x)=2\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)+2.$$

3. Notice the first three terms inside the parentheses form a perfect square trinomial:

$$p(x)=2\left(\underbrace{x^2+x+\frac{1}{4}}_{\text{perfect square}}-\frac{1}{4}\right)+2.$$

Factor that perfect square:

$$p(x)=2\left(\left(x+\frac{1}{2}\right)^2-\frac{1}{4}\right)+2.$$

4. Distribute the 2 and combine like terms:

$$\begin{aligned} p(x) &= 2\left(x+\frac{1}{2}\right)^2 - \frac{1}{2} + 2 \\ &= 2\left(x+\frac{1}{2}\right)^2 + \frac{3}{2}. \end{aligned}$$

At this point, we have successfully completed the square on the polynomial $p(x)$. If you like, you can then set that equal to zero and solve for the roots, which should match what we obtained via the quadratic formula. Trying this out, we have

$$\begin{aligned} 2\left(x+\frac{1}{2}\right)^2 + \frac{3}{2} = 0 &\iff 2\left(x+\frac{1}{2}\right)^2 = -\frac{3}{2} \\ &\iff \left(x+\frac{1}{2}\right)^2 = -\frac{3}{4} \\ &\iff x+\frac{1}{2} = \pm\sqrt{-\frac{3}{4}} \\ &\iff x+\frac{1}{2} = \pm\frac{\sqrt{3}}{2}i \\ &\iff x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \end{aligned}$$

Exercise 0.4.9. Try Some! ☕

Complete the square on the following polynomials.

- $p(x) = x^2 - 1$
- $p(x) = x^2 - x + 1$
- $p(x) = 3x^2 - 6x + 1$

Sums and Differences of Squares and Cubes

The following formulas come up so often they are worth simply memorizing.

- **Difference of Two Squares:**

$$A^2 - B^2 = (A-B)(A+B)$$

- **Sum of Two Squares:**

$$A^2 + B^2 = (A-Bi)(A+Bi)$$

- **Difference of Two Cubes:**

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

- **Sum of Two Cubes:**

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

Exercise 0.4.10. Testing the Formulas ☕☕☕

- For each of the above four formulas, multiply out the right-hand side. Verify that it does in fact equal the left-hand side.
- Use the above formulas to factor the polynomial $p(x) = 4x^4 - 9$.
- Factor the polynomial $x^6 - 1$ in two ways:
  - Start with the difference of two squares formula, rewriting the polynomial as $\left(x^3\right)^2 - 1^2$.
  - Start with the difference of two cubes formula, rewriting the polynomial as $\left(x^2\right)^3 - 1^2$.

Rational Root Theorem/Polynomial Long Division

The Rational Root Theorem gives us a list of educated guesses as to what a root of our polynomial might be.

Theorem 0.4.11. Rational Root Theorem

Let $p(x)$ be a polynomial with integer coefficients and let a and b be integers with b nonzero. If the rational number a/b is a root of $p(x)$, then a must divide the constant term of $p(x)$ and b must divide the leading coefficient.

The statement of the theorem might be a bit of a mouthful, but it is quite easy to apply.

Example 0.4.12. Applying the Rational Root Theorem

Consider the polynomial $p(x) = 5x^2 - 7x - 6$. The only integer divisors of the leading coefficient 5 are $1, 5$, and their negatives. The only integer divisors of the constant term -6 are $1, 2, 3, 6$, and their negatives. The Rational Root Theorem tells us that any rational root must have numerator that divides the constant term and denominator that divides the leading term. Thus, the only possible rational roots of $p(x)$ are

$$1, 2, 3, 6, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{6}{5},$$

and their negatives.

If we plug these numbers into the polynomial $p(x)$, we find that $p(-3/5) = 0$ and $p(2) = 0$. By the Factor Theorem, we have that $(x + 3/5)$ and $(x - 2)$ must be factors of $p(x)$. By the Fundamental Theorem of Algebra, we know that there are no other roots, since we already found two roots, and the polynomial has degree two. Thus, the polynomial factors as

$$p(x) = 5(x + 3/5)(x - 2).$$

One note is that often it is cleaner to put the leading coefficient (or some part of it) into one of the factors to clean up the fractions. Here, if we put multiply the leading coefficient through the first factor, we get

$$p(x) = (5x + 3)(x - 2),$$

which is quite a bit cleaner.

One aspect of the above example that is perhaps a bit unsatisfying is that it would be extremely tedious to plug that whole list of possible roots into $p(x)$, desperately trying to find two roots. What is easier is after you find one root, just use *long division* to divide off the corresponding factor. We illustrate this below.

Example 0.4.13. Same Factorization with Long Division

Suppose we wanted to factor that same $p(x)$ from the previous example. We start by just plugging in easy numbers from the list, and find that $x = 2$ is a root. Not wanting to keep plugging in numbers, we recall from the Factor Theorem that $(x - 2)$ must be a factor of our polynomial. We now apply long division to find the quotient. In particular, we have the following:

$$\begin{array}{r|rr} & & 5x + 3 \\ \hline x - 2 & 5x^2 & -7x - 6 \\ & -5x^2 + 10x & \\ \hline & & 3x - 6 \\ & & -3x + 6 \\ \hline & & 0 \end{array}$$

which shows us that

$$5x^2 - 7x - 6 = (5x + 3)(x - 2).$$

Here is an example of using this same process on a larger degree polynomial.

Example 0.4.14. Factoring a Cubic

Suppose we wish to factor

$$p(x) = 4x^3 - 36x^2 + x - 9$$

using the Rational Root Theorem and long division. The list of possible roots is

$$1, 3, 9, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{1}{4}, \frac{3}{4}, \frac{9}{4},$$

and their negatives. We try the whole numbers first, and luckily find that $x = 9$ is a root. We proceed with long division:

$$\begin{array}{r}
4x^2 \qquad +1 \\
x-9\overline{)\ 4x^3 - 36x^2 + x - 9} \\
\underline{-\,4x^3 + 36x^2} \qquad\quad \\
x - 9 \\
\underline{-\,x + 9} \\
0
\end{array}$$

which tells us that

$$4x^3 - 36x^2 + x - 9 = (x-9)(4x^2+1).$$

If we are factoring over the real numbers, we are done. If we want to continue to factor using complex numbers, we can, using the Sum of Two Squares Formula. This produces

$$4x^3 - 36x^2 + x - 9 = (x-9)(2x+i)(2x-i).$$

Exercise 0.4.15. RRT/Long Division Factoring ☕☕

Apply the process from the previous two examples to factor the polynomial

$$p(x) = 2x^3 + x^2 - 4x - 3.$$

Specifically, generate a list of possible rational roots. Then, plug those numbers in until you find a root. Use the Factor Theorem to build a corresponding factor, and then use long division to find the quotient.

Exercise 0.4.16. Another Cubic ☕☕

Here we repeat the process of the previous examples, but to save a little tedium, we are given a root.

- Show that the number $x = 25/6$ is a root of the polynomial $p(x) = 6x^3 - 19x^2 - 13x - 50$.
- Note that the Factor Theorem itself tells us that our polynomial must be divisible by $(x - 25/6)$. Though this is true, it is often quite cumbersome to then go through the long division with all the fractions. A helpful strategy is to instead clear fractions (think multiplying both sides by six if the factor was set equal to zero) and instead use $6x - 25$. Perform the long division and find a quotient $q(x)$ such that $p(x) = (6x-25)q(x)$.

Factor by Grouping

Factor by Grouping is a method in which we strategically take the greatest common factor out of different clumps of terms in the hope that we end up with yet another common factor to pull out.

Example 0.4.17. Factoring by Grouping

The polynomial $x^4 + x^3 + 2x^2 + x + 1$ can be factored as follows:

$$\begin{aligned} x^4 + x^3 + 2x^2 + x + 1 &= x^4 + x^3 + x^2 + x^2 + x + 1 \\ &= x^2\left(x^2 + x + 1\right) + 1\left(x^2 + x + 1\right) \\ &= \left(x^2 + x + 1\right)\left(x^2 + 1\right). \end{aligned}$$

We could leave it in that form if we were factoring over the real numbers, or we could continue by using complex roots to obtain

$$\begin{aligned} x^4 + x^3 + 2x^2 + x + 1 &= \left(x^2 + x + 1\right)\left(x^2 + 1\right) \\ &= \left(x - \left(\frac{1 + \sqrt{3}i}{2}\right)\right)\left(x - \left(\frac{1 - \sqrt{3}i}{2}\right)\right)(x + i)(x - i). \end{aligned}$$

Exercise 0.4.18. Revisiting a Previous Example ☕

In Exercise 0.4.14, we factored the polynomial $4x^3 - 36x^2 + x - 9$. Try factoring that same polynomial again, but this time use factor by grouping. Verify the result comes out the same!

Exercise 0.4.19. Factor by Grouping Practice ☕☕

1. Factor the following polynomials by grouping:
   - $x^4 - x^3 + 2x^2 - x + 1$
   - $x^3 - x^2 + 2x - 2$
2. Consider the polynomial
$$x^4 + x^3 - x - 1$$
   - Factor by grouping the degree 3 and 4 terms together, while grouping the degree 1 and 0 terms together.
   - Factor by grouping the degree 4 and 0 terms together, while grouping the degree 3 and 1 terms together.

Pascal's Triangle

Pascal's Triangle can be thought of simply as a table of numbers. One starts with two diagonals of 1's, and then adds two numbers above to produce the number below.

$$\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

For example, in row $n = 4$, you can see the 6 and 4 add together to create the 10 below. The table goes on indefinitely, but we show just rows zero through six above.

Why is this in our section on polynomials? Well, it turns out that Pascal's Triangle gives you the coefficients upon expanding a power of a binomial. More specifically, the coefficients of the polynomial

$$(A + B)^n$$

can be found in row n of Pascal's Triangle. Notice that any term in the expansion of $(A + B)^n$ will have exactly n combined occurrences of A and B. Thus, the possible terms are

$$A^n, A^{n-1}B, A^{n-2}B^2, A^{n-3}B^3, \dots, B^n,$$

which is to say one can start from A^n and then just decrease the power of A as you increase the power of B, ending at B^n. Then, the coefficients come from Pascal's Triangle.

This may sound like a complicated process, but in an example one will see it is quite quick and easy to implement!

Example 0.4.20. Expanding Using Pascal's Triangle

Suppose we wish to expand the polynomial

$$(x - 2)^3.$$

We could just write out three copies of $(x - 2)$ and multiply everything out, but that gets rather tedious. Instead, we apply Pascal's Triangle as described above.
First, we list the terms that will be produced:

$$x^3, -2x^2, (-2)^2x, (-2)^3.$$

Now, we attach the coefficients from row $n = 3$ of Pascal's Triangle (which are $1, 3, 3, 1$) to those terms, and add them up. This produces

$$(x - 2)^3 = 1 \cdot x^3 + 3 \cdot (-2)x^2 + 3 \cdot (-2)^2x + 1 \cdot (-2)^3 = x^3 - 6x^2 + 12x - 8.$$

Ok, so why is this in a factoring section? Well, in reverse, if you happen to see a polynomial and can recognize the coefficients as coming from some row of Pascal's Triangle, then you can factor the entire polynomial in one step! For example, if we encountered the polynomial $x^3 - 6x^2 + 12x - 8$, we could say

$$x^3 - 6x^2 + 12x - 8 = 1 \cdot x^3 + 3 \cdot (-2)x^2 + 3 \cdot 4x + 1 \cdot (-8) = (x - 2)^3,$$

which is far easier than RRT/long division or even Factor by Grouping.

Exercise 0.4.21. Practice with Pascal's Triangle ☕☕

Use Pascal's Triangle to factor the following polynomials:

- $-x^{10} + 5x^8 - 10x^6 + 10x^4 - 5x^2 + 1$
- $x^6 + 2x^3 + 1$

Rational Functions

A *rational function* is a function that can be written as a ratio (hence "rational") of two polynomials. That is, a rational function $r(x)$ is one expressible as

$$r(x) = \frac{p(x)}{q(x)}$$

for polynomials $p(x)$ and $q(x)$. Most of what one wants to know about rational functions can be determined by polynomial long division and the polynomial methods listed in the previous section.

Long Division

Long division with rational functions is a key step. If one has $r(x) = p(x)/q(x)$ and the degree of $p(x)$ is smaller than the degree of $q(x)$, then there is no need to perform division. For example, the function

$$r(x) = \frac{x+2}{x^2+2}$$

has smaller degree in the numerator than the denominator, so there is no need to use long division. But if it were the other way around,

$$r(x) = \frac{x^2+2}{x+2},$$

then we could perform long division. Specifically, it will allow us to write the function in form

$$r(x) = \texttt{quotient} + \frac{\texttt{remainder}}{\texttt{divisor}}.$$

One can think of this as being analogous to how improper fractions can be handled. For example, if one performs long division on 7 by 3, there is a quotient 2 with remainder 1. Thus, we have

$$\frac{7}{3} = 2 + \frac{1}{2}.$$

Let us now work through the example mentioned above.

Example 0.4.22. Long Division

We perform long division on

$$r(x) = \frac{x^2+2}{x+2}$$

as follows:

$$\begin{array}{r} x-2 \\ x+2\overline{)\,x^2 \qquad\quad +2} \\ -x^2-2x \qquad \\ \hline -2x+2 \\ 2x+4 \\ \hline 6 \end{array}$$

From this, we conclude that

$$r(x) = x - 2 + \frac{6}{x+2}.$$

Notice that performing this long division tells you the *end behavior* of your rational function. The remainder term

$$\frac{6}{x+2}$$

will become arbitrarily small as x becomes a large positive or large negative number. Thus, the graph will converge to the quotient, which in this case is the line $y = x - 2$.

Exercise 0.4.23. Checking Work

To really believe the calculation above, we should check it! Specifically, take the expression

$$x - 2 + \frac{6}{x+2}$$

and get a common denominator of $(x + 2)$ by turning the $x - 2$ into

$$\frac{x-2}{1} \cdot \frac{x+2}{x+2}.$$

Add the resulting numerators and recover the original function $r(x)$.

Roots of the Numerator and Denominator

Roots of the denominator of a rational function will cause division by zero, and thus produce either vertical asymptotes in the graph. Roots of the numerator of a rational function correspond to x-intercepts, since a fraction with zero in the numerator is zero. (Note that if the numerator and denominator share a zero, then it is more complicated and other things can happen. This situation will be explored later in the text.)

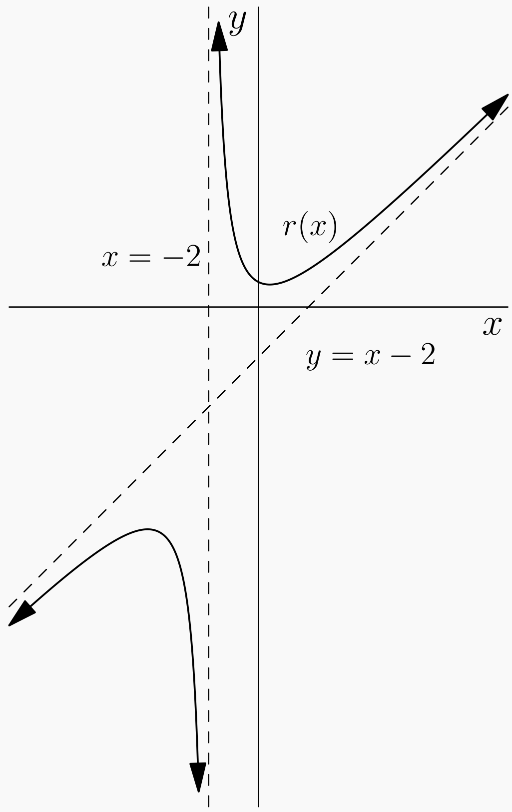
Example 0.4.24. Graphing a Rational Function

Let us combine all the information about the function

$$r(x) = \frac{x^2+2}{x+2}$$

captured above in order to graph it. There are no real roots of the numerator, since the only roots are the complex numbers $\pm\sqrt{2}i$, which are valid roots of course, but they just don't show up on a graph. The denominator has $x = -2$ as its only root, so there is a vertical asymptote at $x = -2$.

It also never hurts to plot a random point or two. A nice one in this case is the y-intercept, $r(0) = 1$. Assembling this information along with the asymptote $y = x - 2$ found in the previous example produces the graph.



Exponentials and Logarithms

Though you were likely exposed to exponentials and logs in your college algebra/precalculus course, to really define exponentials and logarithms properly requires some construction from Calculus! Usually only an intuitive definition is given, something along the following lines:

The expression b^x is x copies of b multiplied together. The function $\log_b(x)$ is defined to be the inverse function of b^x.

The above definition is actually perfectly fine if x is a natural number. For example, one could say

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

and consequently

$$\log_2(8) = 3,$$

since inverse functions reverse the roles of inputs and outputs.

However, what if x is a fraction? Well, it turns out that isn't too bad to define, as one can use a radical. Specifically, if $x = n/m$ for some natural numbers n and m, we have

$$b^{n/m} = \sqrt[m]{b^n}.$$

However however, what if x is an irrational number? For example, what on earth does 2^π mean? To answer such questions, some form of calculus is required. So, in this section we do not dive deep at all, and instead just provide a list of commonly used identities. Note that all log and exponent identities come in pairs, as they are inverse functions: where one has an identity the other must have a corresponding opposite identity. In the table below, b always represents a positive real number.

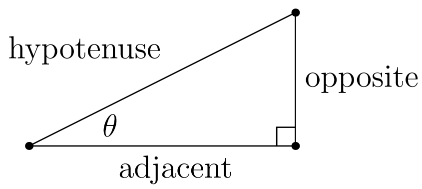
Name of Property	Property of Exponents	Property of Logarithms
Inverse Functions	$b^{\log_b(x)} = x$	$\log_b(b^x) = x$
Product to Sum	$b^x b^y = b^{x+y}$	$\log_b(xy) = \log_b(x) + \log_b(y)$
Difference to Quotient	$b^{x-y} = \frac{b^x}{b^y}$	$\log_b(x) - \log_b(y) = \log_b\left(\frac{x}{y}\right)$
Power to Product	$(b^x)^y = b^{xy}$	$\log_b(x^y) = y\log_b(x)$
Change of base	$b^x = e^{x\ln(b)}$	$\log_b(x) = \frac{\ln(x)}{\ln(b)}$

0.5 Prerequisites from Trigonometry

Trigonometry can be seen very geometrically as the study of triangles; it also can be seen as the study of the six trigonometric functions. Both perspectives will be used frequently throughout the calculus sequence!

Trigonometric Functions as Ratios of Sides

Consider the right triangle below, labelled with angle θ. It has two legs: one of which is *opposite* from the angle θ and one of which is *adjacent* to angle θ. There is only one side which can be called the *hypotenuse*: the side opposite the right angle.



The trigonometric functions are defined as ratios of ordered pairs of distinct sides of that triangle. Thus, there are 3 ways to select the numerator of the ratio and 2 remaining ways to select the denominator, so there are $3 \cdot 2 = 6$ trig functions. We name them and define them below.

- $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$
- $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$
- $\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$
- $\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$
- $\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$

Exercise 0.5.1. Intertwined! ☕

It turns out all six trig functions can be written just in terms of sine and/or cosine. In particular, use the definitions given above to prove that the following four identities are true.

- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
- $\sec(\theta) = \frac{1}{\cos(\theta)}$
- $\csc(\theta) = \frac{1}{\sin(\theta)}$

Because the above exercise provides the other four trig functions for free once you have sine and cosine, the remainder of this section will focus largely on just those two functions rather than all six.

Sine and Cosine as Unit Circle Measurements

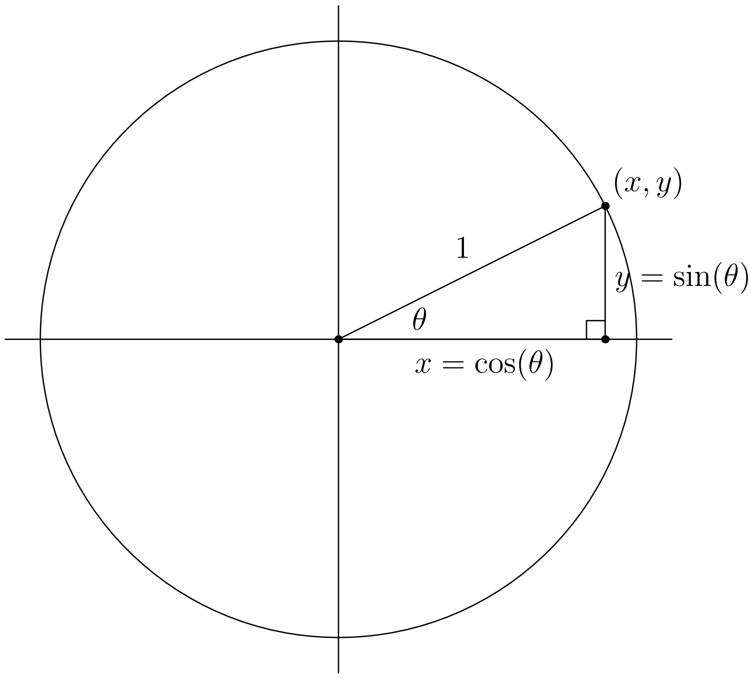
A very nice way to think about sine and cosine is as follows: for simplicity, pick the hypotenuse to equal 1. It is a ratio anyway, so you can always multiply the top and bottom by any nonzero amount. Then, place the adjacent side of the triangle along the positive x-axis with the vertex for angle θ at the origin. This means that if a point (x, y) is distance 1 from the origin, and the corresponding radius makes an angle θ with the positive x-axis, then we have

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x$$

and similarly

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y.$$

This is illustrated below.



On the unit circle, angles are typically measured in *radians* rather than degrees. Radian is simply the measure of arc length along the circumference of the unit circle. Since the circumference of a circle of radius r is

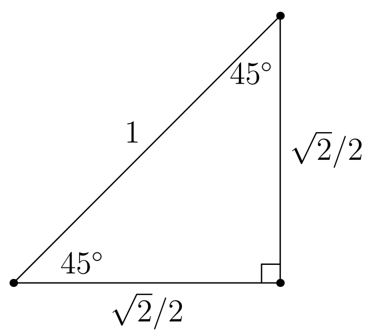
$$C = 2\pi r,$$

here we have a total circumference of 2π (since the radius is 1). Thus, one full lap of $360°$ is 2π radians. One can then scale that ratio up and down to get different equivalences of degrees and radians. Here are some common useful ones.

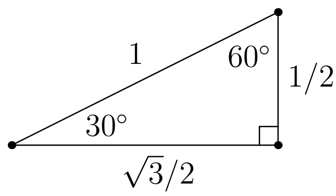
Degree Measure	Equivalent Radian Measure
$360°$	2π
$180°$	π
$90°$	$\pi/2$
$60°$	$\pi/3$
$45°$	$\pi/4$
$30°$	$\pi/6$

The acute angles listed above fit into two particularly special triangles of hypotenuse 1, whose measurements are shown below.

- **The** $45° - 45° - 90°$ **right triangle.** Since two angles are equal, the two legs must also be equal. One can then simply apply the Pythagorean Theorem for x in the equation $x^2 + x^2 = 1^2$ to obtain the measurement of $\sqrt{2}/2$.



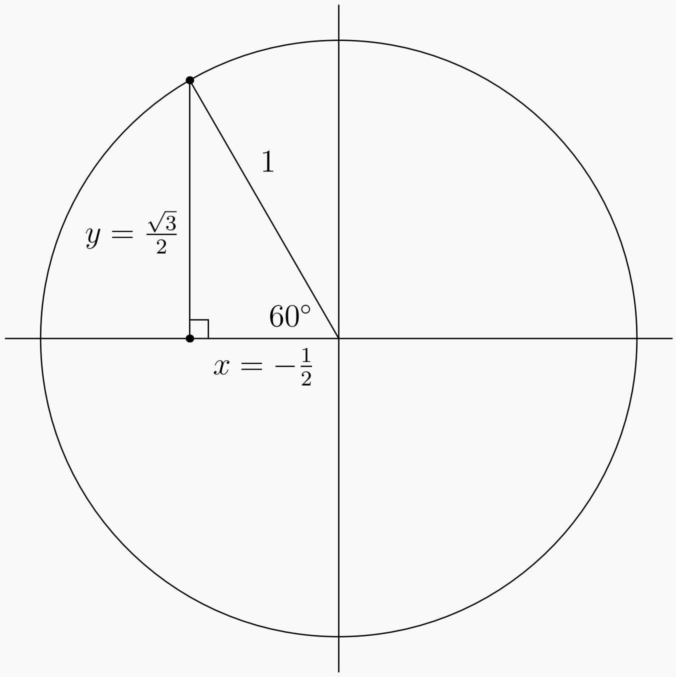
- **The** $30° - 60° - 90°$ **right triangle.** Notice that this triangle is simply half of an equilateral triangle of side length 1. That is how to remember the $1/2$, is it literally half of an equilateral. Then the Pythagorean Theorem provides the $\sqrt{3}/2$ by solving $x^2 + (1/2)^2 = 1^2$ for x.



While it is very common in reference sections like this to then provide a fully labeled unit circle which then students will fastidiously memorize, the author highly recommends against doing so. Instead, simply place the angle in question on the unit circle and find the x and y coordinates of the corresponding point using a special triangle whenever possible.

Example 0.5.2. Calculating Cosine and Sine of an Angle

Here we calculate cosine and sine of the angle $\theta = 2\pi/3$. We first draw the angle on the unit circle. To do so, we notice that since $\pi/3 = 60°$, then twice that must be $2\pi/3 = 120°$. This puts the angle at $60°$ from the negative x-axis. The hypotenuse is always 1, so we can label that side. We then see that we can fit a $30° - 60° - 90°$ triangle perfectly into that angle, which gives us the x and y coordinates.



Lastly, we recall that cosine is simply the x-coordinate and sine is the y-coordinate. We conclude that

$$\cos(2\pi/3) = -\frac{1}{2}$$

and

$$\sin(2\pi/3) = \frac{\sqrt{3}}{2}.$$

Here are a few other notes about unit circle computations:

- If an angle lands on an axis (i.e., is a multiple of ninety degrees) then no special triangle is needed as the x and y coordinates will just be 0, 1, or -1.
- Negative angles can be used; they simply wind clockwise rather than counterclockwise from the positive x-axis.

- Angles with magnitude larger than 2π can be used; this corresponds to taking more than one complete lap around the circle.

Trigonometric Identities

This is by no means a comprehensive list of trigonometric identities, but rather just a list of some that will come up frequently in this course.

- **Pythagorean Identities.**

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$
$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

- **Cosine Angle Sum.**
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

- **Sine Angle Sum.**
$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

- **Cosine Angle Difference.**
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

- **Sine Angle Difference.**
$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

- **Cosine Double Angle.**
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

- **Sine Double Angle.**
$$\sin(2A) = 2\sin(A)\cos(A)$$

- **Cosine Half Angle.**
$$\cos^2(A) = \frac{1 + \cos(2A)}{2}$$

- **Sine Half Angle.**
$$\sin^2(A) = \frac{1 - \cos(2A)}{2}$$

- **Cosine Even.**
$$\cos(-\theta) = \cos(\theta)$$

- **Sine Odd.**
$$\sin(-\theta) = -\sin(\theta)$$

- **Cofunction Identity.**
$$\cos(\pi/2 - \theta) = \sin(\theta)$$

Inverse Trigonometric Functions

The trigonometric functions are not one-to-one, so one must restrict their domains in order to build inverse functions. The table below gives one possible way of restricting the domains of the trigonometric functions and lists the corresponding domains and ranges of the inverse trigonometric functions.

Trig Function on Restricted Domain	Resulting Inverse Trig Function
$\sin : [-\pi/2, \pi/2] \to [-1, 1]$	$\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$
$\cos : [0, \pi] \to [-1, 1]$	$\arccos : [-1, 1] \to [0, \pi]$
$\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$	$\arctan : (-\infty, \infty) \to (-\pi/2, \pi/2)$
$\cot : (0, \pi) \to (-\infty, \infty)$	$\text{arccot} : (-\infty, \infty) \to (0, \pi)$
$\sec : (0, \pi/2) \cup (\pi/2, \pi) \to (-\infty, -1] \cup [1, \infty)$	$\text{arcsec} : (-\infty, -1] \cup [1, \infty) \to (0, \pi/2) \cup (\pi/2, \pi)$
$\csc : (-\pi/2, 0) \cup (0, \pi/2) \to (-\infty, -1] \cup [1, \infty)$	$\text{arccsc} : (-\infty, -1] \cup [1, \infty) \to (-\pi/2, 0) \cup (0, \pi/2)$

To calculate with inverse trig functions once the domains and ranges are known, it is then just a matter of reversing the input and output from a trig function calculation.

Exercise 0.5.3. Inverse Trig Calculation ☕☕

Calculate the values of the inverse trig functions listed below.

- $\arccos\left(-\frac{1}{2}\right)$
- $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

Part I

Limits and Continuity

Chapter 1

Limits and Continuity

1.1 A Graphical and Numerical Approach to Limits

In this section, we approach the idea of a *limit* visually and intuitively. This acts as a gentle introduction to the concept before we bludgeon ourselves over the head in the following section with the actual definition.

One-Sided and Two-Sided Limits

Definition 1.1.1. One-Sided Limits: Left-Hand Limit and Right-Hand Limit

Let $f(x)$ be a function, and let a and L be real numbers. We give the following informal definitions:

- We say *the limit of $f(x)$ as x approaches a from above* and write

$$\lim_{x\to a^+} f(x) = L$$

  for a *right-hand limit*. This means that the function $f(x)$ becomes arbitrarily close to L as x approaches a from the right.

  - The $x \to a^+$ notation can be thought of as an instruction to plug "a plus a little tiny amount" into $f(x)$. As that little tiny amount goes to zero, $f(x)$ will approach L.

- We say *the limit of $f(x)$ as x approaches a from below* and write

$$\lim_{x\to a^-} f(x) = L$$

  for a *left-hand limit*. This means that the function $f(x)$ becomes arbitrarily close to L as x approaches a from the left.

  - Similarly, the $x \to a^-$ notation can be thought of as an instruction to plug "a minus a little tiny amount" into $f(x)$. As that little tiny amount goes to zero, $f(x)$ will approach L.

In the above cases, we say *the limit exists*. If there does not exist such a real number L, we say *the limit does not exist*, often denoted with the abbreviation DNE.

Note that we are not particularly concerned with what happens when $x = a$, but rather we are looking at x values that are very close to a.

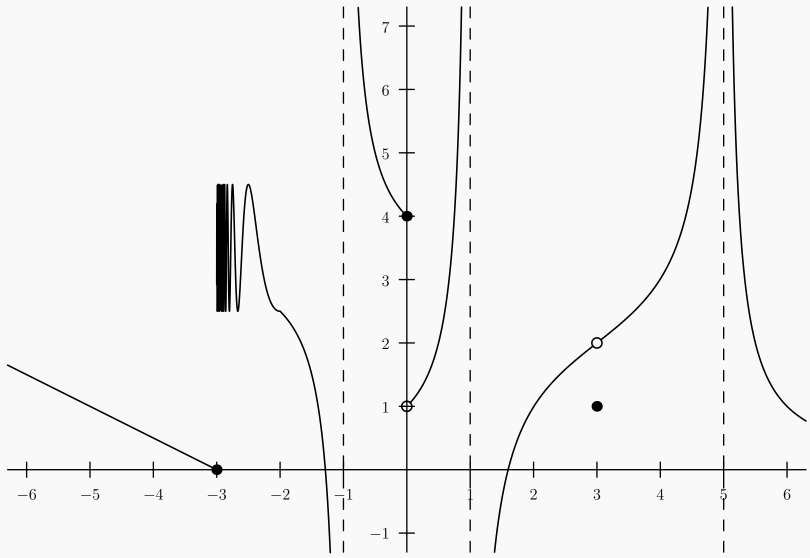
Definition 1.1.2. Infinite Limits

- If the values of $f(x)$ are positive and grow without bound as x approaches a from the right, say the limit is infinity and write
$$\lim_{x\to a^+} f(x) = \infty,$$
and similarly from the left.
- If the values of $f(x)$ are negative and grow in magnitude without bound as x approaches a from the right, say the limit is negative infinity and write
$$\lim_{x\to a^+} f(x) = -\infty,$$
and similarly from the left.

Example 1.1.3. What These Look Like on a Graph

On the graph below, we have the following limits:

- $\lim_{x\to 5^+} f(x) = \infty$
- $\lim_{x\to 3^+} f(x) = 2$
- $\lim_{x\to 0^+} f(x) = 1$
- $\lim_{x\to -1^+} f(x) = \infty$
- $\lim_{x\to -3^+} f(x) = DNE$
- $\lim_{x\to 5^-} f(x) = \infty$
- $\lim_{x\to 3^-} f(x) = 2$
- $\lim_{x\to 0^-} f(x) = 4$
- $\lim_{x\to -1^-} f(x) = -\infty$
- $\lim_{x\to -3^-} f(x) = 0$

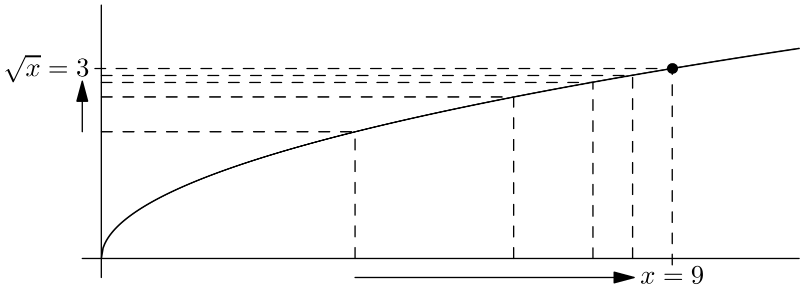


Exercise 1.1.4. One-Sided Limits on a Square Root

Consider the function $f(x) = \sqrt{x}$. Suppose we wish to compute the left-hand limit and right-hand limit at $x = 9$. We consider inputs of the form “9 plus a little” for the right-hand limit and inputs of the form “9 minus a little” for the left-hand limit.

- **Left-Hand Limit:** We choose inputs like $x = 8$, $x = 8.9$, $x = 8.99$, and so on. It is helpful to organize them into a small table of values.

x	8	8.9	8.99	8.999
$\sqrt{x}$	$2.828\ldots$	$2.983\ldots$	$2.998\ldots$	$2.999\ldots$

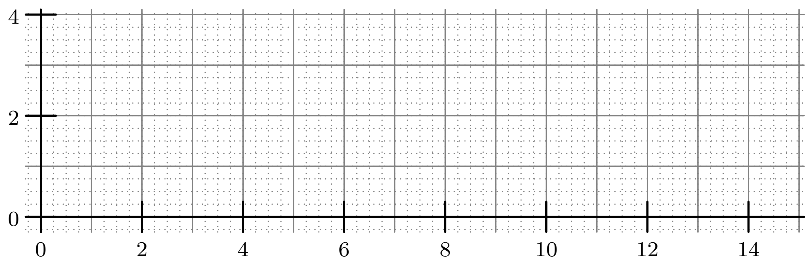


From the data table, we can see that as x approaches 9 from below, the function $\sqrt{x}$ approaches 3. We write this more formally as follows:

$$\lim_{x \to 9^-} \sqrt{x} = 3.$$

- **Right-Hand Limit:** We choose inputs like $x = 10$, $x = 9.1$, $x = 9.01$, $x = 9.001$, and so on. Fill out the data table, draw a corresponding graph, and fill in the blanks in the sentences that follow.

x	10	9.1	9.01	9.001
$\sqrt{x}$				



From the data table, we can see that as x approaches 9 from ___________, the function $f(x) = \sqrt{x}$ approaches ________. We write this more formally as follows:

$$\lim_{x \to 9^+} \sqrt{x} = \underline{\qquad\quad}.$$

When the left- and right- hand limits both approach the same value, we call that a *two-sided limit* or just a *limit* and drop the $+/-$ in the limit notation.

Definition 1.1.5. Two-Sided Limits

Let $f(x)$ be a function, let a be a real number, and let L be a real number, ∞, or $-\infty$.

- We say the *limit of $f(x)$ as x approaches a is L* if and only if

$$\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

  In this case, we write

$$\lim_{x\to a} f(x) = L.$$

- If the left- and right-hand limits do not match, then we write

$$\lim_{x\to a} f(x) = \text{DNE}.$$

Example 1.1.6. Two-sided Limit for the Square Root

Again consider the function $f(x) = \sqrt{x}$ with x approaching 9. Exercise 1.1.4 shows that both the left- and right-hand limits are equal to 3. Thus, the two-sided limit is also equal to 3 and we write

$$\lim_{x\to 9} \sqrt{x} = 3.$$

Example 1.1.7. Two-sided Limit for Our Old Graph

Revisit the graph from Example 1.1.3. The corresponding two-sided limits are as follows:

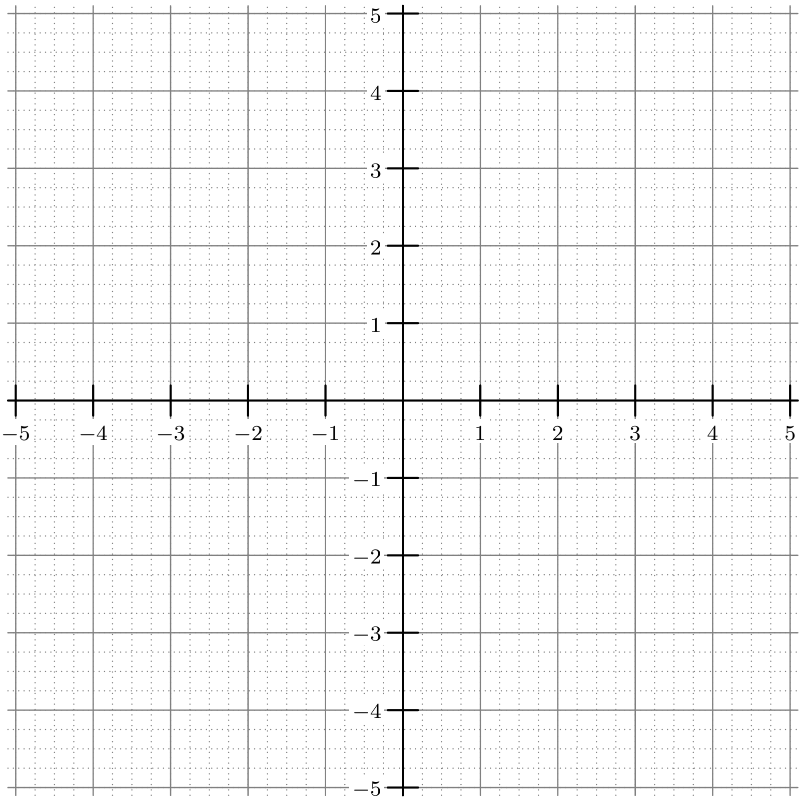
- $\lim_{x\to 5} f(x) = \infty$
- $\lim_{x\to 3} f(x) = 2$
- $\lim_{x\to 0} f(x) = DNE$
- $\lim_{x\to -1} f(x) = DNE$
- $\lim_{x\to -3} f(x) = DNE$

Exercise 1.1.8. Floor Function ☕☕

Recall the *floor function*, the function $f(x) = \lfloor x \rfloor$ that takes a real number and rounds it down to the nearest integer. For example, we show a few values below.

$$f(3) = 3$$
$$f(\pi) = 3$$
$$f(3.9) = 3$$
$$f(4) = 4$$

- Graph $f(x)$.



- Use your graph (and a data table if helpful) to evaluate the following limits:
  - $\lim_{x\to 2.5^-} f(x)$
  - $\lim_{x\to 2.5^+} f(x)$
  - $\lim_{x\to 2.5} f(x)$
  - $\lim_{x\to 2^-} f(x)$
  - $\lim_{x\to 2^+} f(x)$
  - $\lim_{x\to 2} f(x)$

Example 1.1.9. Tangent

As $x \to \pi/2^-$, the function $\tan(x)$ grows without bound, attaining larger and larger positive numbers. Thus, we say

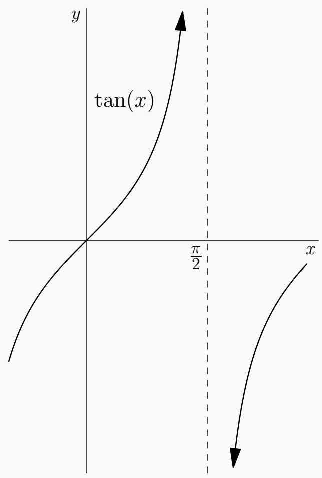
$$\lim_{x \to \pi/2^-} \tan(x) = \infty.$$

From the opposite side, we see the values of the function becoming arbitrarily large in magnitude, but they are all negative. Thus, we say

$$\lim_{x \to \pi/2^+} \tan(x) = -\infty.$$

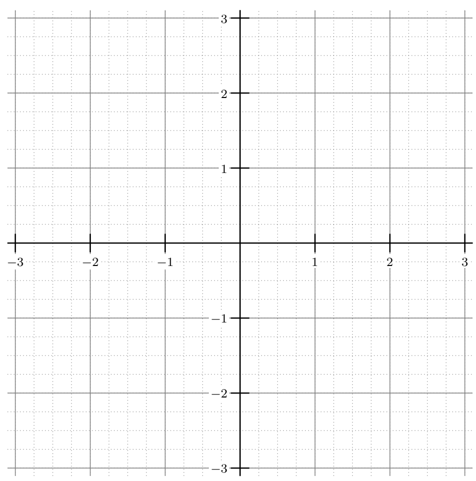
Since the left- and right-hand limits do not agree, we say

$$\lim_{x \to \pi/2} \tan(x) = \text{DNE}.$$



Exercise 1.1.10. A Hyperbola ☕☕

Graph the function $f(x) = \frac{1}{x}$.



Fill out the table of values below.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{1}{x}$							

Use your data and the graph to evaluate the following limits:

- $\lim_{x\to -1^-} f(x)$
- $\lim_{x\to -1^+} f(x)$
- $\lim_{x\to -1} f(x)$
- $\lim_{x\to 0^-} f(x)$
- $\lim_{x\to 0^+} f(x)$
- $\lim_{x\to 0} f(x)$

Limits to Infinity

A *limit to infinity* is just a more formal language for discussing graphical behavior you likely already studied in your College Algebra and/or Precalculus classes. In particular, it reframes the concepts of *horizontal asymptotes* and *end behavior*.

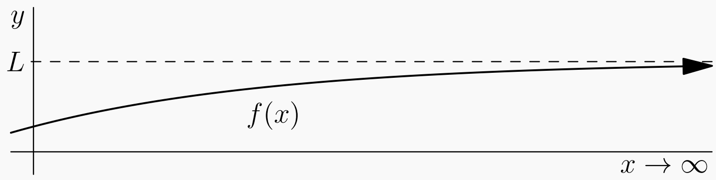
Definition 1.1.11. Limits to Infinity: New Names for Old Ideas

Let $f(x)$ be a function, and let L be a real number. We give the three following intuitive definitions:

- We say *the limit of $f(x)$ as x approaches ∞ is L* and write

$$\lim_{x\to\infty} f(x) = L$$

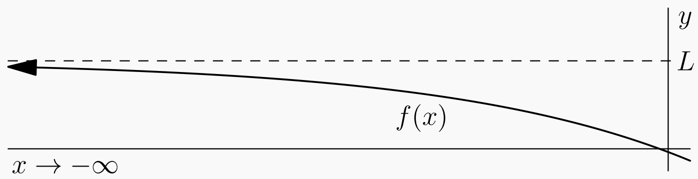
to denote a horizontal asymptote at $y = L$ to the right. This means that the function $f(x)$ becomes arbitrarily close to L as x becomes very large.



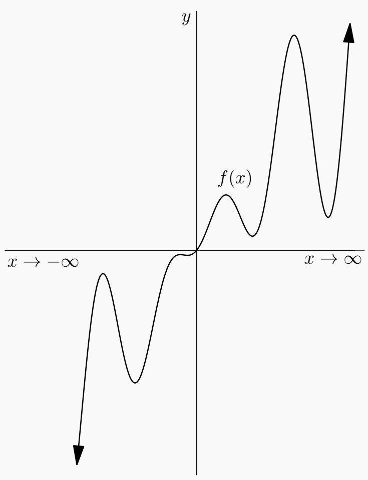
- We say *the limit of $f(x)$ as x approaches $-\infty$ is L* and write

$$\lim_{x\to-\infty} f(x) = L$$

to denote a horizontal asymptote at $y = L$ to the left. This means that the function $f(x)$ becomes arbitrarily close to L as x becomes very large in magnitude but negative.

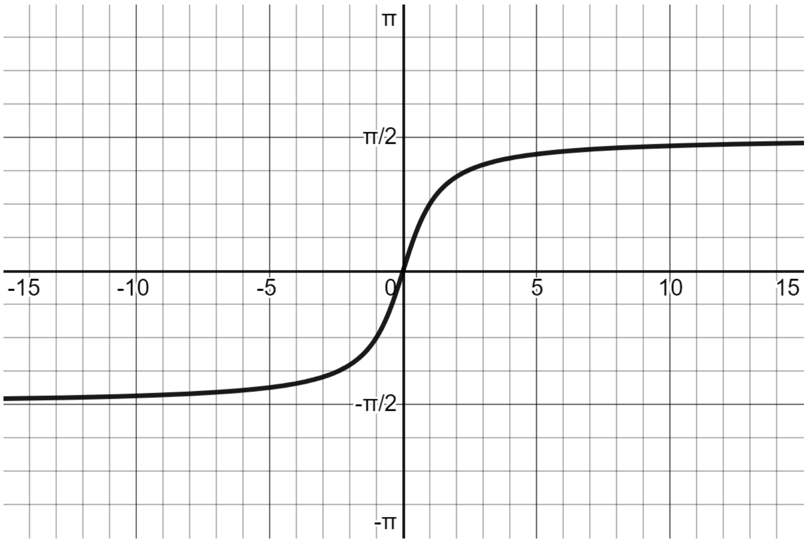


- In either of the above cases, if the values of $f(x)$ are positive and grow without bound, we relax the constraint that L must be a real number and say $L = \infty$ (or $L = -\infty$ if the magnitude grows without bound but the sign is negative). One of these limits approaching ∞ corresponds to *up end behavior*, and one of these limits approaching $-\infty$ corresponds to *down end behavior*.



Exercise 1.1.12. Arctangent ☕

Graph the function $f(x) = \arctan(x)$.



Fill out the table of values below.

x	-100	-10	-1	0	1	10	100
$\arctan(x)$							

Use your data and the graph to evaluate the following limits:

- $\lim_{x\to-\infty} f(x)$
- $\lim_{x\to\infty} f(x)$
- $\lim_{x\to 0^-} f(x)$
- $\lim_{x\to 0^+} f(x)$
- $\lim_{x\to 0} f(x)$

Which of the above limits correspond to horizontal asymptotes?

Exercise 1.1.13. A Rational Function ☕☕

Consider the rational function

$$r(x) = \frac{x^2-9}{x^2-4x+4}.$$

- Perform long division on this fraction. Show that the quotient is $y = 1$.

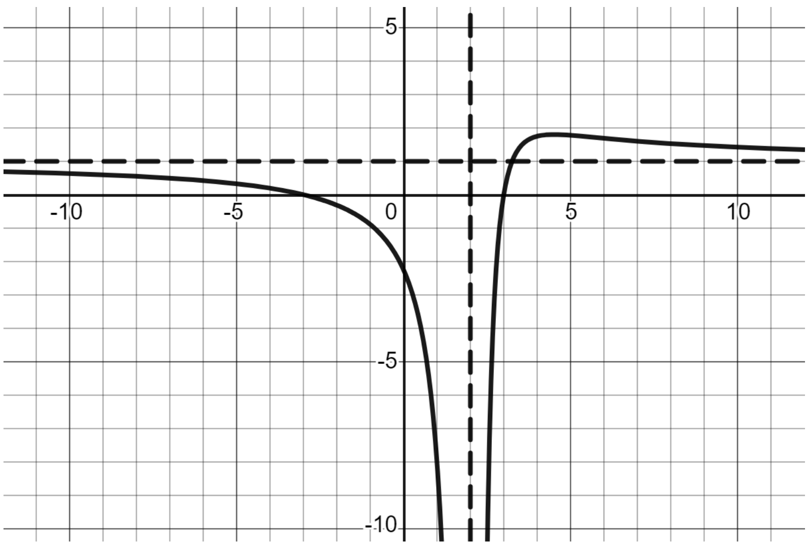
- Let's make a small table of values to see the y-coordinate of $r(x)$ converge upon the y-coordinate of $y = 1$ as x gets very large in either the positive or negative direction.

x	-1000	-100	-10	10	100	1000
$r(x)$						

- Now let's study another feature of this graph! Find the roots of the denominator. Show that there is a root of even multiplicity at $x = 2$. Let us now observe numerically how that affects the graph near $x = 2$.

x	1.9	1.99	1.999	2.001	2.01	2.1
$r(x)$						

- Below is a graph of $r(x)$. It includes dashed lines to represent the graph of the horizontal asymptote and vertical asymptotes.



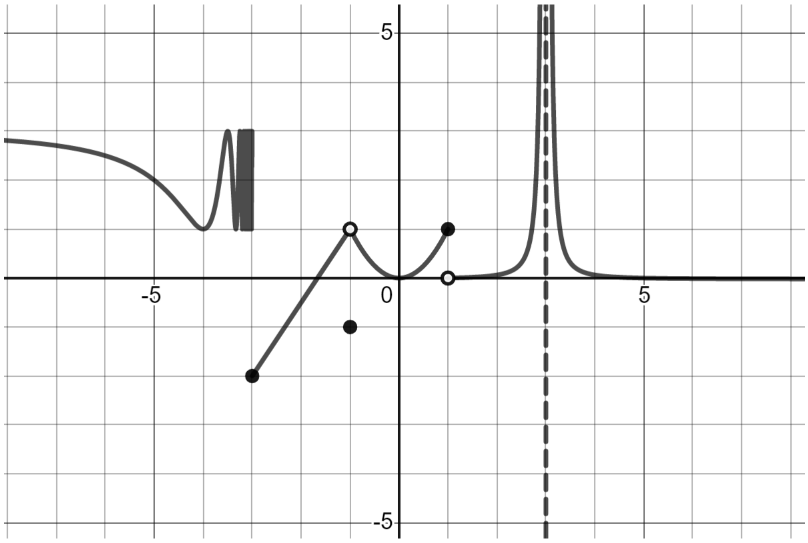
Use your graph and data to evaluate the following limits:

- $\lim_{x \to -\infty} r(x)$

- $\lim_{x\to\infty} r(x)$
- $\lim_{x\to 2^-} r(x)$
- $\lim_{x\to 2^+} r(x)$
- $\lim_{x\to 2} r(x)$

Exercise 1.1.14. A Piecewise Function ☕☕☕

Consider the function $f(x)$ as graphed below.



Use your graph to evaluate the following limits:

- $\lim_{x\to -3^-} f(x)$
- $\lim_{x\to -3^+} f(x)$
- $\lim_{x\to -3} f(x)$
- $\lim_{x\to -1^-} f(x)$
- $\lim_{x\to -1^+} f(x)$
- $\lim_{x\to -1} f(x)$
- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$
- $\lim_{x\to 3^-} f(x)$
- $\lim_{x\to 3^+} f(x)$
- $\lim_{x\to 3} f(x)$

1.2 An Analytic Approach to Limits

While often we can discuss limits just in terms of data tables or graphs, it sometimes becomes difficult to figure out what a limit should be by these intuitive methods. Augustin Louis Cauchy came up with the more precise modern definition in nineteenth century France. The idea is simple, but stating it formally is quite a challenge!

Example 1.2.1. Don't Be Line to Me

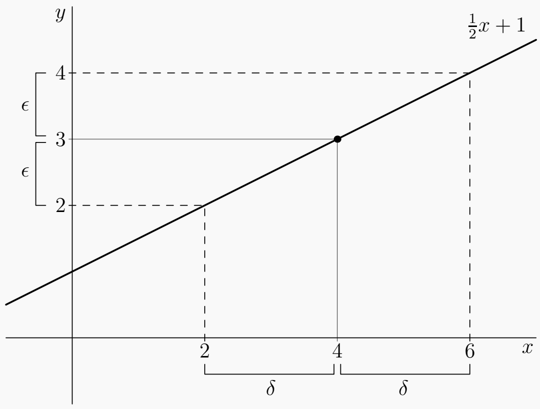
Consider the limit

$$\lim_{x\to 4} \frac{1}{2}x + 1 = 3.$$

We can think about how close the x values must be to 4 in order to promise the y values are within a certain distance of 3. We use the symbol ϵ to represent the allowable distance from 3 to the y-coordinates of the function.

- **Choose $\epsilon = 1$:** If we want the y coordinates of the line $y = \frac{1}{2}x + 1$ to be within 1 of 3, then we can see from the graph that the x values must be between 2 and 6. We rephrase our conclusion in an important way below:

  If the x coordinates are chosen within 2 units to the left or right of $x = 4$, then the y coordinates will be within 1 unit above or below $y = 3$.

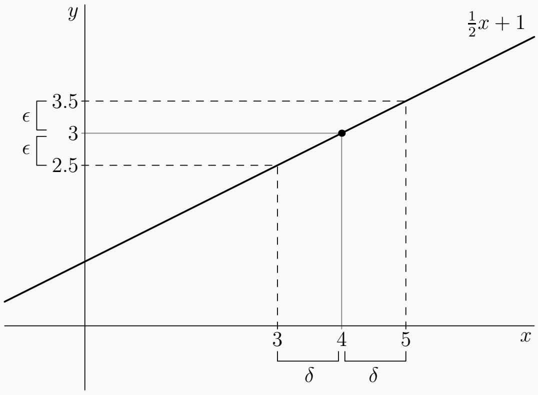
  The allowable horizontal distance from the input is often referred to as δ. The above graph shows the value of $\delta = 2$ as determined.

- **Choose $\epsilon = 1/2$:** If we want the y coordinates of the line $y = \frac{1}{2}x + 1$ to be within 1/2 of 3, then we must restrict to x coordinates that are closer than before. Here the points on the graph maybe aren't quite so clean, so we use a bit of algebra to help. We are searching for some number δ such that $x = 4 + \delta$ will be mapped to 3.5 under the function, and $x = 4 - \delta$ will be mapped to 2.5 under the function. We can see by symmetry that either condition will make the other also hold. Let's solve the first condition for δ:

$$\begin{aligned} \frac{1}{2}(4+\delta) + 1 &= 3.5 \\ \frac{1}{2}(4+\delta) &= 2.5 \\ 4+\delta &= 5 \\ \delta &= 1 \end{aligned}$$

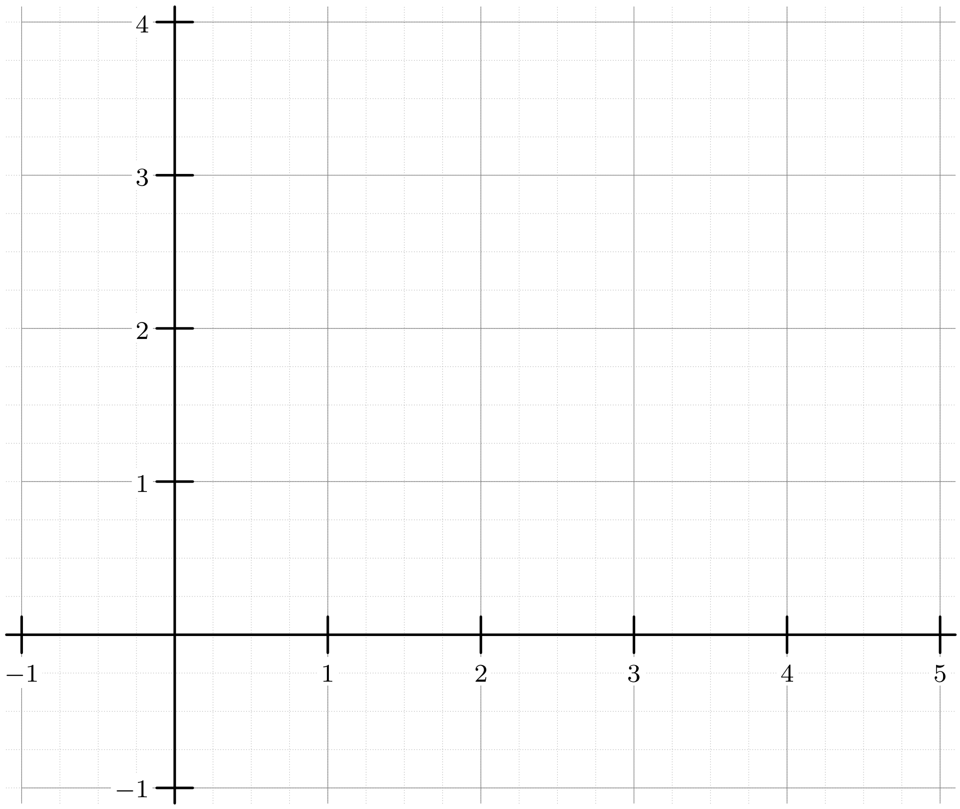
We again rephrase our conclusion in an important way:

If the x coordinates are chosen within $\delta = 1$ unit of 4, the y coordinates will be within $\epsilon = 1/2$ of 3.

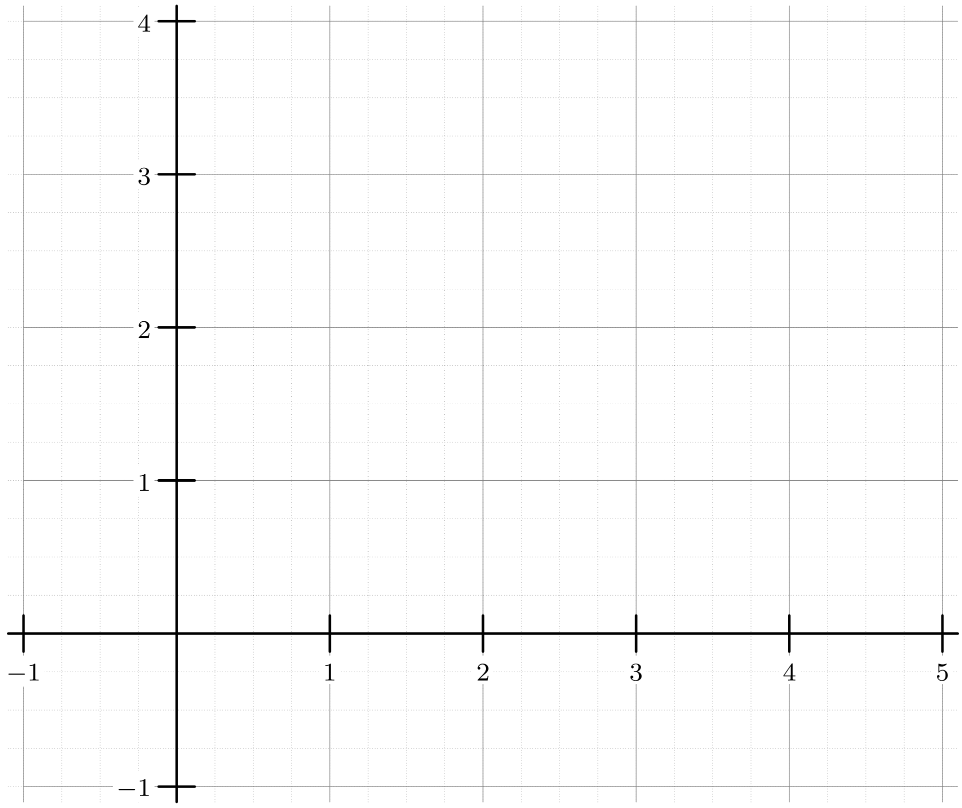


Exercise 1.2.2. An Even Smaller ϵ

- Repeat the above example with $\epsilon = 1/3$. In order to guarantee the y values are within $\epsilon = 1/3$ of 3, how close must the x coordinates be kept to 4? (This distance is what we refer to as δ.) Draw a graph that illustrates your claim similar to the ones above.



- Try yet again with $\epsilon = 1/10$.



- Let's record the list of all gathered ϵ and corresponding δ values in a small table.

ϵ	1	1/2	1/3	1/10
δ	2	1		

Describe the relationship between δ and ϵ in words.

- Describe that same relationship between δ and ϵ in symbols. In particular, give a formula for δ in terms of ϵ.

- Does it seem if we make ϵ too small, we may be unable to find a corresponding δ? Or will there always exist some δ no matter how tiny ϵ becomes? Explain.

Cauchy's Definition of a Limit

We now take the above concept and turn it into a formal definition. We imagine a skeptic coming to us and saying

Hey, I bet you can't guarantee that your function values will be **this** *close to the limit!!! (where* **this** $= \epsilon$*)*

The burden of proof is on us. We must demonstrate to the skeptic that there is some number δ such that if x values are chosen within δ of a, the $f(x)$ values will be within ϵ of the limit L. Since smaller ϵ values will typically require smaller δ values, our only reasonable hope is to choose δ to be some function of ϵ.

Since the formal definition is so long, we often use the mathematical shorthands of $\forall$ to mean *for all* and $\exists$ to mean *there exists*. Also recall that for any real numbers a and b, the distance between a and b can be written as $|a - b|$.

Definition 1.2.3. $\epsilon - \delta$ Definition of a Limit

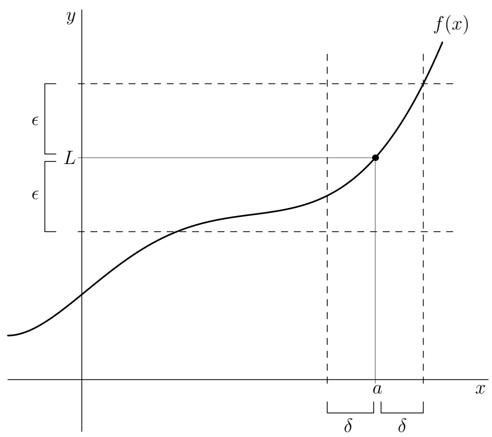
Let $f(x)$ be a function, and let a and L be real numbers. Then

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists \delta, \text{ such that } \forall x \in \mathbb{R}, \text{ if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Since this is quite dense, we unpack the notation and talk through each piece.



- **First Variable Declaration "$\forall \epsilon > 0$":** These symbols say that this definition must hold for all possible choices of positive real numbers ϵ. This can be thought of as ϵ for "ϵrror", as it is specifying how far away $f(x)$ is allowed to be from L.
- **Second Variable Declaration "$\exists \delta > 0$":** Here we state that there must exist a δ for which the following conditions hold. This value will specify how close the x coordinates must be to a.
- **Third Variable Declaration "$\forall x \in \mathbb{R}$, if $0 < |x - a| < \delta$":** This states that the definition must hold for all real numbers x between $a - \delta$ and $a + \delta$, except possibly for $x = a$ itself.

- **Final Condition "$|f(x) - L| < \epsilon$":** This specifies that under the above circumstances the y coordinates of our function must be within ϵ of L.

If a function satisfies the above properties, then we say the limit exists and is equal to L. The process of carefully writing out and verifying the pieces of the definition as listed above is referred to as a $\delta - \epsilon$ proof.

Example 1.2.4. A $\delta - \epsilon$ Proof

Consider again the limit

$$\lim_{x \to 4} \frac{1}{2}x + 1 = 3.$$

We now show that our choice of $\delta = 2\epsilon$ will always work no matter how small ϵ is by writing out a $\delta - \epsilon$ proof.

Proof. Let $\epsilon > 0$, an arbitrary positive real number. Choose $\delta = 2\epsilon$. Let x be a real number and assume

$$0 < |x - 4| < \delta.$$

Under these assumptions, we now wish to demonstrate that the function $f(x) = \frac{1}{2}x + 1$ will be within ϵ of $L = 3$. We now compute:

$$\begin{aligned} |f(x) - L| &= \left|\frac{1}{2}x + 1 - 3\right| \\ &= \left|\frac{1}{2}x - 2\right| \\ &= \frac{1}{2}|x - 4| \\ &< \frac{1}{2}\delta \\ &= \frac{1}{2}(2\epsilon) \\ &= \epsilon \end{aligned}$$

Thus, we have verified that the distance between the function and $L = 3$ is less than ϵ, provided that x is chosen within $\delta = 2\epsilon$ from $a = 4$. By the $\epsilon - \delta$ definition of a limit, we have successfully proven that $\lim_{x\to 4} \frac{1}{2}x + 1 = 3$. □

In the above example, we found δ and the relation to ϵ by just making a table of values and looking at the relationship. Sometimes it is helpful to have a bit more of an algorithmic way to determine δ from ϵ. It can be useful to work backwards and start with the inequality

$$|f(x) - L| < \epsilon$$

and use algebra to turn it into the form

$$|x - a| < g(\epsilon)$$

at which point we can likely choose $\delta = g(\epsilon)$. Still we must write out the proof as shown above, but it sometimes helps establish the relationship between δ and ϵ. We demonstrate this below.

Example 1.2.5. Finding δ without a Data Table

Consider yet again the limit

$$\lim_{x\to 4} \frac{1}{2}x + 1 = 3.$$

We start with the inequality

$$|f(x) - L| < \epsilon$$

and see that in this case it is

$$\left|\frac{1}{2}x + 1 - 3\right| < \epsilon.$$

To turn the left-hand side into our desired $|x - a|$, we must clear the one-half. Thus, we take the inequality and multiply both sides by 2. This produces

$$2\left|\frac{1}{2}x - 2\right| < 2\epsilon$$

which becomes

$$|x - 4| < 2\epsilon.$$

Thus, the expression on the right-hand side should be chosen to be $\delta = 2\epsilon$.

Alright, now you try one!

Exercise 1.2.6. Testing Your Limits ☕☕☕

Consider the limit

$$\lim_{x\to 2} 3x - 4 = 2.$$

- For this limit, what is the relationship between δ and ϵ?

- Write a $\delta - \epsilon$ proof for the limit
$$\lim_{x\to 2} 3x - 4 = 2.$$

Notice that the functions in the above examples were all linear. In each case, the relationship between δ and ϵ was simply

$$\epsilon = \text{slope} \cdot \delta.$$

This will work for the limit of any linear function, as one can see slope as the "stretching factor" between the input error (δ) and the output error (ϵ). In some sense, this is just a restatement of "rise over run". One can think of δ as change in x and ϵ as change in y, which produces the equivalent formulation

$$\text{slope} = \frac{\epsilon}{\delta}.$$

Exercise 1.2.7. Limit of a Line ☕☕☕

Use the principle explained above to write a $\delta - \epsilon$ proof that

$$\lim_{x \to a} mx + b = ma + b.$$

That is, for any linear function, the limit can be computed by just plugging the value for a in for x.

Here is a special case of the above exercise that is worth thinking about on its own.

Exercise 1.2.8. Horizontal Lines ☕☕☕

What if the line is a horizontal line? Does the above argument still hold? Or must something change?

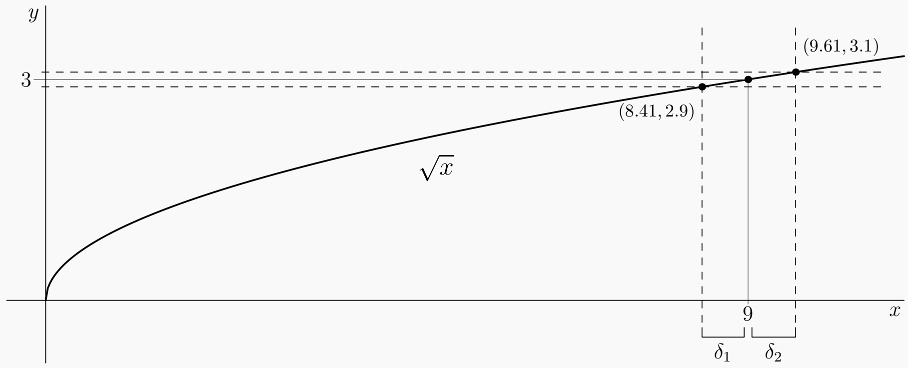
We now consider a nonlinear example, where the relationship between δ and ϵ is not quite so clear.

Example 1.2.9. Back to Our Roots

Recall Example 1.1.4, where we examined the limit

$$\lim_{x\to 9} \sqrt{x} = 3.$$

Suppose we want to find the corresponding δ for $\epsilon = 0.1$. We must find how close the x coordinate must be to 9 in order to guarantee the y coordinate is within one-tenth of 3. What is interesting is that this example lacks the symmetry of the linear examples.



To the left of 9, we have $x = 8.41$ as the first value that has its square root within one-tenth of 3. To the right, we have $x = 9.61$ as the last value that has its square root within one-tenth of 3. Thus, the left-hand side seems to suggest a value of $\delta_1 = 0.59$ whereas the right-hand side seems to suggest a value of $\delta_2 = 0.61$. But which is the correct one? In cases like this, one always wants to choose the *minimum* of the two values to be the correct δ. If we pick the larger, we might have some x values that are within δ of 9 but not within ϵ of 3. However, the smaller value will restrict us to an interval that has corresponding y values that are all within $\epsilon = 0.1$ of 3.
Thus, $\delta = 0.59$ is a correct choice for δ.

Exercise 1.2.10. Seeing Why the Larger δ Fails ☕

Label the graph above to demonstrate why the point $x = 8.4$ is within $\delta_2 = 0.61$ of 9, but the corresponding y coordinate $\sqrt{8.4}$ is not within $\epsilon = 0.1$ of 3.

One place where $\epsilon - \delta$ style arguments come up is in the uncertainty of experimental measurements. If one wishes to have a result with a particular level of uncertainty, there must be a sufficiently small uncertainty in the measurement used for the calculations.

Exercise 1.2.11. Lab Measurements ☕☕

Suppose we need a disc of radius 2 cm whose area is within 0.01 cm^2 of $A = \pi 2^2$. If we had a perfect disc of radius 2 cm, the area would be exactly 4π, but in reality of course it is impossible to have exactly 2 cm for our radius. So perhaps a more accurate way to represent the situation is the following:

As the radius of the circle approaches the value 2, the area will approach 4π.

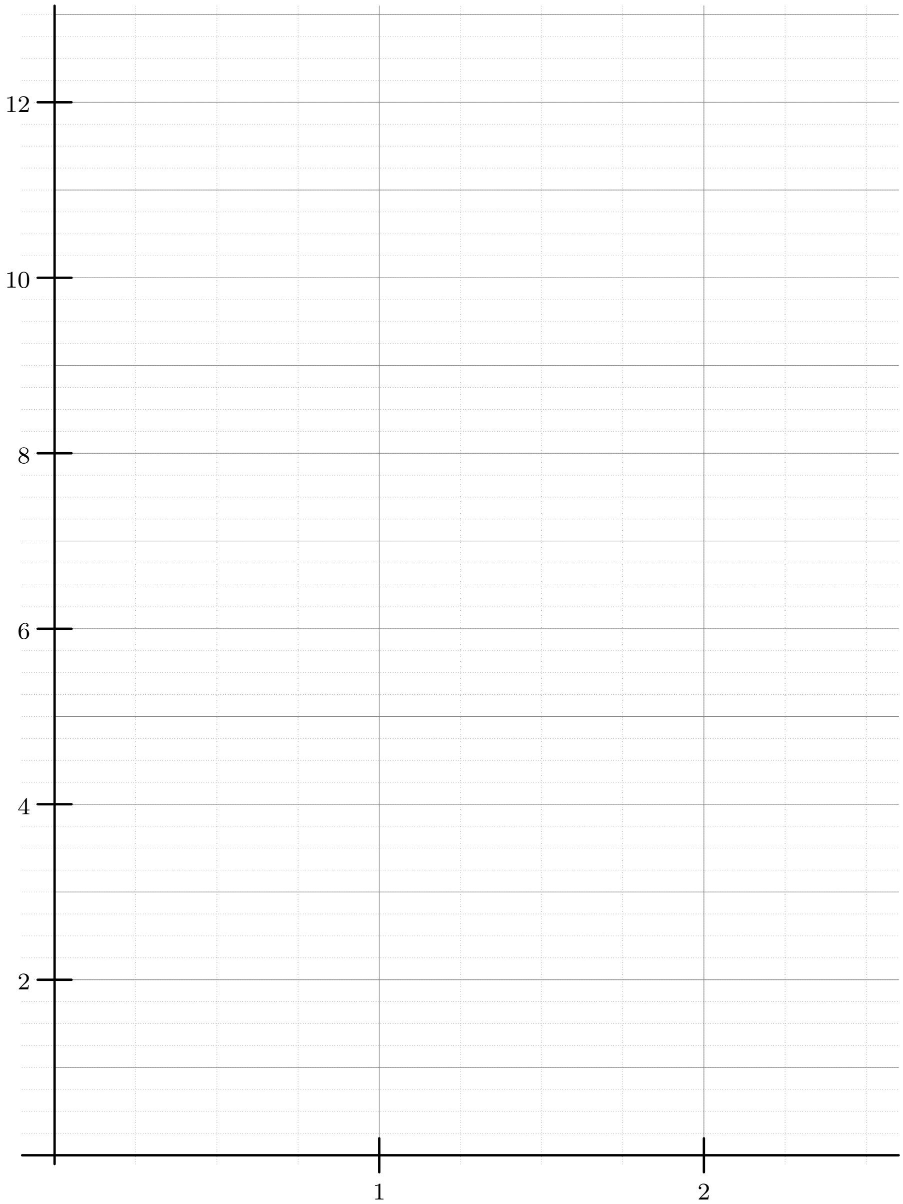
This sentence can then be rewritten in the language of limits and functions. Define the variable r to be the radius of the circle and the function $A(r) = \pi r^2$ to be the area of the circle. Then the sentence above becomes the following mathematical expression:

$$\lim_{r \to 2} A(r) = 4\pi.$$

At this point, we can recast the whole problem as finding a δ for a corresponding ϵ. In particular, we have

- The function $A(r) = \pi r^2$
- The limit $L = 4\pi$
- The input being approached, $a = 2$
- The ϵ value, $\epsilon = 0.01$

Given this setup, draw a graph of the function, the limit, and the ϵ envelope surrounding the line $y = 4\pi$. Use the x axis to represent r, the radius, and the y axis to represent A the area. Find and label the points of intersection with the graph to find the δ value for this situation.



Write a short sentence that reinterprets your δ value as an instruction regarding how precisely the radius of the circle must be measured to achieve the desired level of precision in the circle area.

Limits to Infinity

We would like to establish the same level of rigor for limits to infinity. We can still use ϵ to represent how close we would like the function values to be to L. However, there is no longer an a value to be within δ of. Instead, we will pick some cutoff value N, where $f(x)$ will be within ϵ of L as long as x is greater than N.

Exercise 1.2.12. If You Take Half of a Half of a Half of ...

Consider the function

$$f(x) = \frac{1}{2^x}.$$

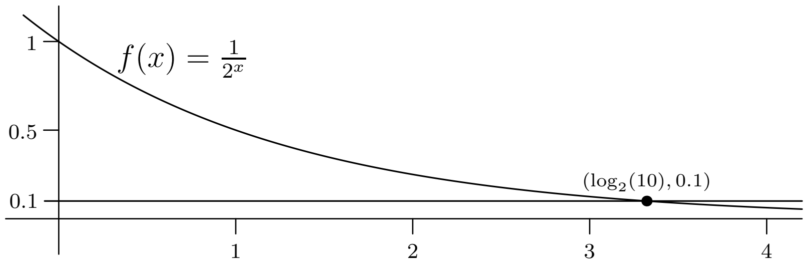
We construct a small input-output table to get a sense of what happens as x approaches ∞.

x	0	1	2	3	4	$\cdots$	∞
$f(x)$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\cdots$	0

From the numbers, we get the impression that the function approaches zero as x becomes larger. Let us begin to analyze this claim with the value $\epsilon = 0.1$. In particular, we ask, "How large would x have to be to guarantee $1/2^x$ is less than one-tenth?" We can solve for this cutoff value using algebra.

$$\begin{aligned} \frac{1}{2^x} &= 0.1 \\ 10 &= 2^x \\ \ln(10) &= \ln(2^x) \\ \ln(10) &= x \ln(2) \\ \frac{\ln(10)}{\ln(2)} &= x \end{aligned}$$

Since the function $f(x) = \frac{1}{2^x}$ is strictly decreasing, as long as x is larger than $\log_2(10)$, we will have $\frac{1}{2^x} < 0.1$.



- How large would x have to be to guarantee $f(x)$ is no more than one-hundredth from zero?

- How large would x have to be to guarantee $f(x)$ is no more than one-thousandth from zero?

Exercise 1.2.13. Log Exercises ☕

In the above argument, we were discussing the quantity $\ln(10)/\ln(2)$ as being the cutoff value for x, but then later changed to talking about $\log_2(10)$. What happened?

No matter how small of a measurement we choose (one-tenth, one-hundredth, one-thousandth, etc), we could always find that after a certain point, all of our sequence terms are no further than that measurement from zero. This is exactly the notion we will reformulate in a more formal manner to define a limit to infinity.

Definition 1.2.14. Limit to ∞

We say the function $f(x)$ converges to a limit $L \in \mathbb{R}$ as x goes to ∞ and write

$$\lim_{x\to\infty} f(x) = L$$

if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{R}, \forall x \in \mathbb{R}, \text{ if } x > N, \text{ then } |f(x) - L| < \epsilon.$$

This complicated definition can be unwound into a to-do list for what one must do to prove that a function converges to a particular limit. In particular, to show that the limit of $f(x)$ as x goes to ∞ is equal to a number L, one must:

- Let ϵ be an arbitrary positive real number.
- Choose N, typically defined as a function of ϵ, since smaller values of ϵ will usually require a larger N to be chosen.
- Let x represent an arbitrary real number greater than N.
- Using the definition of N and the assumption that $x > N$, prove that any corresponding $f(x)$ satisfies $|f(x) - L| < \epsilon$.

Figuring out exactly what N should be in terms of ϵ usually requires a bit of algebra before the proof is written up. If the formula for $f(x)$ is clean enough, you might be able to just work backwards from the inequality $|f(x) - L| < \epsilon$. If you solve it for x, you will find an expression that x must be larger than. Note here we are essentially just finding an inverse function for $f(x)$, but with the mindset of turning one inequality into the other.

Example 1.2.15. Solving for N

Let us solve for N with regards to our function $f(x) = \frac{1}{2^x}$. Since here we suspect $L = 0$, we solve for x in the following inequality:

$$\begin{aligned}
\left|\frac{1}{2^x} - 0\right| &< \epsilon \\
\frac{1}{2^x} &< \epsilon \\
\frac{1}{\epsilon} &< 2^x \\
\ln\left(\frac{1}{\epsilon}\right) &< \ln\left(2^x\right) \\
\ln\left(\frac{1}{\epsilon}\right) &< x\ln(2) \\
\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)} &< x
\end{aligned}$$

Thus we determined our choice of N, namely

$$N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}.$$

Exercise 1.2.16. Justifying Our Work

In words, annotate the above example to indicate why each line follows from the previous.

Now that we found our value for N, we are ready to follow the steps described above and construct our proof.

Example 1.2.17. Writing an $N - \epsilon$ Proof

Prove that

$$\lim_{x\to\infty} \frac{1}{2^x} = 0$$

Proof. Let ϵ be an arbitrary positive real number. Choose $N = \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}$. Let x be a real number such that $x > N$. Under these circumstances, we wish to show that a corresponding $f(x)$ will be less than ϵ away from 0. Proceeding:

$$\begin{aligned}\left|\frac{1}{2^x}-0\right| &= \frac{1}{2^x}\\ &< \frac{1}{2^N}\\ &< \frac{1}{2^{\left(\frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(2)}\right)}}\\ &= \frac{1}{2^{\left(\log_2\left(\frac{1}{\epsilon}\right)\right)}}\\ &= \frac{1}{\frac{1}{\epsilon}}\\ &= \epsilon.\end{aligned}$$

Thus, for inputs x that are larger than our choice of N, the corresponding values of our function $f(x)$ are less than ϵ away from zero as desired. □

Exercise 1.2.18. Justifying Our Work ☕

Once again in words, annotate the above example to indicate why each line follows from the previous. Pay particular attention to identify where we used the starting assumption that $x > N$.

As this course does not go through a general treatment of what constitutes a proof or how to come up with one, the example above could be taken as a template for how an $N - \epsilon$ proof should be written. In a more in-depth study of analysis, you will encounter more complicated situations where the above template may be too simplistic. It will be expanded upon when you have the right tools! For now, follow the above proof template.

Exercise 1.2.19. Verifying a Limit ☕☕☕

Consider the function given by the following formula:

$$f(x) = \frac{2x}{x+1}.$$

- List the terms of the sequence corresponding to $x = 1$, $x = 10$, $x = 100$, and $x = 1000$. What do the terms appear to be converging to as x goes to ∞?

- If you choose $\epsilon = 0.1$, what would the corresponding N be?

- If you choose $\epsilon = 0.05$, what would the corresponding N be?

- Write an $N - \epsilon$ proof that verifies your conjectured limit above is correct.

Exercise 1.2.20. Writing $N - \epsilon$ Proofs ☕☕☕

Write $N - \epsilon$ proofs for each of the following limits:

- $\lim_{x\to\infty} \frac{x}{3x+1} = \frac{1}{3}$

- $\lim_{x\to\infty} \sqrt{9+1/x} = 3$

Nonexistence of a Limit

We now consider a famous pathological example, the function

$$f(x) = \sin\left(\frac{1}{x}\right).$$

It is a good example to have in your pocket; when you want to see if an idea makes sense, try it out on friendly functions like polynomials first. Eventually though, see how it works on the above function!

Exercise 1.2.21. Sine of One over x ☕☕☕

- Fill out the data table below.

x	0.1	0.01	0.001	0.0001
$\sin\left(\frac{1}{x}\right)$				

Based on your values, what does it appear $\sin\left(\frac{1}{x}\right)$ is converging to as x approaches 0 from the right?

- Verify that the input

$$x = \frac{1}{2\pi n + \frac{\pi}{2}}$$

will produce an output of 1 for all $n \in \mathbb{N}$.

- Verify that the input

$$x = \frac{1}{\pi n}$$

will produce an output of 0 for all positive $n \in \mathbb{N}$.

- Use the two above facts to explain why for $\epsilon = 1/2$, there is no δ that will guarantee all function values will be within ϵ of a limit L. Draw a graph that illustrates your answer.

- Since the definition of limit must work for all ϵ, and it failed for $\epsilon = 1/2$, what can we conclude about $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$?

Properties of Limits

As we find more limits, we desire to use old limits that we have already found to find new limits. For our first theorem on limits, we will need a famous property of inequalities called the *triangle inequality*.

Exercise 1.2.22. Triangle Inequality ☕☕

Let A and B be real numbers. We wish to compare the quantity

$$|A+B|$$

to the quantity

$$|A|+|B|.$$

We will consider these quantities in the five cases listed below. In each case, determine if the quantities are equal, or if not then which one is greater. Explain why in each case!

- If at least one of A or B is zero, how do the quantities compare?
- If both are nonzero, we have four subcases.
  1. How do they compare if both A and B are positive?
  2. How do they compare if both A and B are negative?
  3. How do they compare if A is positive and B is negative?
  4. How do they compare if A is negative and B is positive?

Based on the above, explain why the following statement is correct for all real numbers A and B:

$$|A+B| \leq |A|+|B|.$$

Theorem 1.2.23. Limits Distribute over Addition and Multiplication

Let a be a real number, ∞, or $-\infty$. Let $f(x)$ and $g(x)$ be functions. Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are equal to finite real numbers. Under these conditions, the following hold:

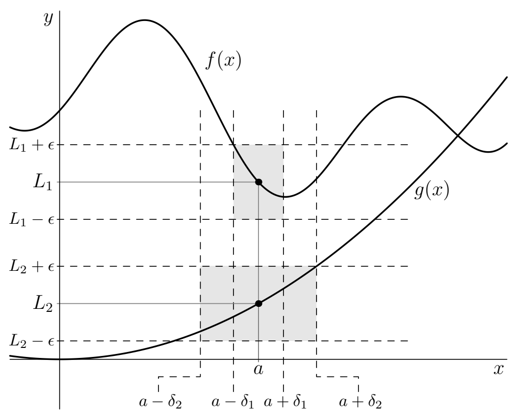
- **Addition:** The $\lim_{x\to a}(f(x)+g(x))$ exists, and furthermore
$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$
- **Multiplication:** The $\lim_{x\to a}(f(x)\cdot g(x))$ exists, and furthermore
$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

Proof. Here we prove the addition property listed above. The proof of multiplication is a similar idea but messier.

Let L_1 and L_2 be the real numbers such that
$$\lim_{x\to a} f(x) = L_1$$
and
$$\lim_{x\to a} g(x) = L_2.$$

We wish to prove that
$$\lim_{x\to a}(f(x)+g(x)) = L_1 + L_2.$$
To do this, we need to show that no matter how small of an ϵ we select, we can restrict the function $f(x)+g(x)$ to some δ radius of a such that any x value in that interval will have a corresponding y value within ϵ of L_1+L_2. We proceed down this path!

Let $\epsilon > 0$, an arbitrary positive real number. Since the limits $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, we can find a δ that will work for any desired error tolerance with regards to those functions. In particular, we use $\epsilon/2$. Thus, there exist positive real numbers δ_1 and δ_2 such that for all $x \in \mathbb{R}$,
$$0 < |x-a| < \delta_1 \implies |f(x) - L_1| < \epsilon/2$$
and
$$0 < |x-a| < \delta_2 \implies |g(x) - L_2| < \epsilon/2.$$



Let δ be the smaller of the two values δ_1 and δ_2 (and pick either if they are equal). Another way to say this is to write it with the *min* function, meaning the minimum of the values. That is,

$$\delta = \min(\delta_1, \delta_2).$$

The advantage of choosing the minimum is that it restricts us to an interval on which both of the above inequalities are simultaneously satisfied. Specifically,

$$0 < |x - a| < \delta \implies |f(x) - L_1| < \epsilon/2 \text{ and } |g(x) - L_2| < \epsilon/2.$$

Let x be an arbitrary real number that is within δ of a. We now combine those two inequalities using the triangle inequality to measure the distance from $f(x) + g(x)$ to $L_1 + L_2$ for that arbitrary x. Specifically, we have

$$\begin{aligned}
|(f(x) + g(x)) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)| \\
&< |f(x) - L_1| + |g(x) - L_2| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, we have demonstrated that it is possible to find a δ that will force $f(x)+g(x)$ within ϵ of L_1+L_2 no matter how small it is chosen! In conclusion,

$$\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

□

Exercise 1.2.24. Reading the Proof ☕

- In the above proof, identify which line the triangle inequality was used on. What is A? What is B?

- Briefly explain why it would not have worked to select δ as the maximum of δ_1 and δ_2 rather than the minimum.

Example 1.2.25. Limit of a Quadratic Function

Consider the limit

$$\lim_{x\to 3} 2x^2 - x - 1.$$

By Exercise 1.2.7, we have the following three limits:

- $\lim_{x\to 3} 2x - 1 = 5$
- $\lim_{x\to 3} x = 3.$
- $\lim_{x\to 3} -1 = -1.$

By Theorem 1.2.23, we can distribute limits across multiplication. In particular,

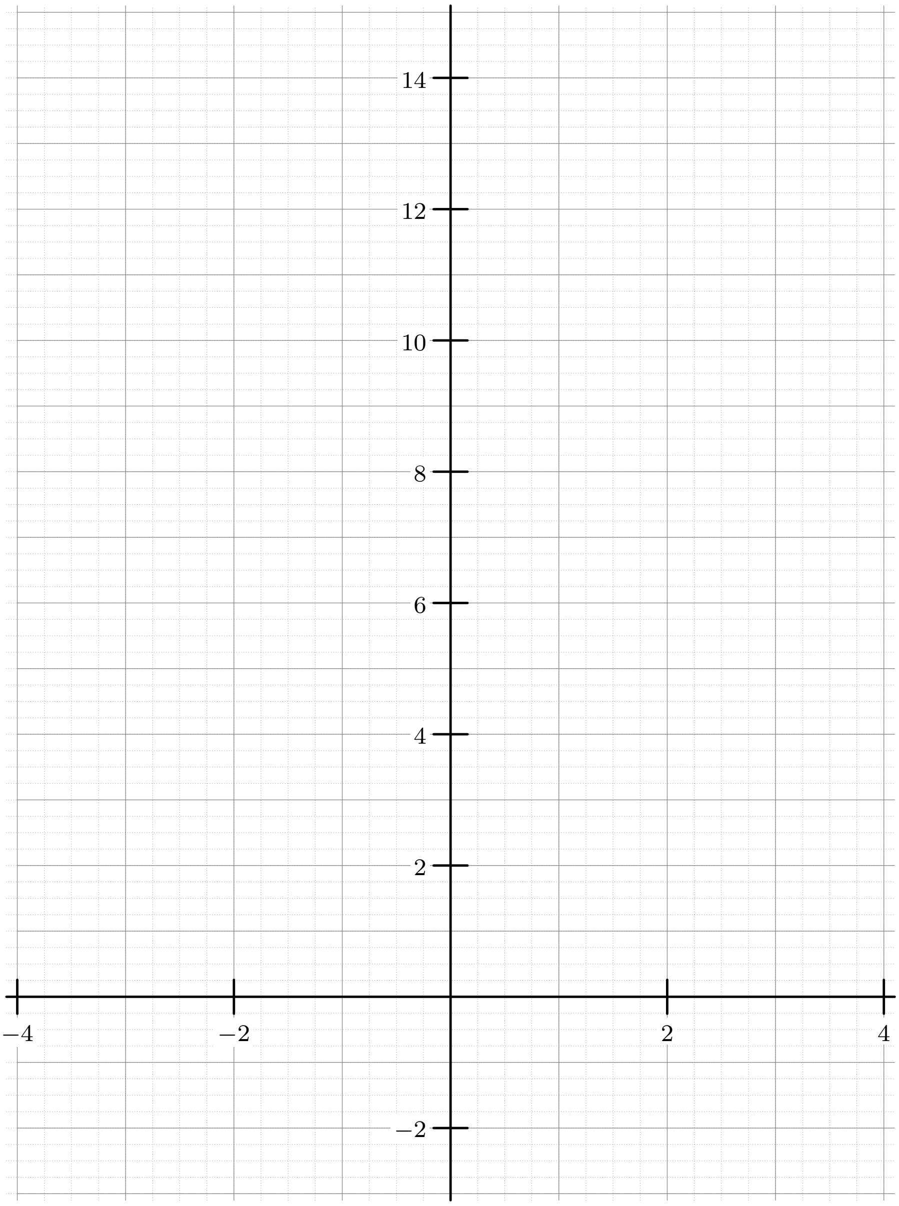
$$\begin{aligned}\lim_{x\to 3} 2x^2 - x &= \lim_{x\to 3}(x)(2x-1)\\ &= \lim_{x\to 3}(x)\cdot\lim_{x\to 3}(2x-1)\\ &= 3\cdot 5\\ &= 15.\end{aligned}$$

Again by Theorem 1.2.23, we can distribute limits across addition. In particular,

$$\begin{aligned}\lim_{x\to 3} 2x^2 - x - 1 &= \lim_{x\to 3}\left(\left(2x^2 - x\right) + (-1)\right)\\ &= \lim_{x\to 3}\left(2x^2 - x\right) + \lim_{x\to 3}(-1)\\ &= 15 + -1\\ &= 14.\end{aligned}$$

Exercise 1.2.26. Checking with the Graph

Graph the function from the above problem, $f(x) = 2x^2 - x - 1$. Include labels for all important features of the graph, including x- and y-intercepts. Verify that the limit we computed using limit properties above is the same as the y-coordinate above the value $x = 3$ on the graph.



Exercise 1.2.27. A Cubic ☕☕

Follow the method of Exercise 1.2.25 to calculate

$$\lim_{x \to -1} x^3 + 1.$$

Be sure to clearly state what linear functions you are piecing together and how!

Exercise 1.2.28. Polynomials ☕☕

Use the method of the above examples to explain why the limit of any polynomial can always just be computed by plugging the value of a in for x. That is, justify the claim that

$$\lim_{x \to a} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = a_n a^n + a_{n-1} a^{n-1} + \cdots + a_2 a^2 + a_1 a + a_0$$

where $a, a_0, a_1, a_2, \ldots, a_n$ all represent arbitrary real numbers.

1.3 The Sandwich and Monotone Convergence Theorems

In this section we continue our analytic treatment of limits by presenting two very important theorems regarding limits. The tldr (but please do read) on the theorems is as follows:

- **Sandwich Theorem** If a gross function is trapped between two clean functions, and the two clean functions approach the same limit, then the gross function (which is sandwiched in the middle, hence the name) does as well. This theorem is used to prove two very important limits involving sine and cosine!
- **Monotone Convergence Theorem** This says that if a function is strictly increasing, but bounded above, its limit exists. Similarly, if a function is strictly decreasing, but bounded below, its limit exists.

The Sandwich Theorem

Here is a strange way to obtain new limits from old limits, not with equality but rather with inequality! Many sources call it the *Squeeze Theorem*, but I like food. So here is the...

Theorem 1.3.1. Sandwich Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be functions. Let c and d be real numbers with $c < d$. Suppose the inequality

$$f(x) \leq g(x) \leq h(x)$$

holds for all $x \in (c, d)$. Let $a \in (c, d)$ and let L represent a real number, ∞, or $-\infty$. Furthermore, suppose

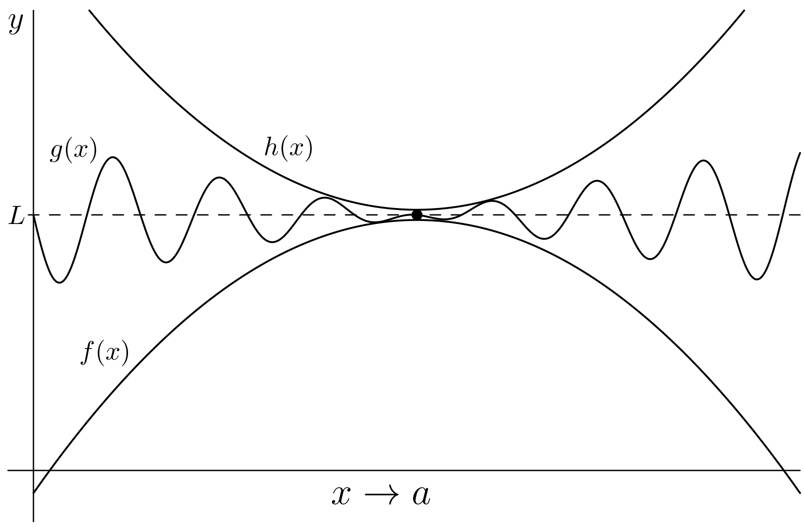
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then

$$\lim_{x\to a} g(x) = L$$

as well.

In short, the Sandwich Theorem says that if one function is trapped between two other functions that have the same limit L, then the middle function must also converge to L, because it's being sandwiched towards L by the top and bottom functions.



Note that the Sandwich Theorem also works for one-sided limits (in which case the inequality only needs to hold for x values on the corresponding side of a) and limits to infinity. Below, we show a diagram illustrating the case where

$$f(x) \leq g(x) \leq h(x)$$

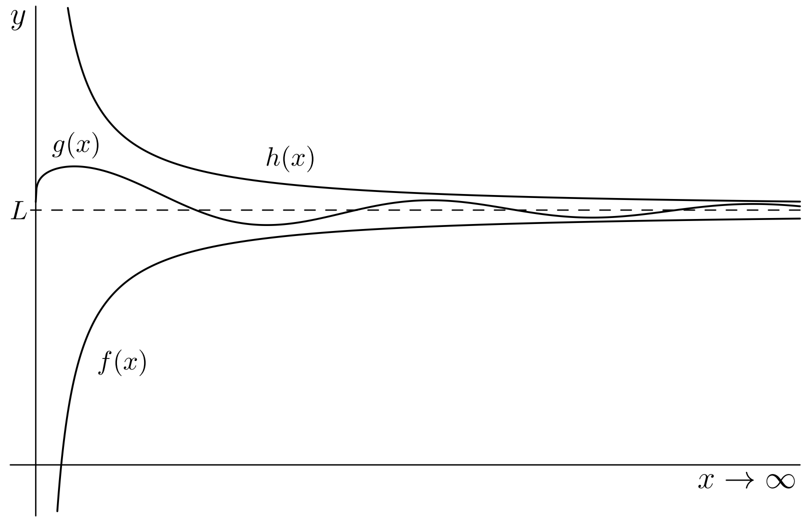
for all $x > 0$, and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = L.$$

By the Sandwich Theorem, (modified appropriately for infinite limits) we conclude that

$$\lim_{x\to\infty} g(x) = L$$

as well.



Exercise 1.3.2. Ordering a Sine Sandwich with Linear Bread ☕

Consider the limit

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right).$$

Explain why we cannot evaluate this limit by distributing the limit across the multiplication as Theorem 1.2.23 tells us we can.

Example 1.3.3. Eating Half of a Sine Sandwich with Linear Bread

Again consider the limit

$$\lim_{x\to 0^+} x \sin\left(\frac{1}{x}\right).$$

We apply the Sandwich Theorem with the following declarations

$$h(x) = x$$
$$g(x) = x \sin\left(\frac{1}{x}\right)$$
$$f(x) = -x.$$

Since the sine function is always between -1 and 1, we have that

$$-1 \leq \sin(x) \leq 1$$

which for $x > 0$ can be manipulated to become

$$-x \leq x\sin(x) \leq x.$$

We know

$$\lim_{x\to 0+} -x = \lim_{x\to 0+} x = 0$$

by Exercise 1.2.7. By the Sandwich Theorem,

$$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$$

as well.

Exercise 1.3.4. Eating the Other Half of a Sine Sandwich with Linear Bread

- Modify the above argument to show that

  $$\lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = 0$$

  as well.

- What does this let you conclude about $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$?

Special Limits for Sine and Cosine

The following special limits will be needed for our chapter on derivatives.

Formula 1.3.5. Special Limits for Trigonometric Functions

The following limits hold:

- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$

Exercise 1.3.6. Small Angle Approximation for Sine ☕

The special limit for sine, rewritten with θ in place of x, would say

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1,$$

can be interpreted in another way:

For small angles θ, the value of $\sin(\theta)$ is very close to θ itself.

After all, the limit says that the two quantities' ratio approaches 1, and the ratio of two quantities is 1 if and only if the quantities are equal. Let us observe this a bit with some data. Calculate the following values on a calculator or computer algebra system:

- $\sin(0.01)$
- $\sin(0.005)$
- $\sin(0.0006214)$

What do you notice? Do you observe the claimed sine special limit in these numbers? Explain.

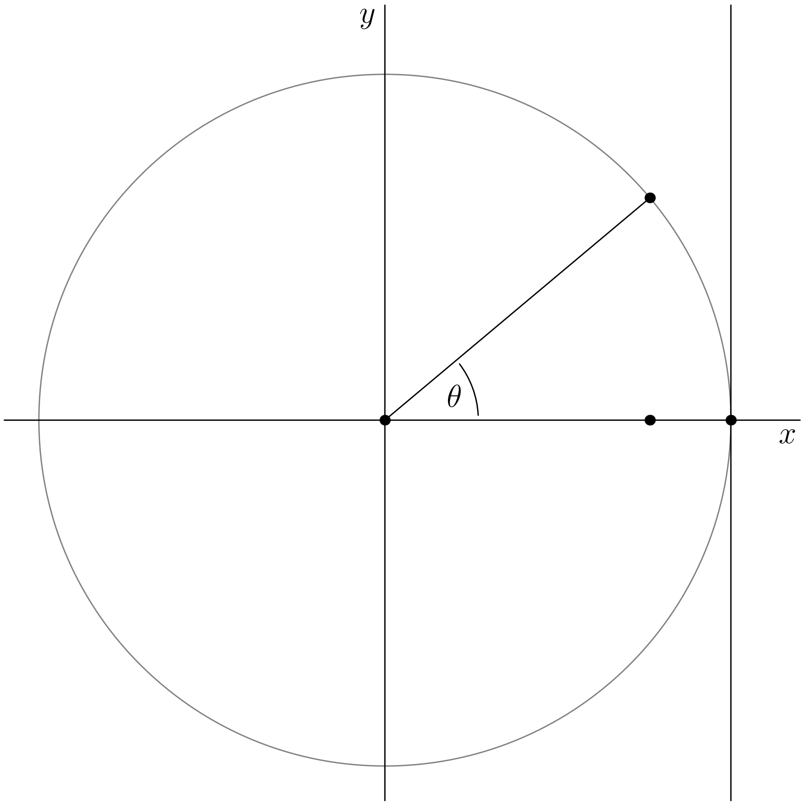
We justify both of these below using the Sandwich Theorem!

Exercise 1.3.7. Special Limit for Sine ☕☕☕

Consider the angle θ, an arbitrary small but positive angle, on the unit circle shown below. Label the following points, segments, and arcs:

- Call the origin A.
- Label the arc of length θ on the unit circle. (Recall that for radian measures on the unit circle, the measure of the angle is equal to the length of the arc it subtends.)
- Label the point $(\cos(\theta), \sin(\theta))$ as B.
- Label the point $(\cos(\theta), 0)$ as C.

- Label the point $(1, 0)$ as E.
- Extend $\overline{AB}$ to meet the line $x = 1$. Label the point of intersection as D.



We now reason our way through this diagram to prove the inequalities we need for the Sandwich Theorem. Since we are considering the positive θ case, we are technically just computing the right-hand limit, but the other side follows similarly.

- First, explain why the arc between points B and E is longer than $\overline{BC}$. Use this relationship to show that $\sin(\theta) \leq \theta$.

- Second, explain why $\triangle ABC$ is similar to $\triangle AED$.

- Use this similarity of triangles to show the measure of $\overline{DE}$ is equal to $\tan(\theta)$.

- Explain why $\tan(\theta)$ must be longer than θ. (**Hint:** Draw the tangent line to the circle at point B.)

- From this inequality, deduce $\theta \cdot \cos(\theta) \leq \sin(\theta)$.

- Explain why we now know that

$$\theta \cdot \cos(\theta) \leq \sin(\theta) \leq \theta.$$

From this, deduce that

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1.$$

- Apply the Sandwich Theorem to compute the limit of $\sin(\theta)/\theta$ as $\theta \to 0^+$.

We use the above inequality for sine and the Pythagorean Identity

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

to deduce the special limit for cosine. Note that this is a common trick; often once you figure something out for one of sine or cosine, you can transfer the information over to the cofunction via the Pythagorean Identity!

Exercise 1.3.8. And Now, Cosine ☕☕☕

Again, consider a small unspecified positive θ value.

- Explain why $0 \leq \frac{1-\cos(\theta)}{\theta}$. Don't work too hard; think about the range of cosine!

- Explain why $\sin(\theta) = \sqrt{1-\cos^2(\theta)}$.

- Recall the key inequality from the previous problem. Specifically, consider

$$\frac{\sin(\theta)}{\theta} \leq 1.$$

Manipulate this inequality according to the following steps:

  - Replace $\sin(\theta)$ in the inequality by $\sqrt{1-\cos^2(\theta)}$.

  - Square both sides.

  - Factor $1-\cos^2(\theta)$ using the difference of two squares formula.

  - Divide both sides by $1+\cos(\theta)$.

  - Explain why $\frac{\theta}{1+\cos(\theta)} \leq \theta$. Again, don't work too hard, just think of what kind of values $\cos(\theta)$ could be.

- Conclude that $\frac{1-\cos(\theta)}{\theta} \leq \theta$.

- Put all of the above information together to construct a Sandwich Theorem argument for the special limit
$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0.$$

The Monotone Convergence Theorem

Let us start by defining the word *monotone*.

Definition 1.3.9. An Increasing Function

Let a and b be real numbers with $a < b$. A function f is *increasing* on an interval (a, b) if and only if for all $x_1, x_2 \in (a, b)$,
$$x_1 < x_2 \implies f(x_1) \leq f(x_2).$$

It should be noted that the term above sometimes is called "weakly increasing", and the word "increasing" is instead taken to mean that under the same conditions, $f(x_1) < f(x_2)$ (which in turn sometimes gets called "strictly increasing"). Some don't like the way we defined it above because it means constant functions are increasing. When in doubt, just check to see if the source you are reading or talking to is using the word increasing to mean weakly or strictly. (We are going with weakly here.)

Exercise 1.3.10. An Example from Trig ☕

- Graph $f(x) = \sin(x)$ on the interval $(0, \pi/2)$. Is it increasing? Explain why or why not.
- Graph $f(x) = \sin(x)$ on the interval $(0, \pi)$. Is it increasing? Explain why or why not.

In each case, if the function is not increasing, you should be able to provide specific numerical examples of points x_1 and x_2 in the given interval such that $x_1 < x_2$ but $f(x_1) \geq f(x_2)$.

Wait, that wasn't the word we were trying to define. But actually, first...

Exercise 1.3.11. Definition of Decreasing ☕☕

Use the above definition of *increasing* to create an analogous definition for the word *decreasing*.

Ok now we're ready to define the word we're trying to define.

Definition 1.3.12. Monotone

If a function is either increasing or decreasing on an interval, we say it is *monotone* on that interval.

Exercise 1.3.13. Our Friend Sine Again ☕

Draw the graph of $f(x) = \sin(x)$. For each of the following intervals, decide whether or not it is monotone. In each case, support your answer with a graph.

- $(-\pi/2, \pi/2)$
- $(0, \pi)$
- $(\pi/2, 3\pi/2)$

Now that we have the word monotone, we are ready to state our theorem of interest!

Theorem 1.3.14. Monotone Convergence Theorem

Let $f(x)$ be a monotone function on the interval (a, ∞) for some real number a. If $f(x)$ is bounded (i.e., never crosses below $y = b_1$ nor above $y = b_2$ for some real numbers b_1 and b_2), then

$$\lim_{x\to\infty} f(x)$$

exists. That is, $\lim_{x\to\infty} f(x) = L$ for some real number L.

Example 1.3.15. Arctangent

The function $f(x) = \arctan(x)$ is monotone on $(0, \infty)$, since it is increasing there. Also, it never goes above $y = \pi/2$ and never below $y = -\pi/2$, and therefore is bounded. Thus, we can apply the Monotone Convergence Theorem to conclude that

$$\lim_{x\to\infty} \arctan(x)$$

exists. (And in fact it is equal to $\pi/2$, but the theorem just implies that it exists as a real number, though it does not tell you what that number is.)

Exercise 1.3.16. Revisiting a Limit

Recall the limit from Exercise 1.2.12, namely $\lim_{x\to\infty} \frac{1}{2^x}$.

- Explain why the function $f(x) = \frac{1}{2^x}$ is bounded on $(0, \infty)$.
- Explain why that same function is monotone on $(0, \infty)$.
- What can one conclude about $\lim_{x\to\infty} \frac{1}{2^x}$ from the Monotone Convergence Theorem?

Exercise 1.3.17. Cases in Which MCT Does Not Apply ☕☕

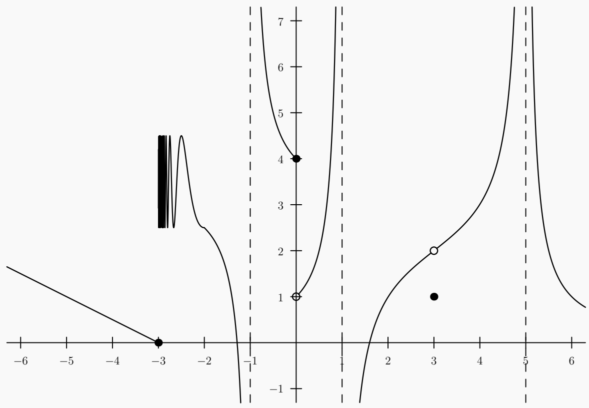
- Consider the limit
$$\lim_{x\to\infty} \sin(x).$$
Why can one not conclude that this limit exists by MCT?
- Consider the limit
$$\lim_{x\to\infty} x^2.$$
Why can one not conclude that this limit exists by MCT?

1.4 Thinking of Continuity Graphically

One of the key ideas built out of limits is the notion of continuity. However, we will first do an intuitive pass on the idea of continuity, and define it more rigorously in terms of limits in the following section! Intuitively, *continuity* is just the idea of a graph being able to be drawn without picking up your pen. If there are no holes or gaps on some interval, then the graph is continuous on that interval. We call these holes or gaps *discontinuities*.

Example 1.4.1. Determining Continuity from a Graph

We revisit the graph of the function $f(x)$ from Example 1.1.3.



The function $f(x)$ is continuous on the set

$$D = (-\infty, -3) \cup (-3, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 3) \cup (3, 5) \cup (5, \infty).$$

It is discontinuous at $-3, -1, 0, 1, 3$, and 5.

We won't formally prove the next theorem, but we will state it and use it. All of the functions from your College Algebra/Precalculus courses that appeared to be continuous, are continuous, at least on the domains on which they appear to be.

Theorem 1.4.2. Continuous Functions

Let b be a positive real number. Let all a_i and b_i be real numbers. Let m and n be natural numbers. The functions below are continuous on the corresponding domains D.

Function Name	Formula	Domain of continuity D
Polynomials	$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$	$\mathbb{R}$
Rational Functions	$f(x) = \frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$	$\mathbb{R}$ without denominator zeros
Absolute Value	$f(x) = \lvert x \rvert$	$\mathbb{R}$
Sine	$f(x) = \sin(x)$	$\mathbb{R}$
Arcsine	$f(x) = \arcsin(x)$	$[-1, 1]$
Cosine	$f(x) = \cos(x)$	$\mathbb{R}$
Arccosine	$f(x) = \arccos(x)$	$[-1, 1]$
Tangent	$f(x) = \tan(x)$	$\mathbb{R}$ except for the zeros of cosine
Arctangent	$f(x) = \arctan(x)$	$\mathbb{R}$
Exponential functions	$f(x) = b^x$	$\mathbb{R}$
Logarithmic functions	$f(x) = \log_b x$	$(0, \infty)$
Even Radicals	$f(x) = \sqrt[n]{x}$, n even	$[0, \infty)$
Odd Radicals	$f(x) = \sqrt[n]{x}$, n odd	$\mathbb{R}$

Exercise 1.4.3. Seeing the Graphs

Sketch a rough graph for each of the functions given in the table above. For functions that are actually a giant family of functions (like rational functions) just pick a simple representative (like $1/x$). Confirm that the graph looks continuous on the specified intervals.

Exercise 1.4.4. You're Gonna Need a Bigger Definition

Perhaps using a graphing utility, graph the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if} x = 0. \end{cases}.$$

Why is it hard to determine whether or not it is continuous based on our intuitive definition of continuity? Reference your work in Exercise 1.3.2 in your explanation!

1.5 Definition and Properties of Continuity

In the previous section, we gave an informal definition of continuity that allowed us to decide if a function was continuous on a given set. However, the language of limits allows us to state a more formal definition that allows us to decide whether a function is continuous or discontinuous at a given point!

Limit Definition of Continuity

Definition 1.5.1. Limit Definition of Continuity

- **Continuous at a Point:** Let $f(x)$ be a function. Then we say f is *continuous at a* if and only if the following are true:
  - $f(a)$ exists.
  - $\lim_{x\to a}$ exists.
  - $\lim_{x\to a} f(x) = f(a)$.
- **Continuous on a Set:** Let D be a subset of the real numbers. If f is continuous for every $a \in D$, we say f *is continuous on* D. If D has left- or right-hand endpoints, we allow continuity to be tested using the corresponding one-sided limit rather than two.
- **Plain Old Continuous:** If f is continuous on $\mathbb{R}$, we say that f is a continuous function or f is *continuous everywhere*.

Example 1.5.2. Translating an Old Example

Recall Exercise 1.2.7. At the time, we simply said that limits of linear functions could be evaluated by plugging the number a in for x. But, we can now restate this using the language of continuity. Let f be a linear function

$$f(x) = mx + b.$$

Note first that for any a in $\mathbb{R}$, $f(a)$ and $\lim_{x\to a} f(x)$ exist. If we wish to calculate the limit at a, we also have

$$\lim_{x\to a} f(x) = ma + b = f(a).$$

Since this works for every a in the real numbers, we can say f is continuous everywhere. Thus, every linear function is continuous everywhere.

Exercise 1.5.3. Polynomials are Continuous ☕

Can we say the same thing about polynomial functions? In particular, let $f(x)$ be a polynomial function, and check the following:

- Does $f(a)$ exist for all a in $\mathbb{R}$?

- Does $\lim_{x\to a} f(x)$ exist for all a in $\mathbb{R}$? Why or why not?

- Does $\lim_{x\to a} f(x) = f(a)$ for all a in $\mathbb{R}$? Why or why not?

Exercise 1.5.4. Settling an Unresolved Question ☕☕

Consider again the question from Exercise 1.4.4. Use our new upgraded formal definition of continuity to settle the previously unsettled issue we had: where is that function continuous?

Continuity as Commutativity

Here is an equivalent and often useful way to think about the limit definition of continuity.

If a function is continuous, you are allowed to swap the limit with the function.

Continuity is in a sense a type of commutative property; if you can interchange the order of lim and f, then it is continuous. This switch of perspective amounts to just adding one extra step in our limit definition of continuity, namely

$$\lim_{x\to a} (f(x)) = f(a) = f\left(\lim_{x\to a} x\right).$$

Note that the last equality holds by Exercise 1.2.7, since x itself is a linear function, so we can evaluate $\lim_{x\to a} x$ by just plugging a in for x.

Claim 1.5.5. Evaluating Limits in Continuous Functions

Let $f(x)$ be a function that is continuous at $x = a$. Then

$$\lim_{x\to a} f(x) = f(a).$$

Example 1.5.6. Using Continuity

Consider the following limit:

$$\lim_{x \to 0} \ln\left(\sin(x)/x\right).$$

Theorem 1.4.2 tells us the function $\ln(x)$ is continuous for positive x. Thus, we use the above formulation of continuity to swap the ln and the lim as follows:

$$\lim_{x \to 0} \ln\left(\sin(x)/x\right) = \ln\left(\lim_{x \to 0} \sin(x)/x\right).$$

Exercise 1.5.7. Finishing the Calculation ☕

- Complete the computation of the limit above by using our special limit for sine to evaluate the inner limit.

- The natural log function is undefined for negative numbers. Yet, here we are using a two-sided limit (rather than just $\lim_{x \to 0^+}$ which would restrict us to only positive x). Is that OK? Did we do something wrong here?

Exercise 1.5.8. Evaluating Limits by Swapping the Order ☕☕☕

Repeat the technique of the above example to evaluate the limits below. In each case, specify which continuous function or functions you are using to perform the swaps.

- $\lim_{x \to 2} \tan(x - 2)$

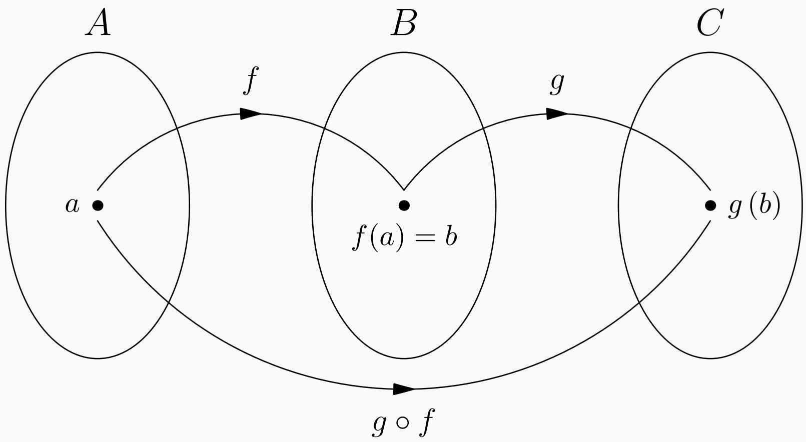
- $\lim_{x \to 2} \frac{\cos(\pi x)}{\cos(\pi x) + 1}$

Composition of Continuous Functions

If you compose two continuous functions, you create another continuous function! This is quite useful, as it means you don't have to swap one function at a time with your limits, but rather can push the input all the way through any composition of functions (assuming something doesn't go wrong on the way).

Theorem 1.5.9. A Composition of Continuous Functions is Continuous

Suppose a function $f(x)$ is continuous at $x = a$. Call $b = f(a)$. Suppose function $g(x)$ is continuous at $x = b$. Then $g \circ f$ is continuous at $x = a$.



Exercise 1.5.10. Proving the Composition Theorem ☕☕

Let's walk through the argument for continuity of a composition.

- We begin with the statement "$f(x)$ is continuous at $x = a$". Write that claim in terms of limits.
- Next consider the statement "$g(x)$ is continuous at $x = b$". Write that claim in terms of limits.
- How are a and b related?
- What does composition notation mean? That is, how do you evaluate the expression $g \circ f$?
- We wish to show "$g \circ f$ is continuous at $x = a$". Write that claim in terms of limits.

- How can you use the first two limits (the given information) to create the third (the desired conclusion)? Show your work below.

One general takeaway from the above; when trying to prove a result in mathematics, the questioning process above is exactly where you want to start. Ask yourself what the definitions are for all the given information; write them down! Then ask yourself what the definition is for the desired conclusion; write it down too! Then, look at how you might be able to manipulate the given information to create the desired conclusion, perhaps using algebra, a theorem, or identity. Many proofs of course become far more complicated and require lots of trickery, but the above is a great starting point.

Example 1.5.11. Revisiting Exercise 1.5.8

Let's analyze the first limit from Exercise 1.5.8 in terms of Theorem 1.5.9. In this case, we had the following functions:

$$\begin{aligned} f(x) &= x - 2 \\ g(x) &= \tan(x) \\ (g \circ f)(x) &= \tan(x-2). \end{aligned}$$

First note that f is continuous at 2, and $f(2) = 0$. The function g is continuous at 0. Thus, the composition $g \circ f$ is continuous at 2. As a consequence, we can all in one step just plug the value $x = 2$ in to evaluate the limit. In particular,

$$\lim_{x \to 2} \tan(x-2) = \tan(2-2) = \tan(0) = 0.$$

Exercise 1.5.12. Revisiting the Second Part of Exercise 1.5.8 ☕☕

Analyze the second limit from Exercise 1.5.8 in terms of Theorem 1.5.9 as we did in the above example.

The above examples illustrate why the composition theorem is so useful. It in essence says that as long as you don't run into anything strange at some intermediate step, you can just plug $x = a$ straight into your function, and it will evaluate your limit correctly.

Exercise 1.5.13. Haven't Seen a Secant in a While ☕☕

Consider the composition of the functions $f(x) = \sec(x)$ and $g(x) = \frac{2x}{x-1}$. Use Theorem 1.5.9 to evaluate the limits given below, or explain why it does not apply.

- $\lim_{x\to\pi/3} \frac{2\sec(x)}{\sec(x)-1}$

- $\lim_{x\to 0} \frac{2\sec(x)}{\sec(x)-1}$

1.6 Types of Discontinuities

There are two main types of discontinuities we come across in our functions, *removable* and *nonremovable*. In essence, the removable discontinuities are kind of harmless and easily fixable. The nonremovables are exactly what they sound like, not easily removed!

Removable Discontinuities

If we have a discontinuity on a function f that we could fix by changing just a single y-coordinate, we consider that discontinuity to be removable. Removable discontinuities are often called "holes". We state this definition more formally below.

Definition 1.6.1. Removable Discontinuity

Let L be a real number, and $f(x)$ be a function such that

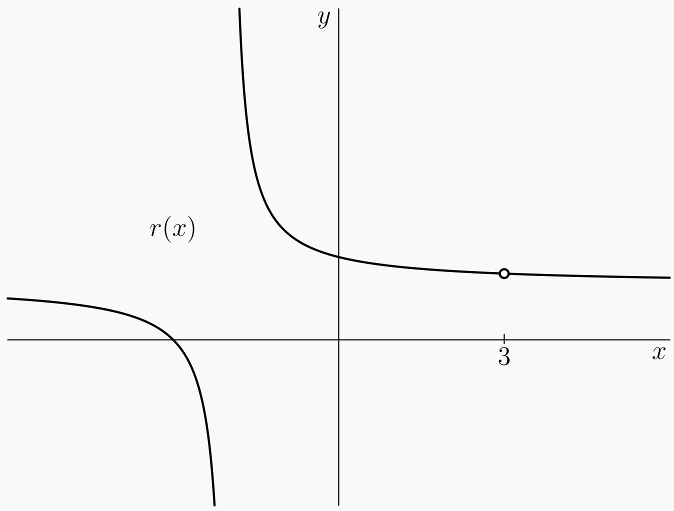
$$\lim_{x\to a} f(x) = L \text{ and } f(a) \neq L.$$

Then $f(x)$ has a *removable discontinuity* at $x = a$.

Example 1.6.2. An Old Rational Function

Consider the rational function,

$$r(x) = \frac{x^2 - 9}{x^2 - x - 6}.$$



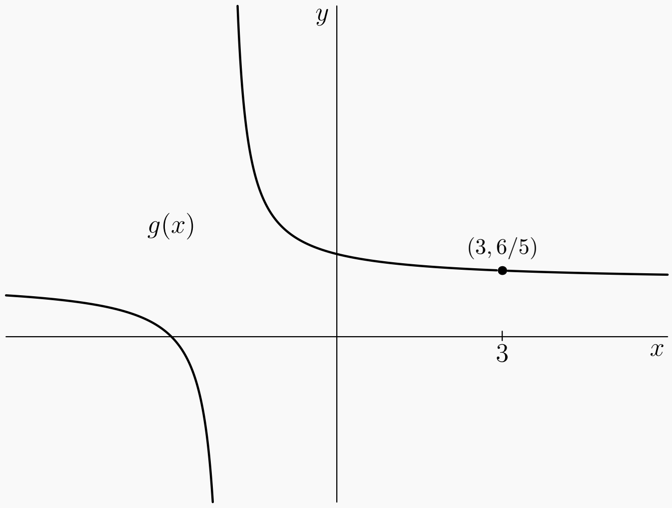
It is discontinuous at $x = 3$, because the limit is not equal to the value of the function at 3. Specifically, $r(3)$ is undefined (division by zero) but

$$\lim_{x\to 3} r(x) = 6/5$$

since the right-hand and left-hand limits both equal that value. Since we cannot say $\lim_{x\to 3} r(x) = r(3)$, we say r is discontinuous at 3.

However, fixing this discontinuity is not hard. If we simply plug the hole with a little dot, the function becomes continuous! In particular, define

$$g(x) = \begin{cases} r(x) & \text{if } x \neq 3; \\ 6/5 & x = 3. \end{cases}$$



Thus, $g(x)$ is just the same function as $r(x)$ but with the hole filled in. So even though we could not say that $\lim_{x\to 3} r(x) = r(3)$ since it is undefined, we can actually say that

$$\lim_{x\to 3} r(x) = \lim_{x\to 3} g(x) = g(3).$$

Exercise 1.6.3. A Few Subtleties to Think About ☕

- Even though r and g are different functions, why is $\lim_{x\to 3} r(x) = \lim_{x\to 3} g(x)$?

- Note that we could perform the algebraic operation

$$\frac{x^2-9}{x^2-x-6} = \frac{(x-3)(x+3)}{(x-3)(x+2)} = \frac{x+3}{x+2}.$$

  Of those three expressions above, which are equal to $r(x)$, which are equal to $g(x)$, and why?

- The function $r(x)$ is also discontinuous at $x = -2$. Explain why that discontinuity is not removable.

Exercise 1.6.4. Our Pathological Path ☕☕

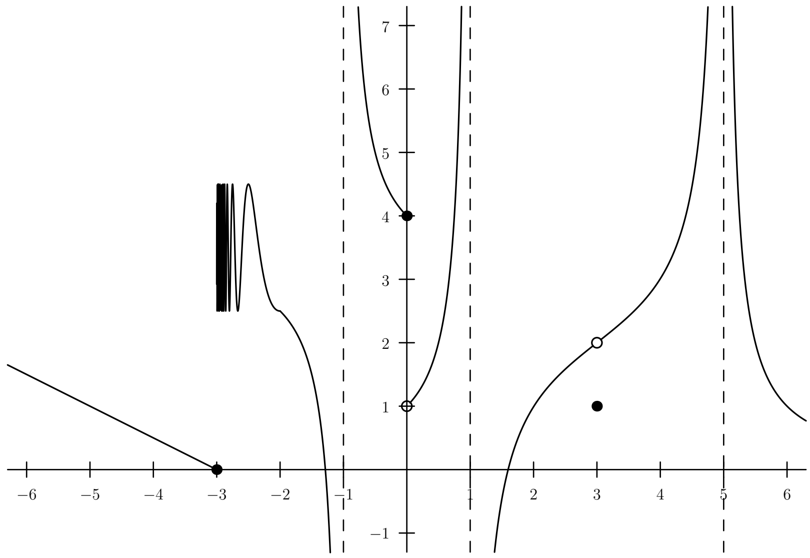
Recall Exercise 1.2.21 where we inspected

$$f(x) = \sin\left(\frac{1}{x}\right).$$

List all x values at which the function is discontinuous. Which of these are removable?

Exercise 1.6.5. Classifying Discontinuities on an Old Graph ☕☕

Recall the function from Example 1.1.3, $f(x)$ which is graphed below.



Find all discontinuities and classify each as removable or not. If it is removable, specify the piecewise function that removes the discontinuity.

Exercise 1.6.6. Classifying Discontinuities ☕☕

Draw a sketch of the graph of each of the following functions. State the interval on which the function is continuous and list all discontinuities. Classify each discontinuity as removable or nonremovable.

- $f(x) = |x|$

- $f(x) = \sqrt{x^2}$

- $f(x) = 1$

- $f(x) = \frac{x}{x}$

- $f(x) = \frac{|x|}{x}$

- $f(x) = \sqrt{x}$

- $f(x) = \sqrt{|x|}$

- $f(x) = \frac{x^2-x-12}{x^2-16}$

- $f(x) = \frac{x+1-\sqrt{x+1}}{x}$

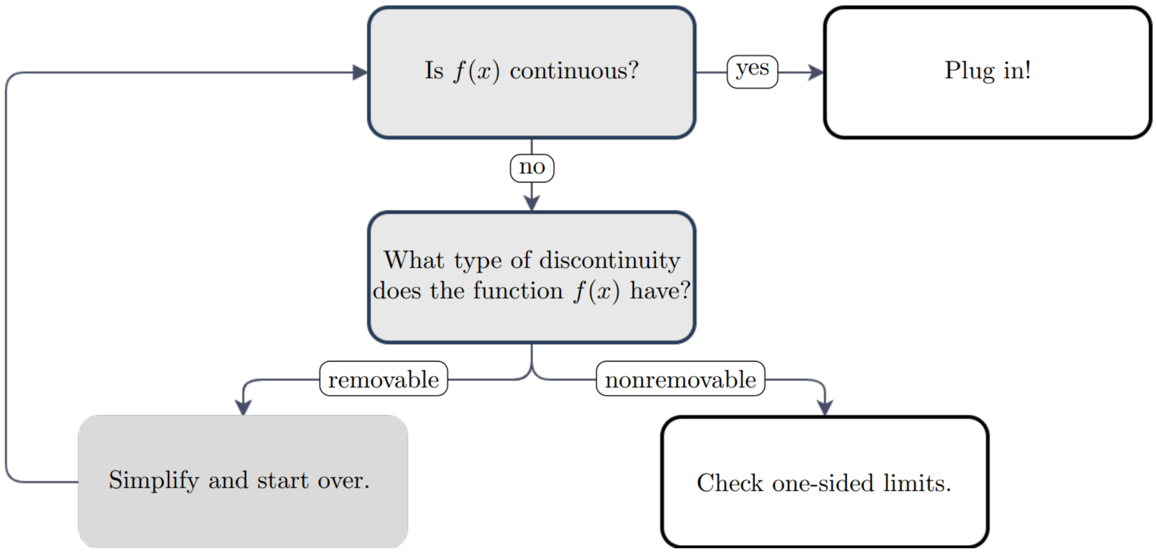
1.7 An Algebraic Approach to Limits

In Section 1.1, we discussed how limits could be approached graphically/numerically by essentially just making data tables or graphing the corresponding points. This is a great approach for getting some intuition as to what is going on, but lacks modern standards of mathematical rigor. ("I mean I think it kinda looks like it's sorta goin' ta 3?")

In Section 1.2, we took the complete opposite approach. With the utmost standards of rigor (brought to us by many mathematicians including Bolzano and Weierstrauss but mostly by Cauchy), we painstakingly worked through the most pedantic of $\delta - \epsilon$ arguments. We also saw the Sandwich Theorem, which can help understand strange functions by bounding them between simpler functions.

In section 1.5, we found a definition of continuity that allowed us to compute limits easily when functions are continuous. Of course, most of the limits we care about exist where functions are discontinuous. Then in section 1.6 we saw that for removable discontinuities, we could find limits still. In fact, many common removable discontinuities can be removed using algebraic techniques. This section is dedicated to some of the more commonly used among those tricks.

In fact, when evaluating limits, we can generally follow a pattern. If the function is continuous, then evaluating the limit is as simple as plugging in the value. If the function has a discontinuity, we need to decide what type of discontinuity it is. If it's removable, we can attempt to remove the discontinuity algebraically, then try again. We can follow this flow chart:



Trick 1: Factor and Cancel

We begin with a motivating example.

Exercise 1.7.1. A Rational Function ☕☕

Consider the function

$$r(x) = \frac{x^2 - x - 6}{x - 3}.$$

- What is the domain of r?

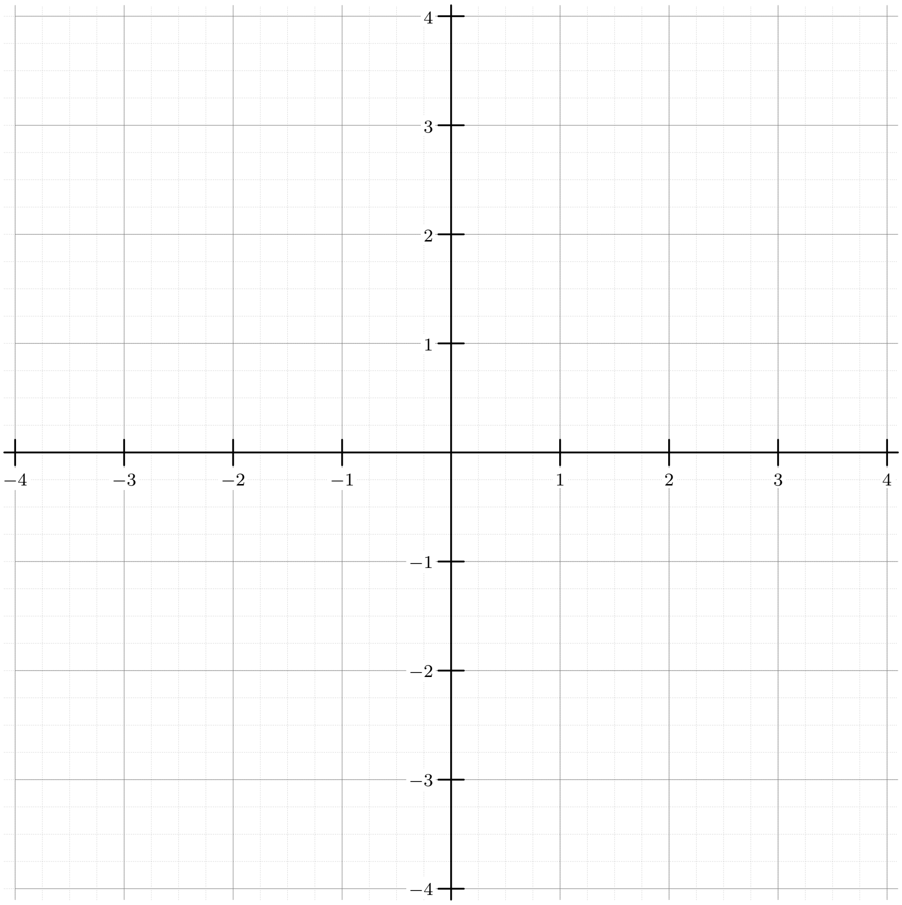
- One thing that is interesting about this particular function is that $x = 3$ is a root of both the numerator and the denominator. Let's study the behavior of the function near this interesting point to see what happens! We start by making a small table of values. We cannot directly plug the number 3 in for x since it is not in the domain, but we can plug in numbers *near* 3. This is fine since it feels like a quite limit-y thing to do anyway.

x	2.9	2.99	2.99	3.001	3.01	3.1
$r(x)$						

- Though technically the graph has a hole at $x = 3$, notice the graph does not go to infinity or minus infinity. instead, what does $r(x)$ approach as x approaches 3? State your answer using the notation of limits.

- Rewrite your function $r(x)$ by factoring the numerator, factoring the denominator, and canceling the common factor $(x - 3)$. Now plug the value $x = 3$ into that new formula. How does this compare with the previous answer?

- Sketch the graph of $r(x)$.



Notice what happened in our example; if there aren't any weird things happening with your function (for example division by zero), then you can just plug $x = a$ into your function to evaluate the limit. If there is something weird happening, we use algebra to rewrite the function until we can just plug in $x = a$.

Exercise 1.7.2. Trying the Trick Out

Evaluate each of the limits given below. Apply our factor and cancel trick if needed, or explain why it is not needed.

- $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2+x-2}$

- $\lim_{x \to 2} \dfrac{x^3 - x^2 - x + 1}{x^3 - 1}$

- $\lim_{x \to -1} \dfrac{x^3 - x^2 - x + 1}{x^3 - 1}$

- $\lim_{x \to 1} \dfrac{x^3 - x^2 - x + 1}{x^3 - 1}$

Trick 2: Use a Conjugate

Often strange limits involving radicals can be cleaned up using conjugates.

Exercise 1.7.3. A Radically Different One ☕☕☕

We now consider the limit $\lim_{x \to \infty} r(x)$ where

$$r(x) = \sqrt{x^2 + x + 1} - x.$$

- Explain why we cannot evaluate the limit by just plugging ∞ straight into the function.

- Fill out the data table below.

x	1	10	100	1000
$r(x)$				

What does it appear the function is approaching as x goes to ∞?

- Simplify the function $r(x)$ by multiplying the top and bottom by $\sqrt{x^2+x+1}+x$. If you say, "But wait, it doesn't have a top and bottom!" that's ok. Everything has a top and bottom, since

$$r(x) = \sqrt{x^2+x+1} - x = \frac{\sqrt{x^2+x+1}-x}{1}.$$

- Now that the function looks like

$$r(x) = \frac{x+1}{\sqrt{x^2+x+1}+x},$$

we simplify further by dividing the top and bottom by x. It's a slightly tricky bit of algebra, so we show how it works out below:

$$\begin{aligned}
r(x) &= \frac{x+1}{\sqrt{x^2+x+1}+x} \\
&= \frac{(x+1)\,\frac{1}{x}}{\left(\sqrt{x^2+x+1}+x\right)\frac{1}{x}} \\
&= \frac{1+\frac{1}{x}}{\frac{\sqrt{x^2+x+1}}{x}+1} \\
&= \frac{1+\frac{1}{x}}{\frac{\sqrt{x^2+x+1}}{\sqrt{x^2}}+1} \\
&= \frac{1+\frac{1}{x}}{\sqrt{\frac{x^2+x+1}{x^2}}+1} \\
&= \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}.
\end{aligned}$$

Annotate each line of algebra with a few words saying what happened.

- Notice that as x approaches infinity, the terms of the form $\frac{1}{x}$ and $\frac{1}{x^2}$ go to zero. Evaluate the limit by sending these terms to zero. That is, finish the calculation below:

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} =$$

Does the resulting limit agree with your values from the data table?

Exercise 1.7.4. Trying the Trick Out ☕☕☕

Evaluate each of the limits given below. Apply our conjugate trick if needed, or explain why it is not needed.

- $\displaystyle\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

- $\displaystyle\lim_{x\to 2} x-\sqrt{x^2+3x-1}$

- $\displaystyle\lim_{x\to \infty} x-\sqrt{x^2+3x-1}$

- $\displaystyle\lim_{x\to -\infty} x-\sqrt{x^2+3x-1}$

It is worth noting that in some sense Trick 1 and 2 are really the same trick just slightly rewritten. Multiplying by the conjugate is just another form of factoring, in that

$$A-B=\left(\sqrt{A}-\sqrt{B}\right)\left(\sqrt{A}+\sqrt{B}\right).$$

Trick 3: Find a Common Denominator

Sometimes it's useful to find common denominators and combine fractions.

Exercise 1.7.5. Too Many Fractions

Consider the limit

$$\lim_{x\to 0} \frac{\frac{1}{x+1} - 1}{x}.$$

Lets slightly rewrite this limit to clean up some fractions

$$\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{1+x} - 1\right).$$

Then we can go ahead and find a common denominator inside the parenthesis:

$$\begin{aligned}\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{1+x} - 1\right) &= \lim_{x\to 0} \frac{1}{x}\left(\frac{1}{1+x} - \frac{1+x}{1+x}\right) \\ &= \lim_{x\to 0} \frac{1}{x}\left(\frac{1-(1+x)}{1+x}\right) \\ &= \lim_{x\to 0} \frac{1}{x}\left(\frac{-x}{1+x}\right)\end{aligned}$$

Now we can cancel terms and end up with

$$\lim_{x\to 0} -\frac{1}{x+1} = -1.$$

Exercise 1.7.6. Why not just plug in?

Why can we not evaluate the limit in the previous by just plugging $x = 0$ straightaway?

Exercise 1.7.7. Denominators

Compute the following limits.

- $\lim\limits_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$.

- $\lim\limits_{x\to 4} \dfrac{\frac{1}{x}-\frac{1}{4}}{x-3}$.

- $\lim\limits_{x\to 4} \dfrac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}$.

Trick 4: Make a Substitution

Often a limit can be cleaned up or simplified by replacing an indexing variable with another expression.

Example 1.7.8. Modified Sine Special Limit

Consider the limit

$$\lim_{x\to 0} \frac{\sin(2x)}{x}.$$

Here it looks so similar to our special limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

that it seems a shame to have to compute this limit from scratch. One way to leverage the old limit is to make the substitution $t = 2x$. Note that $\lim_{x\to 0} t = \lim_{x\to 0} 2x = 0$, so as x approaches

0, t does as well. We can rewrite the limit as

$$\lim_{x\to 0}\frac{\sin(2x)}{x}=\lim_{t\to 0}\frac{\sin(t)}{t/2}=2\lim_{t\to 0}\frac{\sin(t)}{t}=2\cdot 1=2.$$

Exercise 1.7.9. Algebra Carefulness ☕

In big final computation in the above example, why did the x in the denominator become a $t/2$?

Exercise 1.7.10. Algebra Carefulness ☕

Compute the limit below

$$\lim_{x\to 0}\frac{\sin(2x)}{x}.$$

This time, use the sine double angle identity to rewrite the numerator as $2\sin(x)\cos(x)$. Then use the fact that limits distribute across multiplication.

Exercise 1.7.11. Variations on the Sine Special Limit ☕☕☕

Use our special limit for sine and algebraic trickery/trig identities as needed to evaluate the following limits:

- $\lim\limits_{x\to 0}\dfrac{\sin\left(x^2\right)}{x}$

- $\lim\limits_{x\to 0}\dfrac{\sin^2(x)}{x}$

- $\lim\limits_{x\to 0}\dfrac{\sin(-x)}{x}$

- $\lim\limits_{x\to 0}\dfrac{\sin(\tan(x))}{\tan(x)}$

- $\lim_{x\to 0}\frac{\sin(x)}{x^2}$

- $\lim\limits_{x\to 2\pi} \dfrac{\sin(x)}{x-2\pi}$

- $\lim\limits_{x\to \pi} \dfrac{\sin(x)}{x-\pi}$

Exercise 1.7.12. Re-revisiting a Limit ☕☕☕

Recall the limit from Exercise 1.2.12, namely $\lim_{x\to\infty} \frac{1}{2^x}$. Recall that this limit exists by the Monotone Convergence Theorem, as shown in Exercise 1.3.16. Whatever it converges to, call that number L. Here we compute the value of L using a substitution.
Assume the limit $\lim_{x\to\infty} \frac{1}{2^x} = L$. That is, the limit exists and is equal to some number L. Make the substitution $x = t+1$. Since t goes to ∞ as x does, we have

$$L = \lim_{x\to\infty} \frac{1}{2^x} = \lim_{t\to\infty} \frac{1}{2^{t+1}} = \lim_{t\to\infty} \frac{1}{2\cdot 2^t} = \frac{1}{2} \lim_{t\to\infty} \frac{1}{2^t} = \frac{1}{2}L.$$

Notice on the last step, it is valid to put L back in because we have symbol-for-symbol the same limit we started with but just with an x instead of a t!
Ignoring all the middle steps and just taking the very leftmost and very rightmost expressions, we now have an equation we can solve for L, namely $L = \frac{1}{2}L$. Solve it for L.

Often a reciprocal can be useful in a substitution if you want to move from zero to infinity or vice versa.

Example 1.7.13. End Behavior for a Rational Function

Let us once again visit the concept of end behavior for a rational function. Consider the limit

$$\lim_{x\to\infty} \frac{2x^3+x-1}{x^3+1}.$$

We could analyze this by using principles of end behavior from our College Algebra/Precalc class, saying that since the degrees are tied, it should be the ratio of leading coefficients. This is a perfectly good way to look at it.
But, here is an alternative! Make the substitution $t = 1/x$. As x approaches ∞, the quantity t approaches 0 from the right. Thus, we can rewrite the limit as follows:

$$\lim_{x\to\infty} \frac{2x^3+x-1}{x^3+1} = \lim_{t\to 0^+} \frac{2\,(1/t)^3+(1/t)-1}{(1/t)^3+1}.$$

Exercise 1.7.14. Finishing the Limit ☕

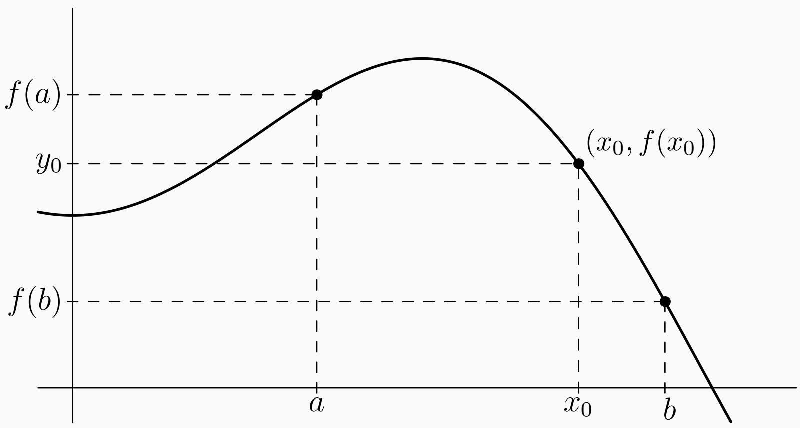
Finish computing the limit above by distributing the exponents, multiplying the top and bottom of the fraction by t^3, and setting $t = 0$.

1.8 Intermediate Value Theorem

The *Intermediate Value Theorem* (IVT) is a result that may seem intuitively obvious, but it is good to state it carefully and see some surprising consequences!

Theorem 1.8.1. Intermediate Value Theorem

Let a and b be real numbers. Let $f(x)$ be continuous on the interval $[a, b]$. Let y_0 be a real number between $f(a)$ and $f(b)$. Then there exists an x_0 between a and b such that $f(x_0) = y_0$.



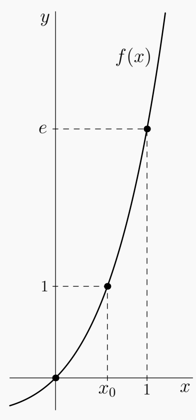
In short, IVT says that for a continuous function, any intermediate y-value (between two other y-values) will have a corresponding x-value that maps to it. That is to say, the function does not skip over any y-values as it increases or decreases. Let us play with the theorem a bit.

Example 1.8.2. Productive but not Constructive

Consider the function

$$f(x) = xe^x.$$

We compute the values $f(0) = 0$ and $f(1) = e$. Since $f(x)$ is continuous everywhere, we can apply IVT to the interval $[0, 1]$. Notice the y-value $y_0 = 1$ is between the y-values at the endpoints of the interval. By IVT, there exists some x_0 such that $f(x_0) = 1$.



Sometimes using IVT may feel a bit disconcerting; it shows existence of the desired x coordinate but it does not provide a formula for finding it! Though we're certain the x_0 in the above example exists based on IVT, we have no idea what the number is! In particular, the equation

$$xe^x = 1$$

has a solution but we don't have an easy way to solve this equation algebraically.

Exercise 1.8.3. Showing a Solution Exists ☕☕

Use IVT to show that the equation

$$e^{x^2+1} + \ln(x) = 2$$

has a solution.

Example 1.8.4. Roots of Polynomials

You may have seen this theorem in the context of finding roots of polynomials in your college algebra or precalculus courses. Since polynomials are always continuous on the entire real number line, we never have to worry about the IVT not applying to a polynomial function.
In the case of polynomials, the IVT could be restated as follows:

If a polynomial $p(x)$ has $p(a) > 0$ and $p(b) < 0$, then p must have a root somewhere between a and b.

Note that in the interpretation here, $y_0 = 0$, since zero is always between a positive number and a negative number.

Exercise 1.8.5. Using IVT on a Polynomial ☕☕

Consider the polynomial function $p(x) = 3x^3 + 10x^2 - 27x - 10$.

- Use the Rational Root Theorem to list all possible rational roots of the function.
- Find the values of the function for $x = -1, 0, 1$.
- Based on the three values you just computed, what interval must contain a root of $p(x)$?

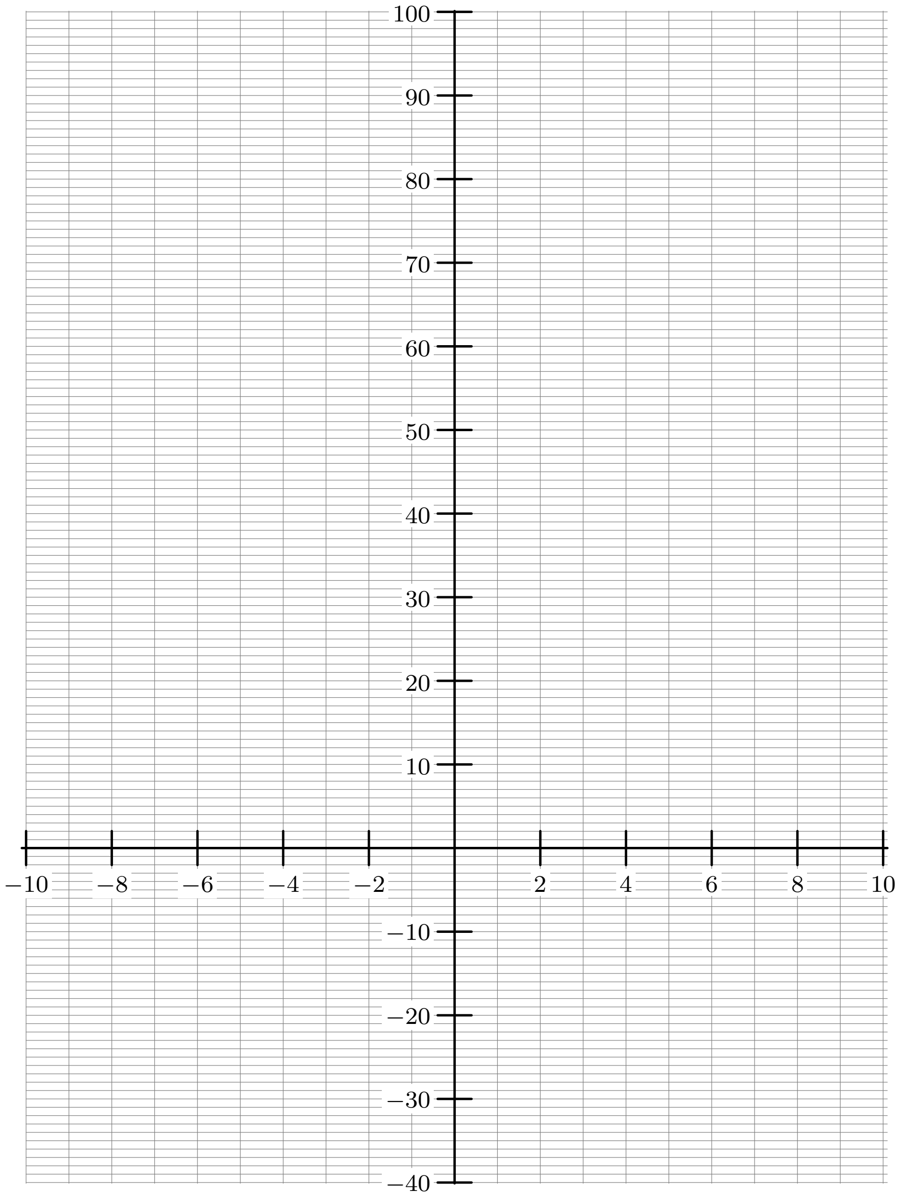
Explain your conclusion in terms of IVT.

- Filter through your RRT list and find all guesses that are in the interval where you know a root lies.

- Plug that filtered list in one at a time into $p(x)$ until you find a root. (Note that in theory the root in that interval could have been irrational in which case none of them will work. But here one will!)

- Use the root you found to fully factor the polynomial $p(x)$.

- Use the factorization to graph the polynomial $p(x)$.

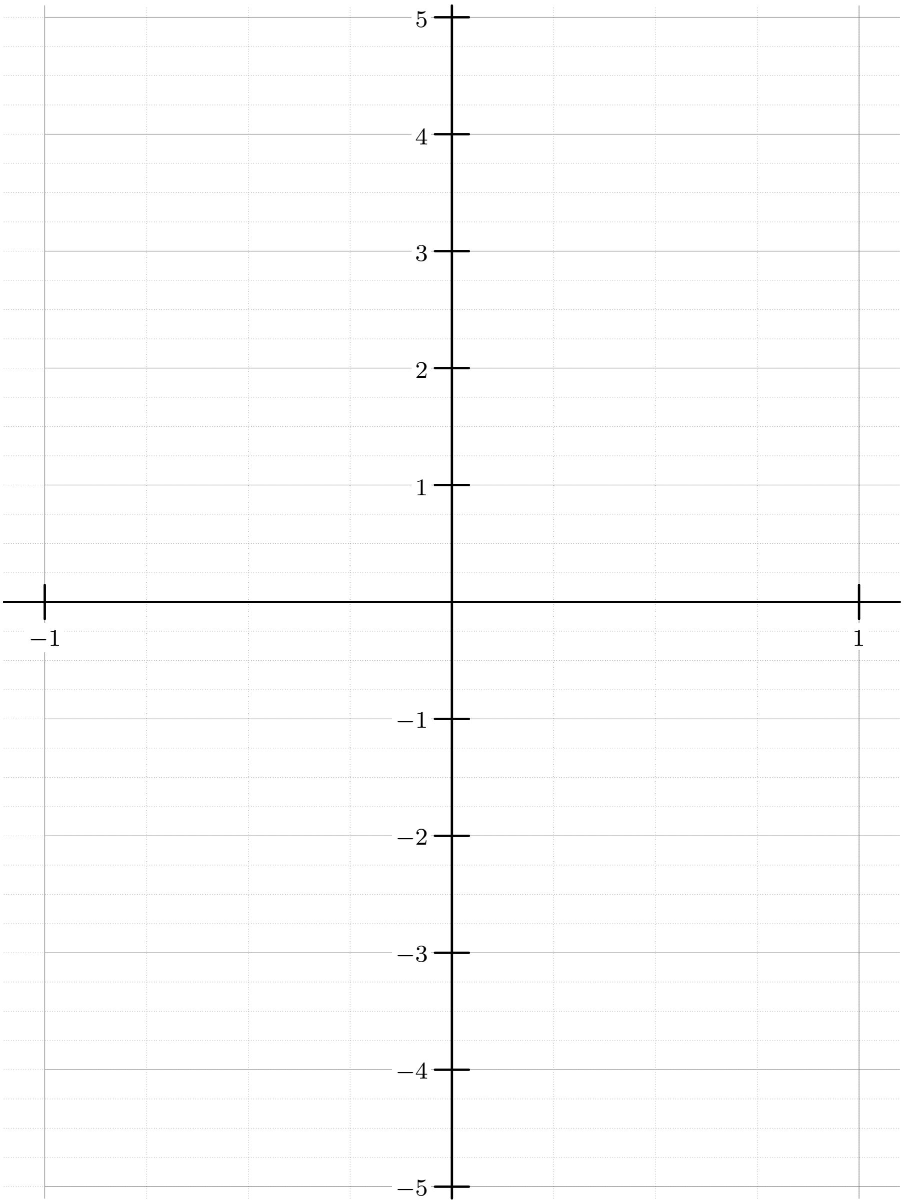


Next, we run through a few examples to see why all preconditions listed in IVT are needed.

Exercise 1.8.6. Showing Continuity is Necessary

Here we show that the conclusion of the IVT may not hold if we drop the precondition of continuity.

- Graph the function
$$f(x) = \frac{1}{x}$$
on the interval $[-1, 1]$.



- Is f continuous on that interval?

- Consider the y-value $y_0 = 1/2$. Does there exist a corresponding x-value such that $f(x_0) = y_0$?

- Consider the y-value $y_0 = 0$. Does there exist a corresponding x-value such that $f(x_0) = y_0$?

Thus, we see that the conclusion of IVT may or may not hold when the function is discontinuous.

Exercise 1.8.7. Another Necessary Condition ☕

Graph the function

$$f(x) = 2x + 1$$

on the interval $[2, 5]$.

- Is f continuous on that interval?

- Consider the y-value $y_0 = 3$. Does IVT guarantee there exists an x_0 such that $f(x_0) = y_0$?

One important observation about IVT is that the converse is false. That is, just because a function takes on every possible intermediate value does not mean it is continuous. The exercise below that illustrates this point.

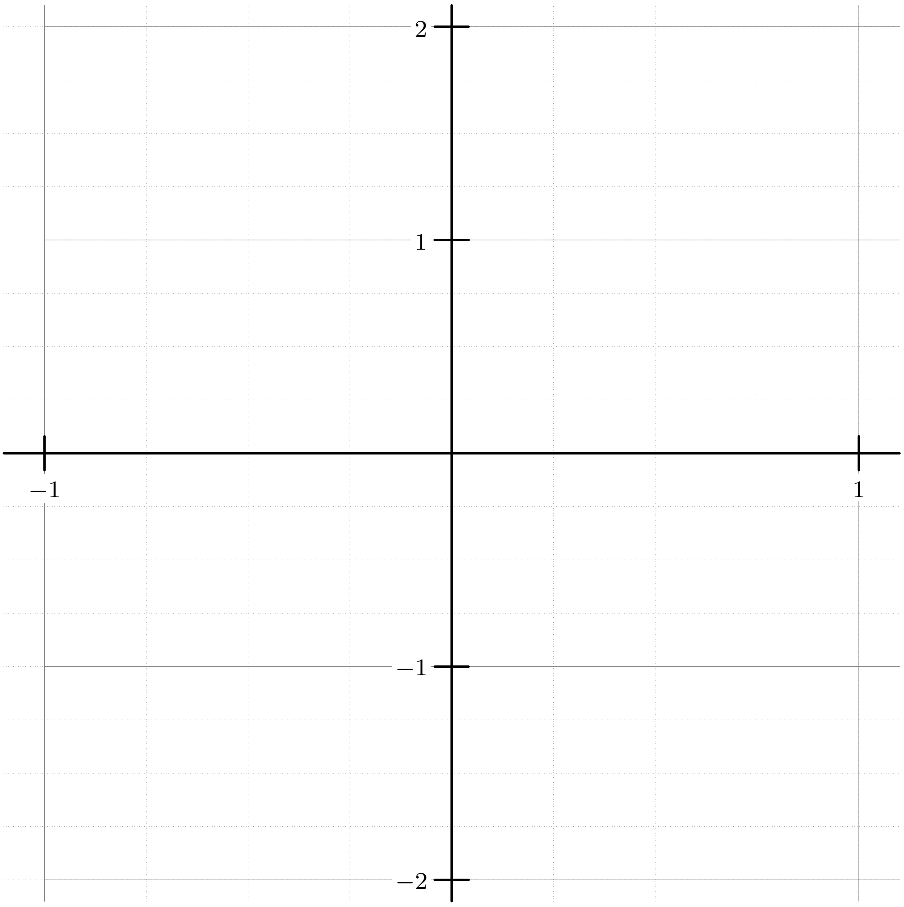
Exercise 1.8.8. Our Pathological Path Once Again ☕☕

Consider the function

$$f(x) = \sin\left(\frac{1}{x}\right)$$

on the interval $[-2/\pi, 2/\pi]$.

- Sketch a rough graph of the function. Be sure to label the x- and y-coordinates of the endpoints.



- Explain why every intermediate y-value has a corresponding x-value that gets mapped to it.

- Explain why f is not continuous, despite taking on every possible intermediate value.

Example 1.8.9. A Hot Theorem

Here is a surprising fact.

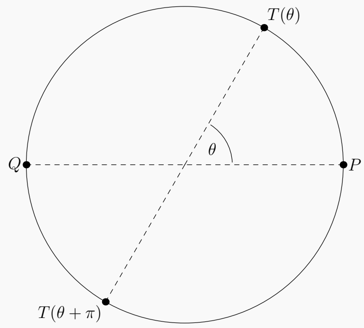
Right now at this very moment, there exist two antipodal points on the equator of the earth that have exactly the same temperature.

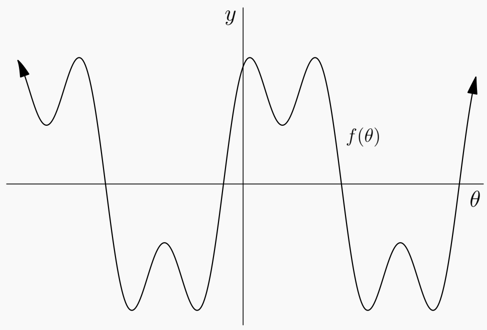
Note that *antipodal* means the points are diametrically opposite one another. Let us prove this fact.

- **Case 1:** All pairs of antipodal points on the equator have the same temperature. In this case, the claim is true, since we are only claiming there exists at least one pair.
- **Case 2:** Not all pairs of antipodal points on the equator have the same temperature. In this case, there exist some pair of antipodal points that have different temperatures. Call these points P and Q, and without loss of generality assume P is the warmer of the two locations. Coordinatize the equator to be a unit circle such that P is at $\theta = 0$ and Q is at $\theta = \pi$. Now define the function $T(\theta)$ to represent the temperature at point the point at angle θ. For example, $T(\pi)$ is the temperature at Q. Now construct the function

$$f(\theta) = T(\theta) - T(\theta + \pi),$$

the difference of the temperature at angle θ vs the temperature of the point opposite angle θ.





We have the setup for the argument written above. Now it is your turn to carry out the last few steps!

Exercise 1.8.10. Finishing Case 2 ☕☕☕

- Explain why $f(0)$ is positive.
- Explain why $f(\pi)$ is negative.

- Explain why there must exist a zero of f on the interval $[0, \pi]$.

- Explain why a zero of f must correspond to a pair of antipodal points of the same temperature!

Exercise 1.8.11. Test This Out with Your GPS ☕☕☕

A hiker starts at a trailhead at 8AM and summits a peak by 11AM. She stays the night in her tent at the top. The following morning, at 8AM, she walks back down the same trail. Prove there exists a point on the trail where she was at the exact same location at the exact same time both days.

Exercise 1.8.12. Sibling Rivalry ☕☕☕

Sven was born sixteen inches long and grew to an adult height of 80 inches. His brother Olaf, five years younger, was born fourteen inches long and grew to an adult height of 82 inches.

- Prove that there was some moment in time where Sven and Olaf were exactly the same height.

- Prove that there exists an age t at which Sven's height when Sven was t years old was exactly equal to Olaf's height when Olaf was t years old.

1.9 Chapter Summary

In this chapter, we built the idea of *limits*. The expression $\lim_{x\to a} f(x) = L$ can be viewed in several ways:

1. **Intuitive/graphical idea:** The limit L is the value that $f(x)$ approaches as x approaches a.
2. **Numerical method:** Build an input-output table where the inputs become closer and closer to a. The limit is the value that the outputs become closer and closer to.

x	Inputs approaching a	$\dots$
$f(x)$	Outputs approaching L	$\dots$

3. **Formal definition:** We adopt the abbreviations $\forall$ to mean "for all" and $\exists$ to mean "there exists". Then the limit can be defined as

$$\forall \epsilon \in \mathbb{R}, \exists \delta \in \mathbb{R}, 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon.$$

This provides the basis for writing an $\delta - \epsilon$ **proof**, which is simply a rigorous verification of the definition of a limit. See below for a template of what such an argument looks like.

Template for an $\delta - \epsilon$ Proof of $\lim_{x\to a} f(x) = L$

Proof. Let $\epsilon > 0$, an arbitrary positive real number. Choose $\delta = \boxed{\text{some function of } \epsilon}$. Let x be a real number and assume

$$0 < |x-a| < \delta.$$

Under these assumptions, we now wish to demonstrate that the function $f(x)$ will be within ϵ of L. We now compute the distance between $f(x)$ and L:

$$\begin{aligned} |f(x) - L| &= \boxed{\text{plug in formulas}} \\ &= \boxed{\text{simplify as appropriate}} \\ &= \boxed{\text{reach expression involving } |x-a|} \\ &< \boxed{\text{replace } |x-a| \text{ by } \delta} \\ &= \boxed{\text{simplify}} \\ &= \epsilon. \end{aligned}$$

Thus, we have verified that the distance between the function and L is less than ϵ, provided that x is chosen within δ from a. By the $\epsilon - \delta$ definition of a limit, we have successfully proven that $\lim_{x\to a} f(x) = L$. □

The trickest step in getting started is often figuring out what function of ϵ we should set δ equal to. In the simple case of a linear function of slope m, one can just use the fact that $\epsilon/\delta = m$ and thus choose $\delta = m/\epsilon$. For nonlinear functions, determining the right choice of δ will require more substantial trickery.

4. This chapter also introduced two key theorems on limits: the **Sandwich Theorem** and the **Monotone Convergence Theorem.** The Sandwich Theorem is particularly important, proving two very fundamental results about sine and cosine:

(a) $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

(b) $\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$

Furthermore, we identified several methods of calculating limits when the graph or a data table may not be tractable or useful. The main strategies to evaluate limits algebraically are to **factor and cancel, use the conjugate,** and **make a substitution**.

This chapter also introduced several notions of continuity.

1. **Informal Idea of Continuity:** If the graph of a function f over a domain D can be drawn in a single stroke without lifting your pen from the page, it is said to be continuous on D.

2. **Formal Definition of Continuity:** If a function $f(x)$ satisfies

$$\lim_{x \to a} f(x) = f(a)$$

then we say the function f is **continuous at** a**.** If the function is continuous at every point $a \in D$, we say the function is continuous on the domain D. Note that at endpoints of D we do allow for the limit to be a one-sided limit.

The incredibly useful consequence of continuity is that **continuous functions can be swapped with limits**. Specifically, $f(x)$ is continuous at a if and only if

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right).$$

We classified two fundamental types of discontinuities, **removable** and **nonremovable.** These can be defined as follows:

1. If $f(x)$ is discontinuous at a, but could be made continuous by changing the value of just the one number $f(a)$, we say that f has a **removable discontinuity** at a.

2. If it is not possible to make the function continuous by just changing the value at one point, we say the discontinuity is **nonremovable.** There are many types of nonremovable discontinuities, including the following:

   (a) **Infinite Discontinuity:** Perhaps the function was discontinuous because there was a vertical asymptote!

   (b) **Jump Discontinuity:** Perhaps the function was discontinuous because the y-values of the function jumped a positive distance across a single x-value!

   (c) **Oscillating Discontinuity:** Perhaps the function was discontinuous because the function oscillated with arbitrarily large frequency!

Lastly, we had an important theorem which applies to continuous functions: the **Intermediate Value Theorem** states that a continuous function on a closed interval $[a, b]$ will achieve every y-value between $f(a)$ and $f(b)$ as an output for some corresponding $c \in [a, b]$.

1.10 Mixed Practice

Exercise 1.10.1. ☕

Consider the following expressions:

$$\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to\infty} f(x) = L.$$

Which of these corresponds to a horizontal asymptote for the graph of $f(x)$ and which corresponds to a vertical asymptote for the graph of $f(x)$ and why?

Exercise 1.10.2. ☕☕

Use parent graphs and transformations to graph the function $f(x) = \ln(x - 3)$. Then, use your graph to evaluate the following limits:

- $\lim_{x\to 2} f(x)$
- $\lim_{x\to 3^-} f(x)$
- $\lim_{x\to 3^+} f(x)$
- $\lim_{x\to 3} f(x)$
- $\lim_{x\to 4^-} f(x)$
- $\lim_{x\to 4^+} f(x)$
- $\lim_{x\to 4} f(x)$
- $\lim_{x\to\infty} f(x)$

Exercise 1.10.3. Evaluating Limits Algebraically ☕☕☕

Consider the limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}.$$

- What are the values of the function $f(x) = \frac{\sqrt{x^2+9}-3}{x^2}$ when evaluated at $x = 0.1$, $x = 0.01$, $x = 0.001$, and $x = 0.0001$?
- What value does $f(x)$ appear to be approaching as x approaches zero? If you try small negative values, does $f(x)$ still seem to approach the same value?
- Evaluate the same limit using algebra; multiply the top and bottom of the fraction by the conjugate $\sqrt{x^2+9}+3$. Simplify the resulting expression until you can just plug in $x = 0$ to evaluate the limit.

Exercise 1.10.4. A $\delta - \epsilon$ proof

Write a $\delta - \epsilon$ proof to verify that

$$\lim_{x \to 2} 2 - x = 0.$$

Exercise 1.10.5. $\delta - \epsilon$ Definition of a Limit

Consider the following limit:

$$\lim_{x \to 2} \left(\frac{1}{3}x + 1 \right).$$

- What does the above limit evaluate to?
- Interpret in words the meaning of the above limit.
- If $\epsilon = 0.1$, what would the corresponding δ be? Sketch a graph below that illustrates this.

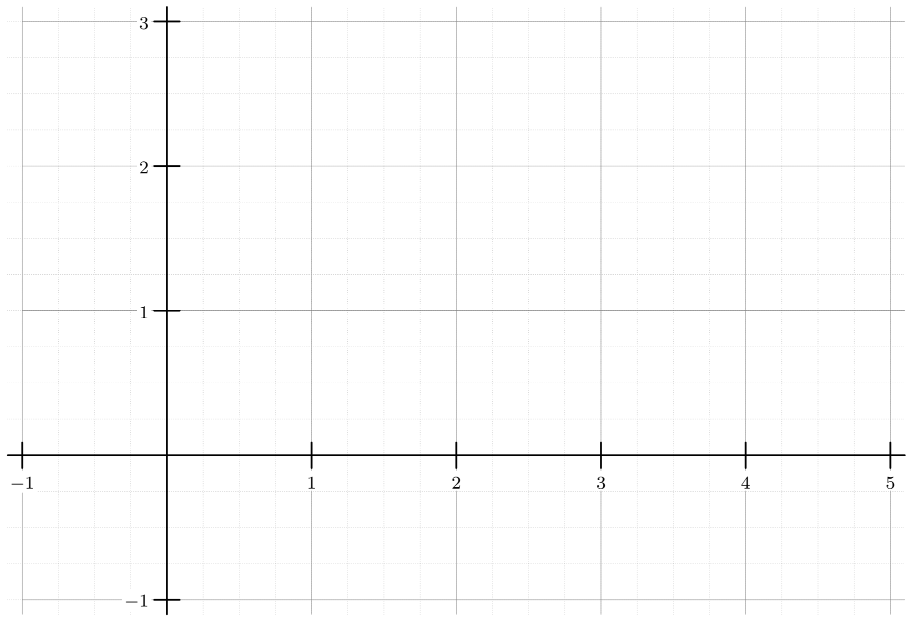
- Write a δ-ϵ proof of the correctness of your limit above.

Exercise 1.10.6. δ-ϵ Definition of a Limit

Consider the following limit:

$$\lim_{x\to 4} \sqrt{x}.$$

- What does the above limit evaluate to?
- Interpret in words the meaning of the above limit.
- If $\epsilon = 0.1$, what would the corresponding δ be? Sketch a graph below that illustrates this. Notice since the graph is not symmetric about the point $(4, 2)$, you will have two different measurements on the left and on the right. Explain why you must choose the smaller measurement as δ rather than the larger.



Exercise 1.10.7.

Decide which of the following domains D the function $f(x) = \tan(x)$ is continuous on, and which of the following domains D the function $g(x) = \arctan(x)$ is continuous on.

- $D = [0, 1]$
- $D = [0, 2]$
- $D = (-\infty, \infty)$

Exercise 1.10.8. ☕☕

Explain why "can be drawn with a single pen stroke" is a insufficient to serve as a rigorous definition of continuity. Why do we need the language of limits to define continuity?

Exercise 1.10.9. ☕☕☕

Graph the function

$$f(x) = \frac{x^2 - 100}{x^3 - 1000}.$$

Where are the discontinuities? Classify the type of each. If any are removable, indicate what value would have to be redefined to produce a continuous function.

Exercise 1.10.10. ☕☕☕

Graph the function

$$f(x) = \frac{x^2 - 100}{x^2 - 100}.$$

Where are the discontinuities? Classify the type of each. If any are removable, indicate what value would have to be redefined to produce a continuous function.

Exercise 1.10.11. ☕☕☕

Graph the function

$$f(x) = \frac{x^2 - 100}{x - 100}.$$

Where are the discontinuities? Classify the type of each. If any are removable, indicate what value would have to be redefined to produce a continuous function at that point.

Exercise 1.10.12. ☕☕☕

- Does the Intermediate Value Theorem imply the function $f(x) = x^2 - x - 1$ has a root on the interval $[0, 1]$? Why or why not?
- Does the Intermediate Value Theorem imply the function $f(x) = x^2 - x - 1$ has a root on the interval $[0, 2]$? Why or why not?
- Does the Intermediate Value Theorem imply the function $f(x) = \frac{1}{x^2 - x - 1}$ has a root on the interval $[0, 2]$?

Part II

Derivatives

Chapter 2

Definition and Properties of the Derivative

In previous courses, you have seen the notion of *slope* of a line. Here we wish to extend that idea to other functions that might not be linear. This is the idea of a *derivative*.

2.1 The Limit Definition of the Derivative

Average Rate of Change vs Instantaneous Rate of Change

Consider the dialogue below, as Mr Plum Tomato and his wife Cherry drive down the highway.

- Plum: *Oh no Cherry! We are entering a heavily policed part of the highway. I better check my speed.*
- Cherry: *Yes, I saw a sign that says there is a 65 mph speed limit.*
- Plum: *Ok, well there are those mile markers on the highway. Ok I see one just now for mile marker 212. What time is it?*
- Cherry: *It's 1'o clock* PM.
- Plum: *Ok, let's watch for the next mile marker.* (Time passes)
- Cherry: *I see the sign for mile marker 213, right there!*
- Plum: *Oh great, what time is it now?*
- Cherry: *It's exactly 1:01*PM.
- Plum: *Ok, so if we want our speed in miles per hour, we can calculate it. We have a change in distance of 1 mile. We have a change in time of* 1 *minute, or* 1/60 *of an hour.*
- Cherry: *Right, so if we want miles per hour, we can find the ratio*

$$\frac{213 - 212 \text{ miles}}{1:01pm - 1:00pm} = \frac{1 \text{ mile}}{1 \text{ minute}} = \frac{1 \text{ mile}}{1/60 \text{ hours}} = 60 \text{ mph}.$$

- Plum: *Phew, ok so for the moment we're going under the speed limit. Safe for now!*

Exercise 2.1.1. ☕

- What strikes you as completely silly about the dialogue above?

- Why might Plum's conclusion that they are safe not be correct?

Instantaneous Rate of Change at a Point

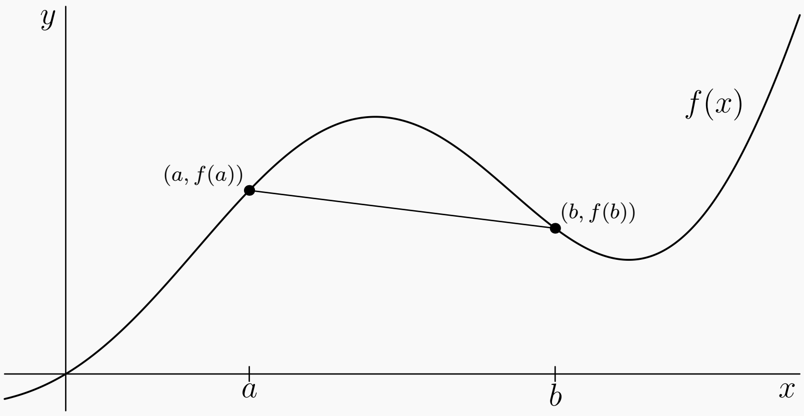
Recall the notion of *average rate of change* of a function. Essentially, we pick two points on a graph and then compute the slope of the secant line connecting them.

Definition 2.1.2. Average Rate of Change: Slope of a Secant Line

Let $f(x)$ be a function and let a and b be real numbers in the domain of f with $a < b$. Then the *average of rate of change* of f on $[a, b]$ is the quantity

$$\frac{\text{Change in } y}{\text{Change in } x} = \frac{f(b) - f(a)}{b - a}.$$

Note this is the slope of the line connecting the points $(a, f(a))$ and $(b, f(b))$.



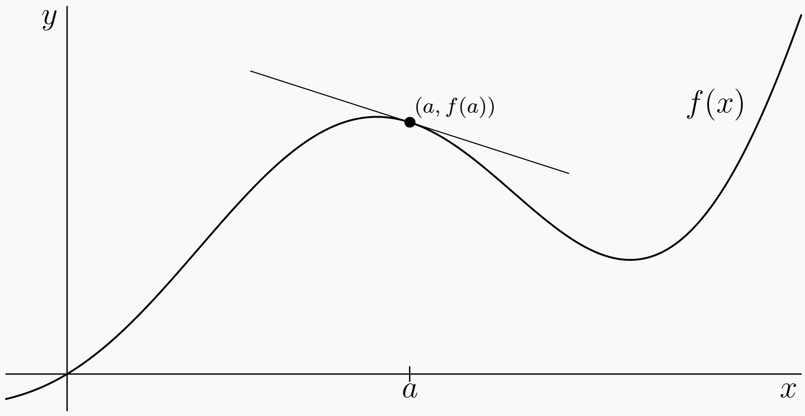
Here we wish to define the corresponding notion of *instantaneous rate of change*, the idea of how the function is changing not over an interval of positive length, but at a particular instant in time. This is the concept that Plum and Cherry of course wished they had. When the police officer tags you, they are not interested in what your average rate of change was across some period of time, but rather what your rate of change was at that very instant (and is why we all have speedometers in our cars instead of just using the milemarker/watch system of Plum and Cherry).

To define instantaneous rate of change, we still compute the average rate of change, but shrink the length of the interval to zero using a limit.

Definition 2.1.3. Instantaneous Rate of Change: Slope of a Tangent Line

Let $f(x)$ be a function and let a be a real number in the domain of f. Then the *instantaneous rate of change* of f at a is written $f'(a)$ (read f prime of a) and is computed as

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$



This quantity is also called the *derivative of f at a*. If the limit exists, we say f is *differentiable* at a. The line with slope $f'(a)$ through the point $(a, f(a))$ is called the *tangent line* to f at a.

Exercise 2.1.4. Why the Limit?

If we wanted to shrink the interval down to length zero, why do we need to use the limit at all? If we want the rate of change at $x = a$, why not just compute the average rate of change where a and b are both the same value?

There is an alternate version of the definition of derivative. If we take the limit

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

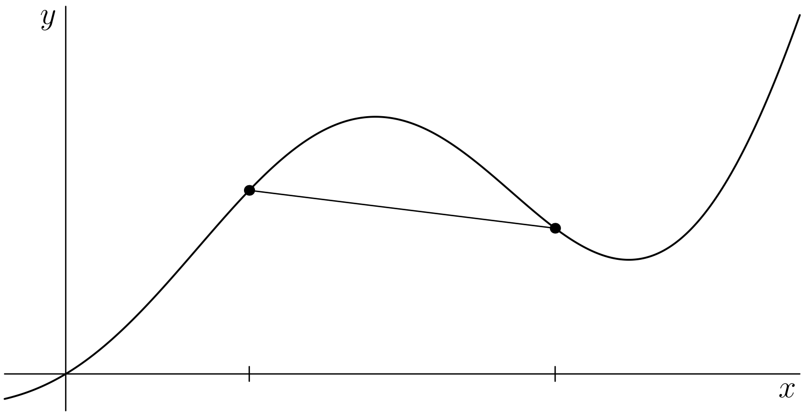
and make the substitution $b = a + h$, we get

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

In theory, it never matters which one you use; they will always produce the same result. But, in practice sometimes one ends up cleaner to evaluate than the other.

Exercise 2.1.5. Relabelling ☕

Take the same diagram from Theorem 2.1.3 but label the same components in terms of a and h rather than a and b.



We first review a factoring trick that we will need in the following example.

Exercise 2.1.6. Difference of Two Cubes Factorization Formula ☕

Recall the identity

$$A^3 - B^3 = (A-B)\left(A^2 + AB + B^2\right),$$

commonly known as the *difference of two cubes factorization.* Verify the formula is correct by multiplying out the right-hand side.

Example 2.1.7. Derivative of a Cubed Root

Let $f(x) = \sqrt[3]{x}$. Suppose we wish to find the derivative at $a = 1$. First, we compute one secant line slope on each side just to get a sense for what the tangent line slope is close to.

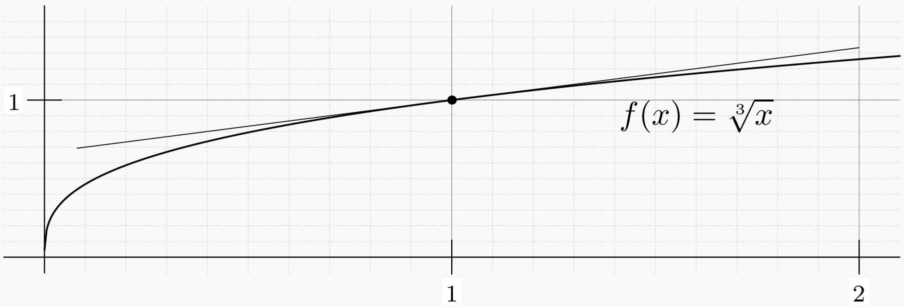
Value of b	0.9	1.1
Average Rate of Change Between 1 and b	$0.345\ldots$	$0.322\ldots$

From the graph, we see that the tangent line should have a slope that is less than 0.345 but greater than 0.322.
To find the exact slope of the tangent line, we set up the corresponding limit. Following the definition of the derivative produces

$$\lim_{b\to 1} \frac{\sqrt[3]{b} - \sqrt[3]{1}}{b-1}.$$

We cannot evaluate this using continuity (i.e., just plugging in $b = 1$) since there is division by zero. This is by design; there will always be division by zero when calculating a derivative via the limit definition. However, we cross our fingers and hope that the discontinuity is removable. In particular, we attempt the trickery of Subsection 1.7. We use the difference of two cubes factorization formula on the denominator as follows:

$$\begin{aligned}\lim_{b\to 1}\frac{\sqrt[3]{b}-\sqrt[3]{1}}{b-1} &= \lim_{b\to 1}\frac{\sqrt[3]{b}-1}{\left(\sqrt[3]{b}\right)^3-(1)^3}\\ &= \lim_{b\to 1}\frac{\sqrt[3]{b}-1}{\left(\sqrt[3]{b}-1\right)\left(\left(\sqrt[3]{b}\right)^2+\left(\sqrt[3]{b}\right)(1)+(1)^2\right)}\\ &= \lim_{b\to 1}\frac{1}{b^{2/3}+b^{1/3}+1}\\ &= \frac{1}{1^{2/3}+1^{1/3}+1}\\ &= \frac{1}{3}.\end{aligned}$$



Note that the true slope of the tangent line is $1/3 = 0.333\ldots$ which is in between the two estimates we computed using secant line slopes above!

Exercise 2.1.8. Cubed Root ☕☕☕

- Compute the average rate of change of the function $\sqrt[3]{x}$ on the interval $[8, 11]$.
- Compute the average rate of change of the function $\sqrt[3]{x}$ on the interval $[8, 10]$.
- Compute the average rate of change of the function $\sqrt[3]{x}$ on the interval $[8, 9]$.

- Compute the average rate of change of the function $\sqrt[3]{x}$ on the interval $[8, 8.1]$.

- Use the limit definition of the derivative to compute the slope of the tangent line to the function $\sqrt[3]{x}$ at the point (8,2).

Exercise 2.1.9. A Cubic ☕☕

- Compute the average rate of change of the function x^3 on the interval $[2, 5]$.

- Compute the average rate of change of the function x^3 on the interval $[2, 4]$.

- Compute the average rate of change of the function x^3 on the interval $[2, 3]$.

- Compute the average rate of change of the function x^3 on the interval $[2, 2.1]$.

- Use the limit definition of the derivative to compute the slope of the tangent line to the function x^3 at the point (2,8).

Exercise 2.1.10. Inverse Functions and Derivatives ☕☕☕

What transformation happens to a graph if you apply an inverse function? How does this relate the previous two exercises?

Exercise 2.1.11. Absolute Value ☕☕

- Compute the average rate of change of the function $|x|$ on the interval $[0, 3]$.
- Compute the average rate of change of the function $|x|$ on the interval $[0, 2]$.
- Compute the average rate of change of the function $|x|$ on the interval $[0, 1]$.
- Compute the average rate of change of the function $|x|$ on the interval $[0, 0.5]$.
- Use the limit definition of the derivative to compute the slope of the tangent line to the function $|x|$ at the point (0,0).

If the graph of a function is not differentiable as a result of the curve coming to a point like seen in the example above, we call that a *cusp*. We will not formally define cusp in this text, but the word will at times be nice to have around for sake of qualitative descriptions.

Exercise 2.1.12. Cosine ☕☕

- Compute the average rate of change of the function $\cos(x)$ on the interval $[0, 1]$.
- Compute the average rate of change of the function $\cos(x)$ on the interval $[0, 0.5]$.
- Compute the average rate of change of the function $\cos(x)$ on the interval $[0, 0.1]$.
- Compute the average rate of change of the function $\cos(x)$ on the interval $[0, 0.05]$.
- Use the limit definition of the derivative to compute the slope of the tangent line to the function $\cos(x)$ at the point (0,1).

Here we state another special limit; this is in regards to the number e.

Special Limit for e

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

Euler himself actually defined e to be the number that satisfied the above relationship, so taking this fact as a starting point doesn't bother us one bit. In Calculus II, you will see why this agrees with other ways of defining the number e.

Exercise 2.1.13. Exponential ☕☕

- Compute the average rate of change of the function e^x on the interval $[1, 2]$.
- Compute the average rate of change of the function e^x on the interval $[1, 1.5]$.
- Compute the average rate of change of the function e^x on the interval $[1, 1.1]$.
- Compute the average rate of change of the function e^x on the interval $[1, 1.01]$.
- Use the limit definition of the derivative to compute the slope of the tangent line to the function e^x at the point $(1, e)$.

Exercise 2.1.14. A Shifted Exponential ☕☕

- Compute the average rate of change of the function $e^x + 3$ on the interval $[1, 2]$.

- Compute the average rate of change of the function $e^x + 3$ on the interval $[1, 1.5]$.

- Compute the average rate of change of the function $e^x + 3$ on the interval $[1, 1.1]$.

- Compute the average rate of change of the function $e^x + 3$ on the interval $[1, 1.01]$.

- Use the limit definition of the derivative to compute the slope of the tangent line to the function $e^x + 3$ at the point $(1, e + 3)$.

Exercise 2.1.15. The Effects of a Vertical Shift ☕

Compare the previous two exercises. How did the "+3" affect the graph of the function? How did it affect the derivative?

The Derivative as a Function

Notice how in Example 2.1.7 and Exercise 2.1.8, we computed fundamentally the same limit, but in one case we had $a = 1$ as our x-value and in the other we had $a = 8$ as our x-value. In order to not have to redo the same computation over and over again, often we just leave the x-coordinate as the variable x itself and then plug in numbers for x later on down the road. This gives us the notion of the derivative as a function. We start with a function $f(x)$ that assigns y-coordinates to x-coordinates. We then think of $f'(x)$ as a new function that maps each x-coordinate to the slope of the tangent line to $f(x)$ at x.

Some notational options to be aware of: $f'(x)$ represents the derivative of the function f. An alternate way of writing it though is $\frac{\mathrm{d}f}{\mathrm{d}x}$ (sometimes called Leibniz notation or differential notation). This notation can be thought of intuitively as a fraction, representing a tiny change in f divided by the corresponding tiny change in x.

Example 2.1.16. The Derivative of the Cubed Root as a Function

Here we recompute the derivative of the cubed root but as a function rather than at a specific point. We use the h definition of the derivative this time just to demo the other formula. Again, the difference of two cubes is the main trick. We multiply the top and bottom by the factor that will complement the numerator and eliminate the radicals for us. We proceed as follows:

$$\begin{aligned}
\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h} &= \lim_{h\to 0}\frac{\left(\sqrt[3]{x+h}-\sqrt[3]{x}\right)\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}\\
&= \lim_{h\to 0}\frac{\left(\sqrt[3]{x+h}\right)^3-\left(\sqrt[3]{x}\right)^3}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}\\
&= \lim_{h\to 0}\frac{x+h-x}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}\\
&= \lim_{h\to 0}\frac{h}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}\\
&= \lim_{h\to 0}\frac{1}{\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}\\
&= \frac{1}{\left(\sqrt[3]{x+0}\right)^2+\sqrt[3]{x+0}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}\\
&= \frac{1}{\left(\sqrt[3]{x}\right)^2+\left(\sqrt[3]{x}\right)^2+\left(\sqrt[3]{x}\right)^2}\\
&= \frac{1}{3\left(\sqrt[3]{x}\right)^2}\\
&= \frac{1}{3x^{2/3}}.
\end{aligned}$$

Thus, we have that if $f(x)=\sqrt[3]{x}$, then

$$f'(x)=\frac{1}{3x^{2/3}}.$$

Again we mention some notational options. The following are all equivalent ways of writing what we computed above:

- $\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{1}{3x^{2/3}}$
- $\left(\sqrt[3]{x}\right)'=\frac{1}{3x^{2/3}}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt[3]{x}\right)=\frac{1}{3x^{2/3}}$.

The first is just Leibniz notation. The second and third are instances where we drop the f and use the explicit formula itself instead of the function name f. Sometimes we think of the "prime" or the $\frac{\mathrm{d}}{\mathrm{d}x}$ as operations that can be applied to formulas that produce a new formula, the derivative of the original formula.

Exercise 2.1.17. Checking the General Against the Specific ☕

Take the general formula for $f'(x)$ obtained in the previous example. Plug in $x = 1$ and $x = 8$ and verify that the formula produces the same numbers we calculated in Example 2.1.7 and Exercise 2.1.8.

Notice in the above example, $f'(x)$ is actually undefined at $x = 0$. This is a result of the tangent line being vertical at $x = 0$; vertical lines have undefined slope.

Exercise 2.1.18. Lines ☕☕

Consider a generic linear function

$$f(x) = mx + b.$$

- Just thinking graphically, what should $f'(x)$ be and why?
- Use the limit definition of the derivative to compute $f'(x)$ and confirm your suspicion from the previous part.

Exercise 2.1.19. Sine and Cosine ☕☕

Use the limit definition of the derivative to compute each of the following:

- $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x))$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x))$

Exercise 2.1.20. Exponential ☕☕

- Use the limit definition of the derivative to show that the derivative of the natural exponential function e^x is itself!

- Are there any other functions whose derivative equals the original function?

2.2 Properties of Derivatives

Power Rule

In this subsection, we come up with one of the most widely used derivative formulas, *the power rule*!

Exercise 2.2.1. Natural Number Exponent ☕☕

- Find the derivative of the function $f(x) = x$ using the limit definition of the derivative.
- Find the derivative of the function $f(x) = x^2$ using the limit definition of the derivative.
- Find the derivative of the function $f(x) = x^3$ using the limit definition of the derivative.
- Find the derivative of the function $f(x) = x^4$ using the limit definition of the derivative.
- Based on your work above, conjecture a pattern for the derivative of x^n for an arbitrary $n \in \mathbb{N}$. Specifically, if $f(x) = x^n$ then

$$f'(x) = \underline{\hspace{6cm}}.$$

Let's see if the conjectured formula still holds for negative integer powers!

Exercise 2.2.2. Negative Integer Powers ☕☕☕

- Find the derivative of the function $f(x) = x^{-1}$ using the limit definition of the derivative.

- Find the derivative of the function $f(x) = x^{-2}$ using the limit definition of the derivative.

- Do negative powers seem to fit the same pattern for their derivative as the positive powers? Explain.

Let's see if the formula above still holds for powers that are not integers!

Exercise 2.2.3. Fractional Powers ☕☕☕

- Find the derivative of the function $f(x) = x^{1/2}$ using the limit definition of the derivative.

- Find the derivative of the function $f(x) = x^{1/3}$ using the limit definition of the derivative.

- Find the derivative of the function $f(x) = x^{1/4}$ using the limit definition of the derivative.

- Do fractional powers seem to fit the same pattern for their derivative as the positive powers? Explain.

It turns out the examples above do in fact work for all real number exponents (though technically we haven't defined what exactly is meant by say x^{π}). We state this below.

Theorem 2.2.4. Power Rule

For all real numbers n, the derivative of $f(x) = x^n$ is

$$f'(x) = nx^{n-1}.$$

That is,

$$\frac{\mathtt{d}}{\mathtt{d}x}(x^n) = nx^{n-1}.$$

Exercise 2.2.5. The Zero Case ☕

What does the above theorem say if $n = 0$? Is it still valid? Or does it no longer make sense?

Linearity

One of the most important properties of derivatives is called *linearity*. This will follow directly from the limit definition of the derivative and Theorem 1.2.23.

Theorem 2.2.6. The Derivative is a Linear Operator

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$(f(x) + g(x))' = f'(x) + g'(x).$$

Furthermore, if $c \in \mathbb{R}$,

$$(c \cdot f(x))' = c \cdot (f'(x)).$$

A nice Tweedledum and Tweedledee way of saying the first property above is as follows:

The derivative a sum is the sum of the derivatives.

The second property can be simply put as

Constants factor out of derivatives.

Here we prove the first property.

Proof. We compute the left-hand side using the limit definition of the derivative and simplify until we reach the right-hand side. Proceeding, we have the following chain of equality:

$$\begin{aligned}
(f(x) + g(x))' &= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\
&= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h} \\
&= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\
&= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right) \\
&= f'(x) + g'(x).
\end{aligned}$$

□

Exercise 2.2.7. Identifying Theorem Use ☕

In the above proof, write a short justification of each step of the computation. Which step required use of Theorem 1.2.23?

Exercise 2.2.8. Proving the Second Property ☕☕

Follow the same process to prove the second linearity property! Be sure to indicate where Theorem 1.2.23 is used.

What is so nice about linearity is that if we encounter a sum of terms we have already found the derivative of, we don't need to take out the limit definition to compute their derivative, but rather just add together the formulas that we already know. Likewise, if we see a constant times something familiar, we can just let the constant sit there and use the familiar derivative.

Example 2.2.9. Differentiating with Linearity

Suppose we wish to compute the derivative of

$$f(x) = 4x^2 + 2x.$$

Using linearity, we reduce this to a power rule computation. Specifically,

$$\begin{aligned} f'(x) &= \left(4x^2 + 2x\right)' \\ &= \left(4x^2\right)' + (2x)' \\ &= 4\left(x^2\right)' + 2\,(x)' \\ &= 4 \cdot 2x^1 + 2 \cdot 1x^0 \\ &= 8x + 2. \end{aligned}$$

Exercise 2.2.10. Verifying with the Limit Definition ☕☕

Use the limit definition of the derivative to directly verify that $\frac{\mathrm{d}}{\mathrm{d}x}\left(4x^2+2x\right)=8x+2$ as claimed above.

Exercise 2.2.11. Power Rule and Linearity Practice ☕☕

For each of the following functions $f(x)$, find $f'(x)$ using linearity and/or the power rule as needed:

- $f(x)=3$
- $f(x)=\pi^2$
- $f(x)=x^2-3x$
- $f(x)=\frac{6}{\sqrt[3]{x}}$
- $f(x)=\frac{x+\sqrt{x}}{x}$
- $f(x)=20x^{5/2}$

Product Rule

Notice that Theorem 2.2.6 did not work for an arbitrary product of two functions together. It worked for sums of functions and for a product of a constant with function. This is because the Tweedledee phrase does not work for products of functions. That is to say, it is not always the case that the derivative of the product of two functions is the product of the individual derivatives.

Exercise 2.2.12. Showing the Product Rule is not Quite as Easy

Let $f(x) = x$ and $g(x) = x$. Show that the relationship

$$(f(x) \cdot g(x))' = f'(x) \cdot g'(x)$$

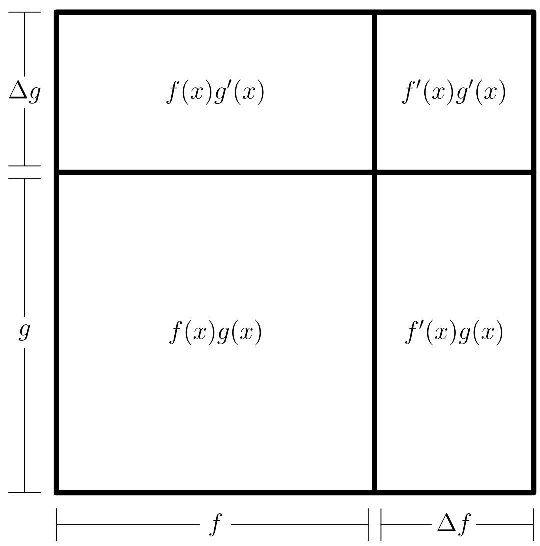
fails for this choice of functions.

A little geometry shows instead what the correct product rule is for derivatives. Think of the function $A(x) = f(x)g(x)$ as representing the area of an f by g box.

To correctly think about $(f(x) \cdot g(x))'$, we need to think about how A changes as x changes. There are essentially two places where it changes:

- As x changes, $f(x)$ will change at the rate of $f'(x)$. This will create a new rectangle of area $f'(x)g(x)$ that gets added to our box.

- As x changes, $g(x)$ will change at the rate of $g'(x)$. This will create a new rectangle of area $f(x)g'(x)$ that gets added to our box.



We summarize (product-ize?) the above geometry in the following formula.

Theorem 2.2.13. Product Rule for Derivatives

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

It may seem like we are missing that $f'(x)g'(x)$ box in the upper-right corner. However, it is in fact safe to discard that term. One way to think of it is that it is an infinitessimal quantity times yet another infinitessimal quantity and thus is of trivial magnitude compared to the other boxes (which have only one infinitessimal). We make this more formal in the proof below; since h goes to 0 in the derivative definition, any term in the numerator with h-degree of 2 or more will vanish in the limit, whereas terms of h-degree 1 will stick around.

Exercise 2.2.14. Proving the Product Rule ☕☕☕

To prove the product rule, we need a bit of trickery, which we walk through here.

- Write out the limit definition of the derivative for $(f(x) \cdot g(x))'$.
- Add and subtract the quantity $f(x+h)g(x)$ from the numerator.
- Regroup terms to create the limit definition of the derivatives for $f'(x) \cdot g(x) + f(x) \cdot g'(x)$, and declare victory.

Example 2.2.15. Using the Product Rule

Suppose we wish to compute the derivative of $e^x \cos(x)$. This would be utterly hideous with the limit definition of the derivative. With the product rule however, it's not too bad! Let e^x be our first function and $\cos(x)$ be our second function, then and then apply the product rule as follows:

first $=$ e^x	second $=$ $\cos(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}$first $=$ e^x	$\frac{\mathrm{d}}{\mathrm{d}x}$second $=$ $-\sin(x)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\cos(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)\cdot\cos(x) + e^x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)\right)\\ &= e^x\cos(x) + e^x(-\sin(x))\\ &= e^x\cos(x) - e^x\sin(x).\end{aligned}$$

Exercise 2.2.16. Practice with the Product Rule ☕☕☕

Use the product rule to evaluate the following derivatives. Note that you may need to combine it with linearity and/or power rule!

- $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x\right)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\sin(x)\right)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x\sin(x)-e^x\cos(x)}{2}\right)$

Exercise 2.2.17. Verifying the Power Rule ☕☕

- What is the derivative of x^2 via power rule? Do you get the same thing if you split it into $x \cdot x$ and use product rule?
- What is the derivative of x^3 via power rule? Do you get the same thing if you split it into $x^2 \cdot x$ and use product rule?
- What is the derivative of x^4 via power rule? Do you get the same thing if you split it into

$x^3 \cdot x$ and use product rule?

- What is the derivative of x^n via power rule? Do you get the same thing if you split it into $x^{n-1} \cdot x$ and use product rule?

Exercise 2.2.18. Ok This is Just Ridiculous ☕☕

- What is the derivative of the constant function 1?

- Is 1 equal to $x \cdot x^{-1}$?

- What is the derivative of $x \cdot x^{-1}$ via product rule? Does it match the derivative of 1?

- Is 1 equal to $\sqrt{x} \cdot \frac{1}{\sqrt{x}}$?

- What is the derivative of $\sqrt{x} \cdot \frac{1}{\sqrt{x}}$ via product rule? Does it match the derivative of 1?

Chain Rule

The chain rule might more appropriately be called the "Composition Rule for Derivatives" for sake of consistency with the product rule. It is a formula that shows how to differentiate a composition of two functions in terms of their individual derivatives. The construction of the formula is the following surprisingly short process:

$$
\begin{aligned}
(f(g(x)))' &= \lim_{h\to 0} \frac{f\left(g(x+h)\right) - f\left(g(x)\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(f\left(g(x+h)\right) - f\left(g(x)\right)\right)}{h} \cdot \frac{(g(x+h)-g(x))}{(g(x+h)-g(x))} \\
&= \lim_{h\to 0} \frac{f\left(g(x+h)\right) - f\left(g(x)\right)}{g(x+h)-g(x)} \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f\left(g(x+h)\right) - f\left(g(x)\right)}{g(x+h)-g(x)} \cdot g'(x).
\end{aligned}
$$

Now notice that if we make the substitutions $b = g(x+h)$ and $a = g(x)$, then as $h \to 0$, $b \to a$. Thus, the above expression becomes

$$
\begin{aligned}
(f(g(x)))' &= \lim_{b\to a} \frac{f\left(b\right) - f\left(a\right)}{b-a} \cdot g'(x) \\
&= f'(a) \cdot g'(x) \\
&= f'(g(x)) \cdot g'(x).
\end{aligned}
$$

Thus, we have come upon the chain rule!

Theorem 2.2.19. Chain Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$(f(g(x)))' = f'(g(x))g'(x).$$

Exercise 2.2.20. Justifying Each Line ☕

Write a short justification next to each line in the construction of the chain rule above.

To make all the symbols a little more memorable, sometimes one will call f the *outer* function and g the *inner*. This gives a nice wordy way to remember the chain rule as follows:

The derivative of a composition of function is outer prime of inner times inner prime.

Here we show how the formula and our little sentence line up with each other.

$$\underbrace{(f\circ g(x))'}_{\substack{\text{derivative of} \\ \text{a composition}}} = \underbrace{f'}_{\substack{\text{outer} \\ \text{prime...}}} \underbrace{(g(x))}_{\substack{\text{...of} \\ \text{inner...}}} \underbrace{\cdot g'(x)}_{\substack{\text{...times} \\ \text{inner} \\ \text{prime}}}$$

In Leibniz notation, the chain rule derivation above can be thought of much more succinctly as

$$\frac{\mathtt{d}f}{\mathtt{d}x} = \frac{\mathtt{d}f}{\mathtt{d}g}\frac{\mathtt{d}g}{\mathtt{d}x},$$

where we can imagine the truth of the statement coming from just canceling the $\mathtt{d}g$ in the numerator with the $\mathtt{d}g$ in the denominator. This isn't truly a fraction but rather a limit of fractions, so it isn't necessarily rigorous to perform such algebraic manipulations on those expressions. However, the intuition is essentially correct, and it provides an easy way to remember the formula.

Example 2.2.21. Applying the Chain Rule

Suppose we want the derivative of the function $\cos^2(x)$. Note that this is really a composition of two functions, namely our outer function,

$$f(x) = x^2$$

and inner function

$$g(x) = \cos(x)$$

then

$$f \circ g(x) = f\left(g(x)\right) = \left(\cos(x)\right)^2 = \cos^2(x).$$

To take the derivative, we apply the chain rule, treating the square as the outer function and cosine as the inner.

outer $=$ x^2	inner $=$ $\cos(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}$outer $=$ $2x$	$\frac{\mathrm{d}}{\mathrm{d}x}$inner $=$ $-\sin(x)$

Proceeding, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos^2(x)\right) = 2 \cdot (\cos(x)) \cdot (-\sin(x)) = -2\cos(x)\sin(x).$$

Exercise 2.2.22. Checking Our Work with the Product Rule ☕

Treat the function $\cos^2(x)$ as a product by splitting it as $\cos(x) \cdot \cos(x)$. Differentiate it using the product rule. Do you get the same answer as we did in the previous example?

Exercise 2.2.23. Verifying the Chain Rule ☕☕

Consider the function $f(x) = (2x + 3)^2$.

- Find $f'(x)$ by expanding the polynomial and using power rule term-by-term.

- Find $f'(x)$ by using the chain rule.

- Verify that your answers match.

Here is some more fun with polynomials. There are just so many things you can do with those.

Exercise 2.2.24. Squares of Cubes and Cubes of Squares ☕☕

- Find the derivative of the function x^6 by the power rule.

- Find the derivative of the function x^6 by rewriting as $\left(x^2\right)^3$ and then applying chain rule. Verify it matches your previous answer.

- Find the derivative of the function x^6 by rewriting as $\left(x^3\right)^2$ and then applying chain rule. Verify it matches your previous answers.

Note also that the chain rule can be iterated if there is a composition of three or more functions. When you get to the "inner prime" step of one chain rule application, you might need to use the chain rule again!

Example 2.2.25. An Iterated Chain Rule

Suppose we wish to differentiate the function

$$e^{e^{e^x}}.$$

Here the outer function is clearly arctangent. Nah. Hehe. The outer function is e^x. The inner function is e^{e^x}.

outer $=$ e^x	inner $=$ e^{e^x}
$\frac{d}{dx}$outer $=$ e^x	$\frac{d}{dx}$inner $=$

To take the derivative of our inner function, we need to do another chain rule! This time the outer function is e^x and the inner function is also e^x!

outer $= e^x$	inner $= e^x$
$\frac{d}{dx}$outer $= e^x$	$\frac{d}{dx}$inner $= e^x$

So we get that

$$\frac{d}{dx}\left(e^{e^x}\right) = e^{e^x}e^x.$$

Then we can complete our previous table:

outer $= e^x$	inner $= e^{e^x}$
$\frac{d}{dx}$outer $= e^x$	$\frac{d}{dx}$inner $= e^{e^x}e^x$

Finally, we put together our chain rule:

$$\begin{aligned}\frac{d}{dx}\left(e^{e^{e^x}}\right) &= e^{e^{e^x}}\frac{d}{dx}\left(e^{e^x}\right)\\ &= e^{e^{e^x}}e^{e^x}e^x\\ &= e^{\left(e^{e^x}+e^x+x\right)}.\end{aligned}$$

Exercise 2.2.26. Chain Rule Practice ☕☕☕

Use the chain rule (along with linearity, power rule, and product rule as needed) to find the derivative of each of the following functions:

- $\cos(\sin(x))$
- $\sin(\cos(x))$
- $e^{(e^x-1)}$
- $\cos(\sqrt{x}e^x)$

- $e^{e^{e^{e^x}}}$

Exercise 2.2.27. Derivative of Tangent ☕☕

Recall that the function tangent is the ratio of sine to cosine. Find the derivative of tangent by using the product rule, power rule, and chain rule via the equality

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \sin(x) \cdot (\cos(x))^{-1}.$$

Express your answer in terms of secant.

Exercise 2.2.28. The Other Trig Functions ☕☕☕

Use the technique of the previous exercise to find derivatives of the remaining three trig functions. Express the derivative of secant in terms of secant and tangent. Express the derivatives of cosecant and cotangent in terms of cosecant and/or cotangent.

- $\frac{\mathrm{d}}{\mathrm{d}x}(\sec(x))$

- $\frac{\mathrm{d}}{\mathrm{d}x}(\csc(x))$

- $\frac{\mathrm{d}}{\mathrm{d}x}(\cot(x))$

Quotient Rule

In theory, we don't need a separate quotient rule for derivatives, since we already have a product rule and power rule (as the tangent exercise above demonstrates). However, it comes up often enough that it is sometimes efficient to give it its own dedicated rule. We apply our method from the tangent example to a quotient of generic functions and then simplify the answer. We compute as follows:

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(f(x)\cdot(g(x))^{-1}\right)' \\
&= f'(x)\cdot(g(x))^{-1} + f(x)\cdot(-1)\,(g(x))^{-2}\,g'(x) \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)} \\
&= \frac{f'(x)g(x)}{g^2(x)} - \frac{f(x)g'(x)}{g^2(x)} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.
\end{aligned}$$

Theorem 2.2.29. Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

A little sing-songy way this is commonly remembered is built by calling $f(x)$ *hi* (since it is high up in the numerator) and calling $g(x)$ *lo* (since it is down low in the denominator). Also abbreviate "derivative of" as just *dee*. This produces the following phrase for the quotient rule:

lo dee hi minus hi dee lo all over lo squared

though the author recommends just thinking of it via the construction above. It looks like the product rule but the -1 exponent spits out a minus sign and creates a -2 exponent when the power rule is applied.

Example 2.2.30. Gratuitious Quotient Rule

Just for sake of demonstrating the quotient rule, we use it to differentiate the function $x/2$. Note that this is totally unnecessary, as we can pull constants out of derivatives, so it is much easier to simply do

$$(x/2)' = \left(\frac{1}{2}\right)' = \frac{1}{2}\,(x)' = \frac{1}{2}1 = \frac{1}{2}.$$

The quotient rule however will also produce a correct result. We use x as the numerator, the constant function 2 as the denominator, and go for it! Proceeding, we have

$$(x/2)' = \frac{(x)'2 - (x)(2)'}{2^2} = \frac{1\cdot 2 - (x)(0)}{4} = \frac{2}{4} = \frac{1}{2}.$$

Exercise 2.2.31. Even More Gratuitous ☕☕

Use the quotient rule to differentiate the function x/x. Since it equals 1, the derivative must be what?

Exercise 2.2.32. Tangent Again ☕☕

Use the quotient rule to again find the derivative of tangent, where $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Verify the result matches the answer from Exercise 2.2.27.

Exercise 2.2.33. Comparing to Chain Rule ☕☕

- Differentiate the function e^{-x} using the chain rule.
- Differentiate the function e^{-x} using the quotient rule by rewriting as $1/e^x$. Verify your answers match! They will, since you can't spell match without m-a-t-h.

Exercise 2.2.34. Quotient Rule Practice ☕☕☕

Use the quotient rule (along with linearity, power rule, product rule, and chain rule as needed) to find the derivative of each of the following functions:

- $\frac{\cos(x)\sin(x)}{e^x}$

- $\frac{1-\sqrt{1-4x}}{2x}$

2.3 But What If We Don't Have an Explicit Formula?

Up until this point, we always assumed we had an explicit formula for the function whose slope we were interested in. That is, we had the graph as given by some formula of the form

$$y = f(x)$$

where the right-hand side was expressed in finitely many operations using just the variable x and any constants we like. But what if we don't have an explicit formula for the graph at hand? Well, perhaps the inverse graph is for some reason easier to understand! Or, perhaps we have an equation for the graph, but it just isn't solved for y all nice and neat. These two situations are explored in this section.

Inverse Function Theorem

This section generalizes the relationship we noticed in Exercise 2.1.10. In particular, we would like to be able to express the derivative of an inverse of a function in terms of the original function. In that example, we noticed that the slope got reciprocated when an inverse function was applied, since x and y coordinates got interchanged. Essentially we are just noticing that the following quantities are reciprocals of each other:

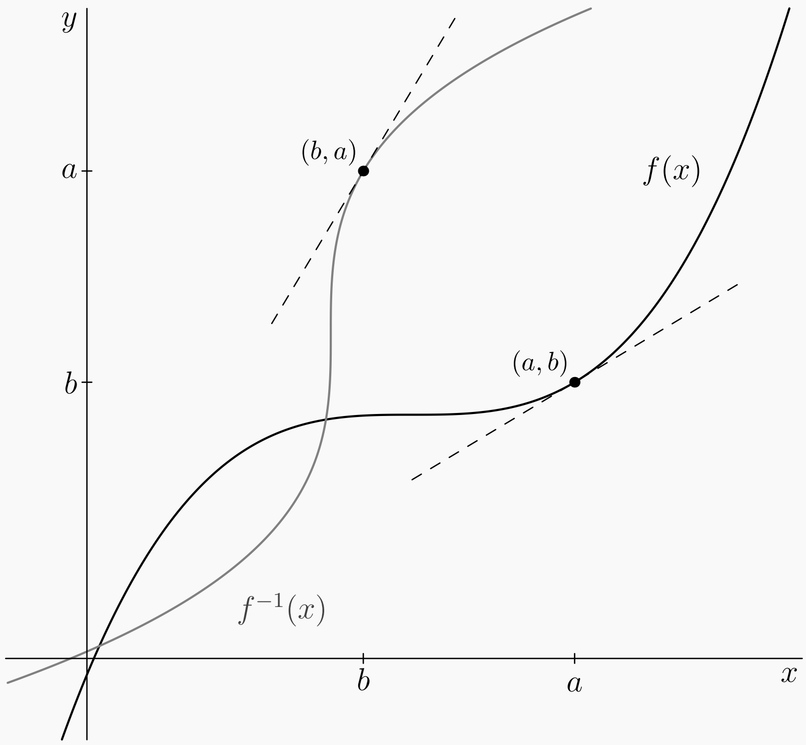
$$\frac{\text{Change in } y}{\text{Change in } x} \quad \underset{\text{reciprocal}}{\overset{\substack{\text{swap} \\ x \text{ and } y}}{\longleftrightarrow}} \quad \frac{\text{Change in } x}{\text{Change in } y}$$

We now state this idea more formally.

Theorem 2.3.1. Inverse Function Theorem (IFT)

Let $f(x)$ be an invertible function and let $f^{-1}(x)$ denote its inverse. Let the point (a, b) represent a point on the graph of $f(x)$. Then (b, a) is a point on the graph of $f^{-1}(x)$, and

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}.$$



Note that the notation f^{-1} here does not mean "one over f" but rather is being used in the sense of an inverse function. For example, if $f(x) = \cos(x)$, then $f^{-1}(x) \neq \sec(x)$ but rather $f^{-1}(x) = \arccos(x)$.

Exercise 2.3.2. Rewriting the Inverse Function Theorem ☕

- Explain why in the above theorem, $f(a) = b$.
- As a consequence, explain why $f^{-1}(b) = a$.
- Replace all occurrences of a with $f^{-1}(x)$ and all occurrences of b with x.

We show the result of the previous manipulations in a nice fancy box because it deserves one.

Formula 2.3.3. Formula for the Derivative of an Inverse Function

If $f^{-1}(x)$ is the inverse of an invertible differentiable function $f(x)$, then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

One caveat of course is that $f'\left(f^{-1}(x)\right)$ must not be zero in order for the above formula to be true. However, in this case it still tells you that there is a vertical tangent line at that point.

Exercise 2.3.4. Chain Rule Derivation of IFT ☕☕

Rather than the very geometric argument we gave above for IFT, one can also do a very algebraic argument. In particular, we can just start with the definition of an inverse function as a function satisfying

$$f\left(f^{-1}(x)\right) = x.$$

Differentiate both sides with respect to x, applying the chain rule. Solve for $\left(f^{-1}\right)'(x)$ and verify the resulting formula matches what we stated in IFT.

Before trying out a new tool in an unfamiliar situation, it is wise to test it in a situation where you already know the answer.

Exercise 2.3.5. Reciprocal ☕☕☕

- What is the derivative of $1/x$?

- Verify that $1/x$ is its own inverse function. That is, if $f(x) = 1/x$, show that $f^{-1}(x) = 1/x$ as well.

- Use IFT to compute the derivative of $f^{-1}(x) = 1/x$. Verify it matches your result from above.

Derivatives of Logarithms

Since we already know how to differentiate the natural exponential function, IFT is the perfect tool for finding the derivative of the natural logarithm!

Example 2.3.6. Derivative of the Natural Logarithm

Recall the function $\ln(x)$, defined as the inverse function of e^x. We apply the IFT to find the derivative of the natural logarithm. In particular, we take

$$f(x) = e^x$$

and

$$f^{-1}(x) = \ln(x).$$

This declaration also implies

$$f'(x) = e^x$$

since the natural exponential function is its own derivative. Plugging all of these components into IFT, we have

$$\begin{aligned}(\ln(x))' &= \left(f^{-1}(x)\right)' \\ &= \frac{1}{f'\left(f^{-1}(x)\right)} \\ &= \frac{1}{f'\left(\ln(x)\right)} \\ &= \frac{1}{e^{(\ln(x))}} \\ &= \frac{1}{x}.\end{aligned}$$

Thus, the derivative of the natural logarithm is $1/x$. That is,

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\ln(x)\right) = \frac{1}{x}.$$

The same trick works for other bases!

Exercise 2.3.7. Base 2 ☕☕☕

- Explain why the function $f(x) = 2^x$ can be rewritten as
$$f(x) = e^{\ln(2)x}.$$

- Use that rewritten form and the chain rule to find the derivative of $f(x) = 2^x$.

- Use your answer above and IFT to find the derivative of $f^{-1}(x) = \log_2(x)$.

- Was there anything special about the base 2? Suppose 2 was replaced by some other positive real number a. What formulas hold for the derivatives of a^x and $\log_a(x)$?

Derivatives of Inverse Trigonometric Functions

We now apply the same framework to find the derivatives of our inverse trigonometric functions.

Example 2.3.8. Derivative of Arcsine

Recall the function $\arcsin(x)$, defined as the inverse function of $\sin(x)$. We apply IFT to find the derivative of the natural logarithm. In particular, we take

$$f(x) = \sin(x)$$

and

$$f^{-1}(x) = \arcsin(x).$$

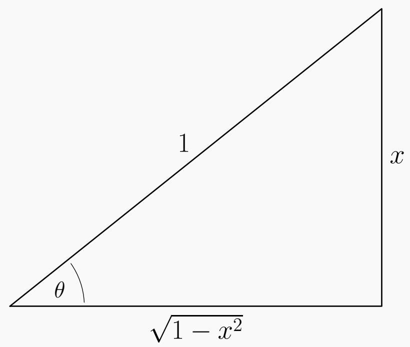
This declaration also implies

$$f'(x) = \cos(x).$$

Plugging all of these components into IFT, we have

$$\begin{aligned}(\arcsin(x))' &= \left(f^{-1}(x)\right)' \\ &= \frac{1}{f'\left(f^{-1}(x)\right)} \\ &= \frac{1}{f'\left(\arcsin(x)\right)} \\ &= \frac{1}{\cos\left(\arcsin(x)\right)}.\end{aligned}$$

It would not be mathematically incorrect to leave the answer as written above, however it is morally incorrect. We have a trigonometric function (cos) of an inverse trigonometric function (arcsin) so we can clean these up. Specifically, think of the $\arcsin(x)$ as the phrase "the angle whose sine is x" and draw a triangle with hypotenuse on the unit circle representing this.



We can then use Pythagorean Theorem to solve for the other sides of the triangle. This allows us to evaluate cosine as the adjacent side length divided by the hypotenuse (which is just 1 in this case). Therefore,

$$\cos\left(\arcsin(x)\right) = \sqrt{1-x^2}.$$

Thus, the derivative of the inverse sine function is $\frac{1}{\sqrt{1-x^2}}$. That is,

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}.$$

Exercise 2.3.9. Some Details ☕

- In the above example, why were we able to assume the hypotenuse was length one?
- Show the algebra that was used to compute the adjacent side as length $\sqrt{1-x^2}$.

Exercise 2.3.10. Derivative of Arctangent ☕☕☕

Follow the method of the above example to find the derivatives of the other five inverse trig functions!

- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos(x)\right)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan(x)\right)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{arcsec}(x)\right)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{arccsc}(x)\right)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{arccot}(x)\right)$

Derivatives of Hyperbolic Trig Functions and Their Inverses

Recall (or if you haven't seen them in your precalculus course, then "meet!") the *hyperbolic trigonometric functions.*

Definition 2.3.11. Exponential Formulas for Hyperbolic Trig Functions

Define the hyperbolic trig functions as follows:

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$ • $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ • $\operatorname{sech}(x) = \frac{1}{\cosh(x)}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$ • $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$ • $\operatorname{csch}(x) = \frac{1}{\sinh(x)}$.

Here we explore the properties of these functions and note many parallels between them and the standard trigonometric functions. In Calculus II and III you'll see even more parallels, including what they have to do with hyperbolas!

Exercise 2.3.12. Derivatives of the Hyperbolic Trig Functions ☕☕

Find the derivatives of the six hyperbolic trig functions. Express your answers not in terms of exponential functions (though you may need those at intermediate steps), but rather in terms of the hyperbolic trig functions.

- $\frac{d}{dx}(\sinh(x))$
- $\frac{d}{dx}(\cosh(x))$
- $\frac{d}{dx}(\tanh(x))$
- $\frac{d}{dx}(\operatorname{sech}(x))$
- $\frac{d}{dx}(\coth(x))$
- $\frac{d}{dx}(\operatorname{csch}(x))$

For each of those six functions, we write arc $***$h to denote the inverse of $***$ h. Since the original hyperbolic trig functions are defined in terms of exponentials, it is sensible to think the inverses would have formulas using logarithms. We show one of these formulas and verify it below.

Example 2.3.13. Logarithmic Formula for arcsech

Consider the inverse function of hyperbolic secant, arcsech. We claim it has the following formula in terms of logarithms:

$$\operatorname{arcsech}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

To verify this formula is valid, we compute the composition of it with

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}.$$

The author recommends fully hydrating before proceeding. The composition is

$$\begin{aligned}
\operatorname{sech}(\operatorname{arcsech}(x)) &= \operatorname{sech}\left(\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right) \\
&= \frac{2}{e^{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)} + e^{-\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}} \\
&= \frac{2}{e^{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)} + e^{-1\cdot\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}} \\
&= \frac{2}{e^{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)} + e^{\ln\left(\left(\frac{1+\sqrt{1-x^2}}{x}\right)^{-1}\right)}} \\
&= \frac{2}{\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \left(\frac{1+\sqrt{1-x^2}}{x}\right)^{-1}} \\
&= \frac{2}{\frac{1+\sqrt{1-x^2}}{x} + \frac{x}{1+\sqrt{1-x^2}}} \\
&= \frac{2}{\frac{\left(1+\sqrt{1-x^2}\right)^2}{x\left(1+\sqrt{1-x^2}\right)} + \frac{x^2}{x\left(1+\sqrt{1-x^2}\right)}} \\
&= \frac{2}{\left(\frac{\left(1+\sqrt{1-x^2}\right)^2+x^2}{x\left(1+\sqrt{1-x^2}\right)}\right)} \\
&= \frac{2x\left(1+\sqrt{1-x^2}\right)}{\left(1+\sqrt{1-x^2}\right)^2+x^2} \\
&= \frac{2x\left(1+\sqrt{1-x^2}\right)}{1+2\sqrt{1-x^2}+(1-x^2)+x^2} \\
&= \frac{2x\left(1+\sqrt{1-x^2}\right)}{2+2\sqrt{1-x^2}} \\
&= \frac{2x\left(1+\sqrt{1-x^2}\right)}{2\left(1+\sqrt{1-x^2}\right)} \\
&= x
\end{aligned}$$

which is just the identity map!

Exercise 2.3.14. Annotating the Computation ☕

Write a short phrase next to each line above, explaining what is being done on each phase of the algebra.

Though the above calculation shows the logarithmic formula for hyperbolic secant is correct, it does not give any indication how one might have come up with it themselves. Here is one possible way to construct such a formula.

Exercise 2.3.15. Finding the Log Formula for Hyperbolic Secant ☕☕☕

Recall the exponential formula for hyperbolic secant, defined as the reciprocal of hyperbolic cosine as follows:

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}.$$

We now carry out the standard process for finding a formula of an inverse function.

- Consider the formula $y = \frac{2}{e^x+e^{-x}}$. Interchange the roles of y and x in order to come up with an equation that all the points on the graph of the inverse function (arcsech) satisfy.

- To find the formula for the inverse function arcsech, we need to solve that equation for y. To do this, carry out the following algebraic steps:
  - Multiply both sides by $e^y + e^{-y}$.
  - Multiply both sides by e^y.
  - Make the substitution $w = e^y$ to eliminate all occurrences of y in the equation.
  - Notice that the equation is quadratic in w and apply the quadratic formula to find w. How do you know whether to choose the plus or minus?
  - Explain why $y = \ln(w)$. Plug your formula for w into that relationship to find y, thus finding the formula for $\operatorname{arcsech}(x)$.

We now do a bit of algebra that will be useful in the following example.

Exercise 2.3.16. Alternate Formula for Hyperbolic Tangent ☕

Verify that hyperbolic tangent can be written as

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

And finally, back to derivatives!

Example 2.3.17. Using IFT on Inverse Hyperbolic Secant

Let us apply IFT to the inverse hyperbolic secant function. In particular, we take

$$f(x) = \operatorname{sech}(x)$$

and

$$f^{-1}(x) = \operatorname{arcsech}(x).$$

This declaration also implies

$$f'(x) = -\operatorname{sech}(x)\tanh(x).$$

Plugging all of these components into IFT, we have

$$\begin{aligned}
(\operatorname{arcsech}(x))' &= \left(f^{-1}(x)\right)' \\
&= \frac{1}{f'\left(f^{-1}(x)\right)} \\
&= \frac{1}{f'\left(\operatorname{arcsech}(x)\right)} \\
&= \frac{1}{-\operatorname{sech}\left(\operatorname{arcsech}(x)\right)\tanh(\operatorname{arcsech}(x))}.
\end{aligned}$$

Once again, we in theory could just leave it written as above, but the denominator will clean up enormously with a little simplification. The hyperbolic secant and its inverse cancel to just be a single x. The second factor is a bit more complicated, but can be simplified using the exponential and logarithmic formulas for the hyperbolic and inverse hyperbolic trig functions, respectively! We carry this simplification out as follows:

$$\begin{aligned}
\tanh(\operatorname{arcsech}(x)) &= \frac{e^{2\operatorname{arcsech}(x)} - 1}{e^{2\operatorname{arcsech}(x)} + 1} \\
&= \frac{e^{2\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)} - 1}{e^{2\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)} + 1} \\
&= \frac{e^{\ln\left(\left(\frac{1+\sqrt{1-x^2}}{x}\right)^2\right)} - 1}{e^{\ln\left(\left(\frac{1+\sqrt{1-x^2}}{x}\right)^2\right)} + 1} \\
&= \frac{\left(\frac{1+\sqrt{1-x^2}}{x}\right)^2 - 1}{\left(\frac{1+\sqrt{1-x^2}}{x}\right)^2 + 1} \\
&= \frac{\left(1+\sqrt{1-x^2}\right)^2 - x^2}{\left(1+\sqrt{1-x^2}\right)^2 + x^2} \\
&= \frac{1 + 2\sqrt{1-x^2} + \left(1-x^2\right) - x^2}{1 + 2\sqrt{1-x^2} + \left(1-x^2\right) + x^2} \\
&= \frac{2 - 2x^2 + 2\sqrt{1-x^2}}{2 + 2\sqrt{1-x^2}} \\
&= \frac{\left(2 - 2x^2 + 2\sqrt{1-x^2}\right)\left(2 - 2\sqrt{1-x^2}\right)}{\left(2 + 2\sqrt{1-x^2}\right)\left(2 - 2\sqrt{1-x^2}\right)}.
\end{aligned}$$

Again, we could leave the expression as is, however we clean it up just a bit more using the conjugate of the denominator. In particular,

$$\begin{aligned}
\tanh(\operatorname{arcsech}(x)) &= \frac{\left(2-2x^2+2\sqrt{1-x^2}\right)\left(2-2\sqrt{1-x^2}\right)}{\left(2+2\sqrt{1-x^2}\right)\left(2-2\sqrt{1-x^2}\right)} \\
&= \frac{4-4x^2+4\sqrt{1-x^2}-4\sqrt{1-x^2}+4x^2\sqrt{1-x^2}-4\left(1-x^2\right)}{4-4\left(1-x^2\right)} \\
&= \frac{4x^2\sqrt{1-x^2}}{4x^2} \\
&= \sqrt{1-x^2}.
\end{aligned}$$

Returning now to the original computation, we have

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\operatorname{arcsech}(x)\right) = -\frac{1}{x\sqrt{1-x^2}}.$$

Exercise 2.3.18. An Alternate Approach ☕☕☕

Recall once more our logarithmic formula for inverse hyperbolic secant, namely

$$\operatorname{arcsech}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

Rather than using IFT as we did in the previous example, find the derivative directly by differentiating the formula above via chain rule and quotient rule.

It would not be hyperbolic to say the following exercises will likely take hours.

Exercise 2.3.19. Inverse Hyperbolic Sine ☕☕☕

Do all of the above for $\operatorname{arcsinh}(x)$! By all of the above, we mean the following:

- Use the "swap x and y" trick on the exponential formula for $\sinh(x)$ to find a logarithmic formula for $\operatorname{arcsinh}(x)$.

- Compose the logarithmic formula for $\operatorname{arcsinh}(x)$ with the exponential formula for $\sinh(x)$ to verify they are inverses (at least in one direction).

- Find the derivative of $\operatorname{arcsinh}(x)$ by using IFT.

- Find the derivative of $\operatorname{arcsinh}(x)$ by differentiating your logarithmic formula for $\operatorname{arcsinh}(x)$.

Verify the two methods produce the same derivative!

Exercise 2.3.20. Inverse Hyperbolic Tangent ☕☕☕

Once again, do all of the above for $\operatorname{arctanh}(x)$! By all of the above, we mean the following:

- Use the "swap x and y" trick on the exponential formula for $\tanh(x)$ to find a logarithmic formula for $\operatorname{arctanh}(x)$.

- Compose the logarithmic formula for $\operatorname{arctanh}(x)$ with the exponential formula for $\tanh(x)$ to verify they are inverses (at least in one direction).

- Find the derivative of $\operatorname{arctanh}(x)$ by using IFT.

- Find the derivative of $\operatorname{arctanh}(x)$ by differentiating your logarithmic formula for $\operatorname{arctanh}(x)$. Verify the two methods produce the same derivative!

In the context of calculus, most of the time when we work with a function, we write it as an *explicit formula*, a formula that has finitely many symbols that gives some fixed number of operations to be performed on a given input. For example,

$$y = \sin(x)$$

is an explicit formula for y in terms of x. This is in contrast to an *implicit formula* for such a function; one that describes a relationship between the inputs and outputs without necessarily providing a clear instruction set as to how one can take an input and obtain a corresponding output. For example, every point on the graph of $y = \sin(x)$ satisfies the relationship

$$y^2 + \cos^2(x) = 1.$$

This would be an implicit formula for the sine function. Sometimes, one can obtain an explicit formula from an implicit formula by algebra.

Exercise 2.3.21. Explicit to Implicit ☕

Make the substitution $y = \sin(x)$ in the equation $y^2 + \cos^2(x) = 1$ to verify the above claim, that the explicit formula satisfies that implicit formula. Specifically, replace each occurrence of y with

$\sin(x)$ and explain why the resulting equation is true for all x.

Implicit Differentiation

Usually we differentiate explicit formulas, something of the form $y = f(x)$. But we can also perform *implicit differentiation*, that is, differentiating an implicit formula. To do so, perform the following steps:

- Choose which variable you want to consider as your independent variable.
- Choose which other variables are dependent on that independent variable.
- Differentiate both sides of your equation with respect to the independent variable, treating all dependent variables as unknown unspecified functions of the dependent variable. This means we apply a chain rule whenever we see an independent variable with any function applied to it.

Example 2.3.22. Implicit Differentiation of Sine

Let us find the derivative of $y = \sin(x)$ by using implicit differentiation on the equation

$$y^2 + \cos^2(x) = 1.$$

We imagine y as an unknown function of x and differentiate both sides with respect to x. We then solve the resulting equation for $\frac{\mathtt{d}y}{\mathtt{d}x}$. This produces the following:

$$\begin{aligned} 2y\frac{\mathtt{d}y}{\mathtt{d}x} + 2\cos(x)\left(-\sin(x)\right) &= 0 \\ \frac{\mathtt{d}y}{\mathtt{d}x} &= \frac{2\cos(x)\sin(x)}{2y} \\ \frac{\mathtt{d}y}{\mathtt{d}x} &= \frac{\cos(x)\sin(x)}{y} \end{aligned}$$

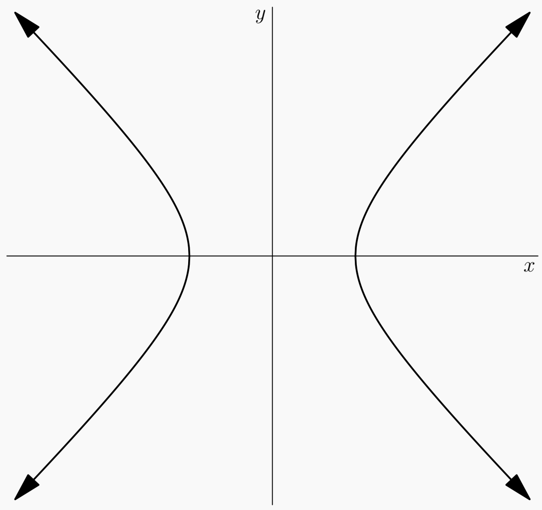
Exercise 2.3.23. Verifying the Example ☕

In the above example, what do you get for $\frac{\mathtt{d}y}{\mathtt{d}x}$ if you plug in $y = \sin(x)$ in the denominator of the final formula? Does this make sense?

Example 2.3.24. A Hyperbola

Consider the graph below of the hyperbola

$$x^2 - y^2 = 1.$$



Since it fails the vertical line test, this is not the graph of a function. Thus, this is a nice context for implicit differentiation (since it still makes sense to ask for the slope of a tangent line to the graph).

We apply implicit differentiation to find a formula for $\frac{dy}{dx}$ as follows:

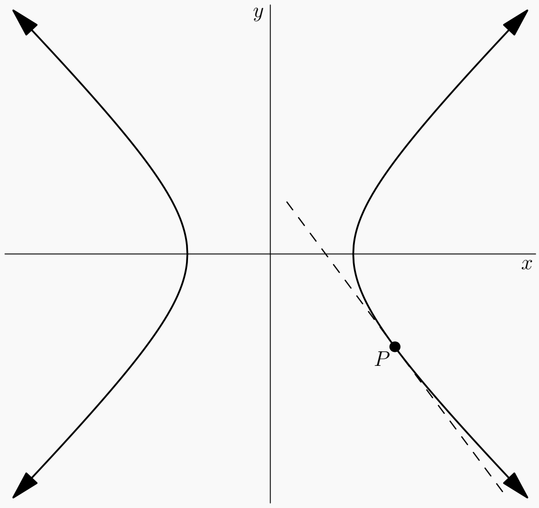
$$\begin{aligned} 2x - 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y} \end{aligned}$$

Suppose we wish to find the equation of the tangent line at the point $P = \left(3/2, -\sqrt{5}/2\right)$. We can simply plug these x and y values into the formula for $\frac{dy}{dx}$, as follows:

$$\begin{aligned} \frac{dy}{dx} &= \frac{3/2}{-\sqrt{5}/2} \\ &= -\frac{3\sqrt{5}}{5} \end{aligned}$$

Using point-slope form for a line, we now identify the tangent line as

$$y + \frac{\sqrt{5}}{2} = -\frac{3\sqrt{5}}{5}\left(x - \frac{3}{2}\right).$$



Exercise 2.3.25. Verifying the Implicit Differentiation

Let us check our work in the above example by constructing the same tangent line in a different manner.

- Solve the equation $x^2 - y^2 = 1$ for y, choosing the negative square root. What portion of original hyperbola graph do we now have?

- Use ordinary differentiation to find y'.

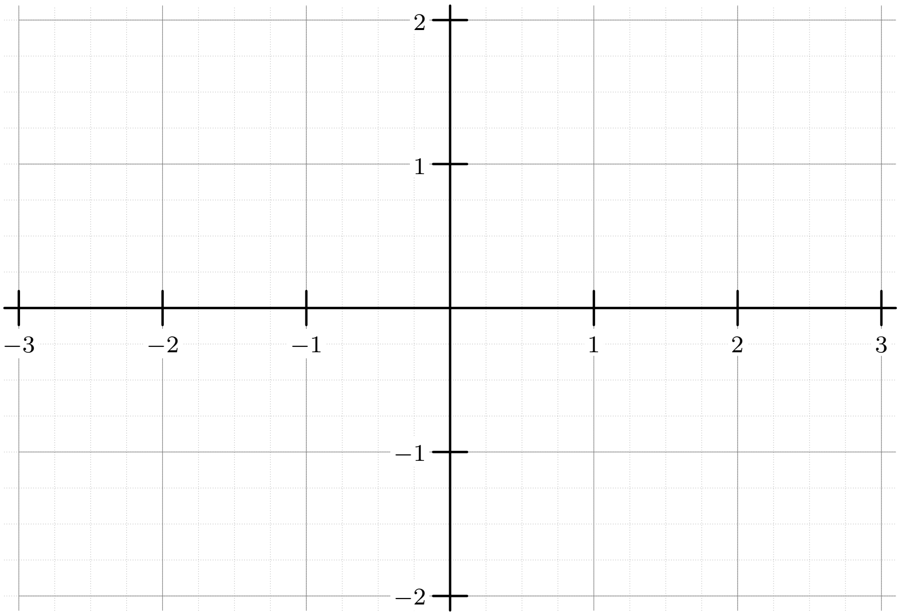
- Plug in the value $x = 3/2$ into your formula for y' and verify the resulting slope matches what we obtained via implicit differentiation.

Exercise 2.3.26. An Ellipse

Consider the equation

$$\frac{x^2}{4} + y^2 = 1.$$

- Draw the graph of the equation below.



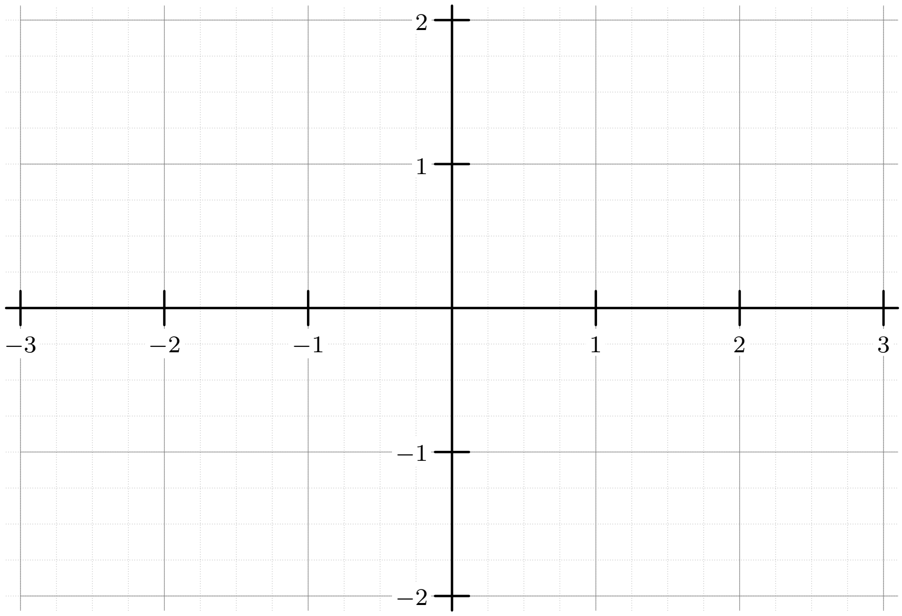
- Verify the point $P = \left(\sqrt{2}, \sqrt{2}/2\right)$ is on the ellipse by showing that the coordinates satisfy the equation. Plot and label this point on your graph above.
- Use implicit differentiation to find $\frac{dy}{dx}$.
- Evaluate your your derivative at point P. Use this information to find the equation of the tangent line to the graph at P.
- Plot the tangent line to the graph at point P on your graph above.
- Solve the ellipse equation for y to obtain an explicit formula for y (choosing the positive sign on the square root).
- Use ordinary differentiation to again calculate $\frac{dy}{dx}$. Plug in the value $x = \sqrt{2}$ and verify that it returns the same slope as implicit differentiation.

Exercise 2.3.27. A Cubic

Consider the cubic equation $y - y^3 = x$.

- Explain why the graph of that equation must be the reflection of the graph of $f(x) = x - x^3$ through the line $y = x$.

- Use the trick mentioned above to graph the solution to the original cubic equation.



- Use implicit differentiation to find a formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

- Use your formula to find the equation of the tangent line at the point $P = (0, 1)$.

- Find the same line by finding the tangent line to $f(x) = x - x^3$ at the point $(1, 0)$ and then applying a reflection through $y = x$. Verify your results match!

2.4 Differentiable Implies Continuous

Here we look at an important connection between two of the big concepts in calculus: differentiability and continuity. Loosely put, we say that differentiability implies continuity. More formally this result is stated as the theorem below.

Theorem 2.4.1. Differentiable Implies Continuous

Let $f : D \to \mathbb{R}$ be a function and $a \in D$. If f is differentiable at a, then f is also continuous at a.

Because continuity and differentiability are both defined using limits, the properties of limits will be critical in the proof of the theorem. For ease of reference during the proof, here are some commonly used ones listed below:

- Additive: If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a}(f(x) + g(x))$ exists and is equal to $\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$.
- Multiplicative: If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a}(f(x) \cdot g(x))$ exists and is equal to $\lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
- Limit of a Constant Function is That Constant: If $c \in \mathbb{R}$, then $\lim_{x\to a} c = c$.

Exercise 2.4.2. Filling in Holes in the Proof ☕☕☕

Read and fill in the blanks on the following proof of our theorem.

Proof. First recall the definition of continuity. To show that f is continuous at a, we must show that

$$\underline{\hspace{6cm}}.$$

We do this by showing the difference $f(x) - f(a)$ approaches ______________ as x approaches a. The given information is that f is differentiable at a. The limit definition of the derivative tells us that

$$f'(a) = \underline{\hspace{6cm}}.$$

To connect the given information to the unknown, we multiply by one in the form of $\frac{x-a}{x-a}$, which is valid as long as $x \neq a$. Proceeding, we have

$$\begin{aligned}
\lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) && (2.1)\\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a}(x - a) && (2.2)\\
&= f'(a) \lim_{x\to a}(x - a) && (2.3)\\
&= f'(a) \cdot 0 && (2.4)\\
&= 0. && (2.5)
\end{aligned}$$

On line (4.1) we don't need to worry about the case where $x = a$ because

____.

Line (4.2) is valid because

____.

Line (4.3) is just using the definition of

____

and the assumption that f is ______________. Line (4.4) uses the fact that the function $x - a$ is ______________. Line (4.5) we're not too worried about.
To use what we have just computed, we now start with the limit of $f(x)$ and we add zero in the form of $f(a) - f(a)$.

$$\lim_{x\to a} f(x) = \lim_{x\to a} (f(a) + f(x) - f(a)) \tag{2.6}$$
$$= \lim_{x\to a} f(a) + \lim_{x\to a} (f(x) - f(a)) \tag{2.7}$$
$$= f(a) + \lim_{x\to a} (f(x) - f(a)) \tag{2.8}$$
$$= f(a) + 0 \tag{2.9}$$
$$= f(a). \tag{2.10}$$

Line (4.6) is valid because we ______________. Line (4.7) is valid because ____________________________. On line (4.8) we use the fact that ____________________________. Line (4.9) is just a restatement of our computation above. Line (4.10), we again aren't too worried about.
Thus, f is continuous at a, since we verified that ____________________________.

□

An interesting followup is to note that the converse of the theorem is not true.

Exercise 2.4.3. Counterexample ☕

Provide an example of a function f that is continuous everywhere but not differentiable at some point. Explain why this is the case for your function.

Let us look at a family of functions that further illustrates this one-way relationship between differentiability and continuity.

Exercise 2.4.4. A Piecewise Function ☕☕

Consider the piecewise function

$$f(x) = \begin{cases} 6 + x - x^2 & \text{if } x \leq 2; \\ ax + b & \text{if } x > 2. \end{cases}$$

Graph the function for each of the following given values of a and b. In each case, determine whether or not $f(x)$ is continuous as well as whether or not $f(x)$ is differentiable at $x = 2$.

- $a = 1$ and $b = 0$

- $a = 1$ and $b = 2$

- $a = -3$ and $b = 10$

- $a = -3$ and $b = 9$

What do the results above have to do with the major theorem we proved in this section?

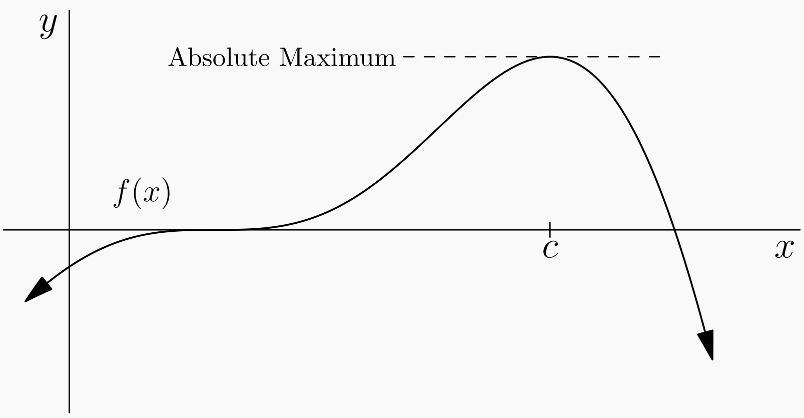
2.5 Fermat's Theorem and EVT

The *Extreme Value Theorem* (EVT) is a theorem of great consequence for our purposes in this course. First we carefully define a type of maximum and minimum.

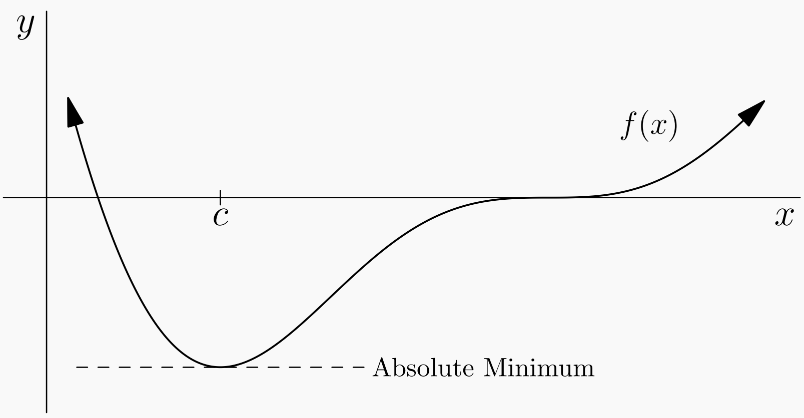
Definition 2.5.1. Absolute Max and Absolute Min

Let c be a real number and $f(x)$ be a function with domain D.

- We say $f(x)$ has an *absolute maximum* at c if and only if $f(x) \leq f(c)$ for all $x \in D$.



- We say $f(x)$ has an *absolute minimum* at c if and only if $f(x) \geq f(c)$ for all $x \in D$.



Theorem 2.5.2. Extreme Value Theorem

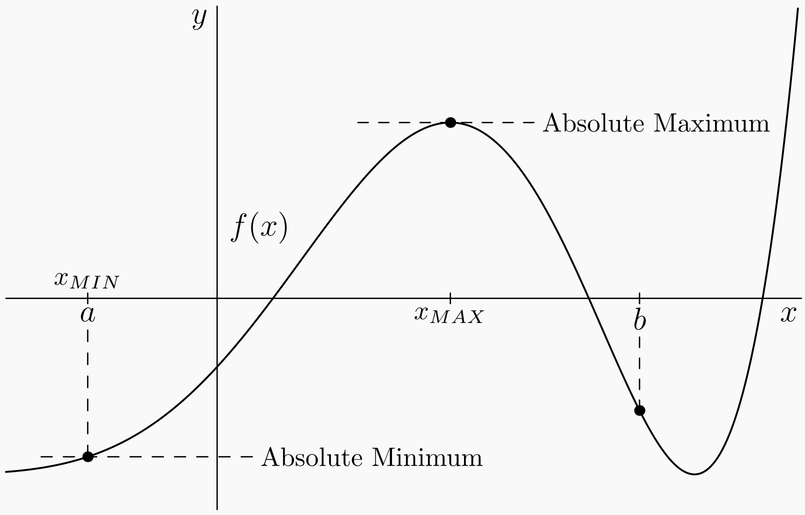
Let a and b be real numbers. Let $f(x)$ be continuous on the interval $[a, b]$. Then f attains an absolute maximum and an absolute minimum on that interval. That is, there exist real numbers x_{MIN} and x_{MAX} between a and b (including perhaps a or b itself) such that

$$f(x_{\text{MIN}}) \leq f(x)$$

and

$$f(x) \leq f(x_{\text{MAX}})$$

for all $x \in [a, b]$.



We will not formally prove EVT in this course. At the moment, we will accept it as something that is intuitive enough to work with. To prove it formally requires quite a bit of thorough analysis of the real numbers, which will happen later in your mathematical adventures!

Exercise 2.5.3. A Parabola ☕

Explain why the function $f(x) = x^2$ satisfies the preconditions of EVT on the interval $[-2, 2]$. What are x_{MIN} and x_{MAX}? Support your answer with a graph.

Exercise 2.5.4. Not a Parabola ☕☕

Graph the function

$$f(x) = \frac{x^2}{|x|}.$$

Explain why it does not satisfy the preconditions of EVT on the interval $[-2, 2]$. Does it attain an absolute max? Does it attain an absolute min?

Exercise 2.5.5. A Hyperbola ☕☕

Graph the function

$$f(x) = \frac{1}{x}.$$

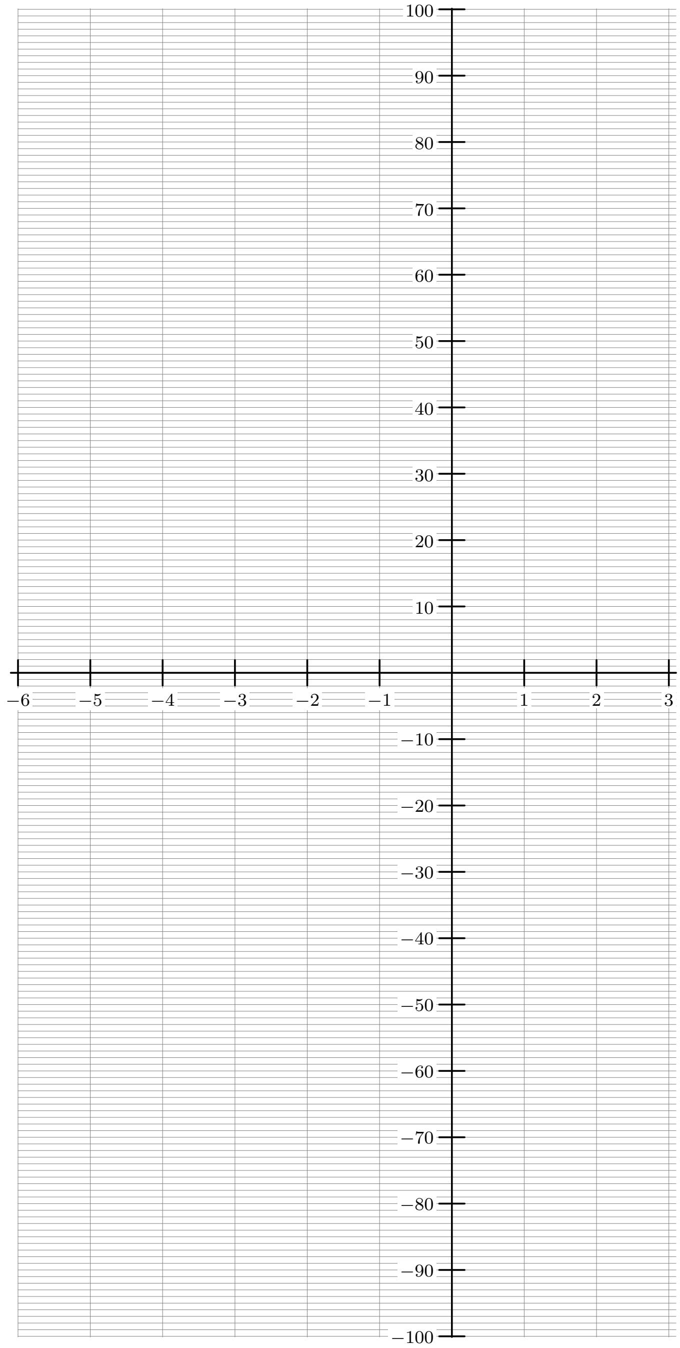
Explain why it does not satisfy the preconditions of EVT on the interval $[0, 2]$. Does it attain an absolute max? Does it attain an absolute min?

Exercise 2.5.6. A Polynomial ☕☕

We revisit the function

$$p(x) = 3x^3 + 10x^2 - 27x - 10$$

from Exercise 1.8.0.5. Graph the function on the interval $[-5, 2]$ and explain why EVT applies.



Mark on your graph roughly where you might guess x_{MAX} and x_{MIN} occur.

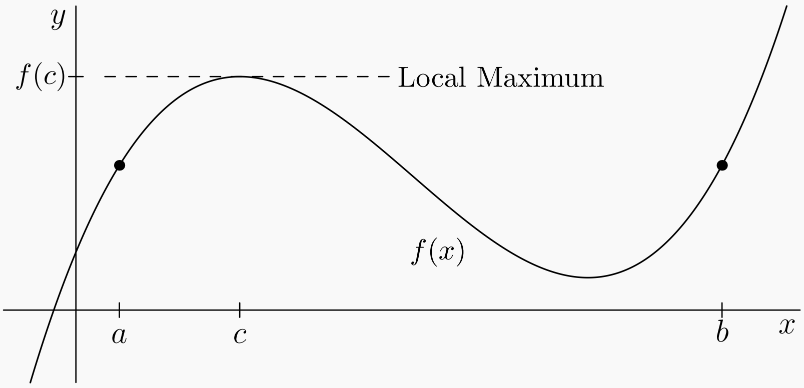
Note that at the moment we don't have the means to find exactly where those points occur. EVT, much like IVT, simply says they exist but does not indicate how to find where they are. In the next part, we will learn techniques which will often let us find exact values for those points!

Fermat's Theorem beautifully connects the concepts of differentiability and continuity in yet another way! In particular, we link the idea of derivatives together with EVT (Section 2.5). Fermat's Theorem says that the derivative of a function will be zero at a local max/min. We state this definition carefully and then prove Fermat's Theorem below.

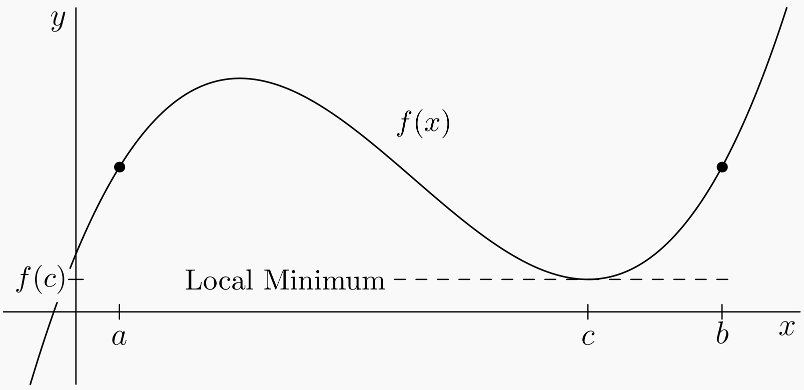
Definition 2.5.7. Local Maximum and Local Minimum

Let c be a real number and $f(x)$ be a function with domain D.

- We say $f(x)$ has a *local maximum* (also sometimes called a *relative maximum*) at c if and only if there exists some interval $[a, b] \subset D$ where $f(x) \leq f(c)$ for all $x \in [a, b]$. That is, if we can find some restriction of the domain on which $f(x)$ has an absolute maximum at c, then c is a local maximum.

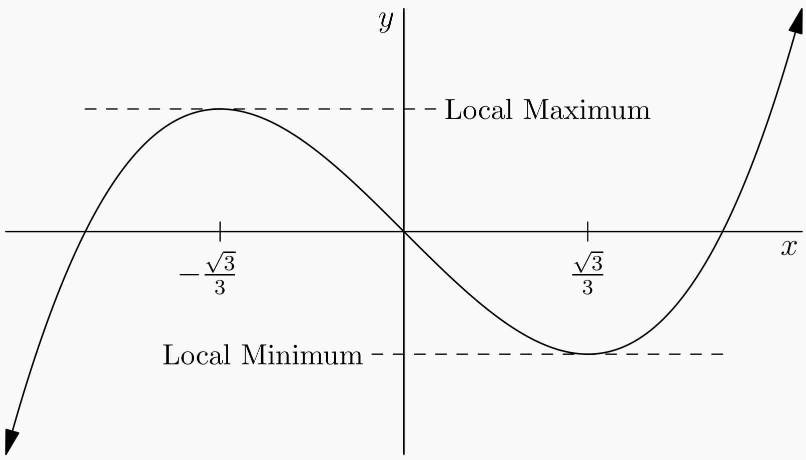


- We say $f(x)$ has a *local minimum* (also sometimes called a *relative minimum*) at c if and only if there exists some interval $[a, b] \subset D$ where $f(x) \geq f(c)$ for all $x \in [a, b]$. That is, if we can find some restriction of the domain on which $f(x)$ has an absolute minimum at c, then c is a local minimum.



Example 2.5.8. A Cubic with a Local Max and Local Min

The function $f(x) = x^3 - x$ has a local max at $-\frac{\sqrt{3}}{3}$ and a local min at $\frac{\sqrt{3}}{3}$. Neither is absolute.



Exercise 2.5.9. Domain Restriction

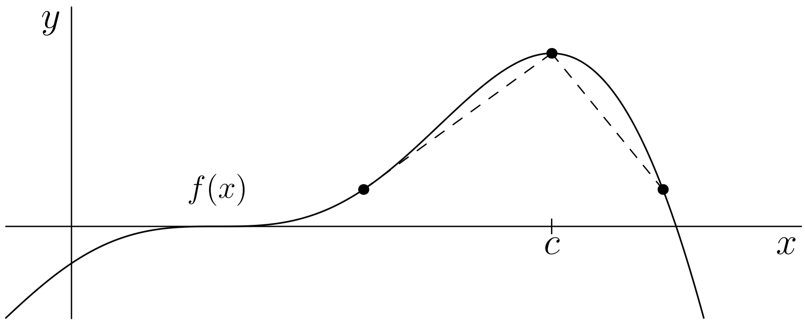
What could we restrict the domain to in order to find $-\frac{\sqrt{3}}{3}$ as a local max? What could restrict the domain to in order to find $\frac{\sqrt{3}}{3}$ as a local min?

Now that we have illustrated the definition of local max or min, we proceed to state Fermat's Theorem.

Theorem 2.5.10. Fermat's Theorem

Let $f(x)$ be differentiable at some number c. If $f(x)$ has a local maximum or local minimum at c, then $f'(c) = 0$.

Proof. We handle the case where $f(x)$ has a local maximum at c. The local minimum case is left as an exercise to the reader.



Since $f(x) \leq f(c)$ for all x in an interval surrounding c, we can say that the quantity $f(x) - f(c)$ is always negative or zero. That is,

$$f(x) - f(c) \leq 0$$

for all sufficiently close x. We now consider the derivative as the limit of the difference quotient

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since we assumed f is differentiable at c, we know the limit exists. Since a limit exists if and only if it is equal to both the left- and right-handed limits, we know

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}.$$

If we consider the right-handed limit, $x > c$ so

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

since a negative number divided by a positive number is negative. If we consider the left-handed limit, $x < c$ so

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

since a negative number divided by a negative number is positive. The first inequality tells us that $f'(c) \leq 0$ and the second tells us that $f'(c) \geq 0$. The only number that is simultaneously less than or equal to and also greater than or equal to zero is zero itself! Thus, $f'(c) = 0$. $\square$

Exercise 2.5.11. Left as an Exercise to the Reader ☕☕

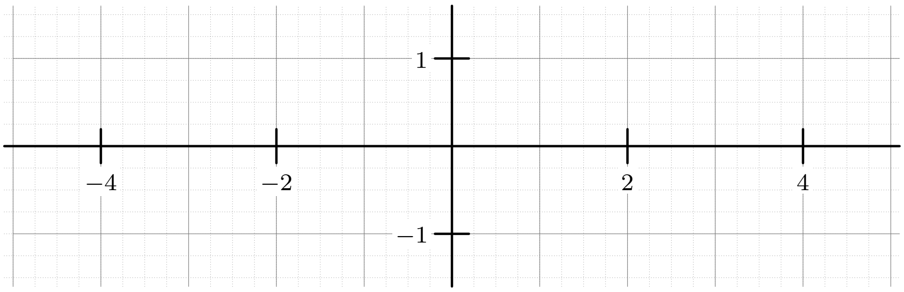
Above, we handled the case where $f(x)$ had a local maximum at c but not the case where it had a local minimum. Show that it also works in the case where the function has a min!

Exercise 2.5.12. Checking the Cubic

Return to Example 2.5.0.8 and verify that $f'(c) = 0$ for the two c values given. Verify it by using the power rule to compute the derivative of $f(x) = x^3 - x$ and then plugging in the two values of c.

Exercise 2.5.13. Fermat's Theorem on Cosine

Graph the function $\cos(x)$ on the interval $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

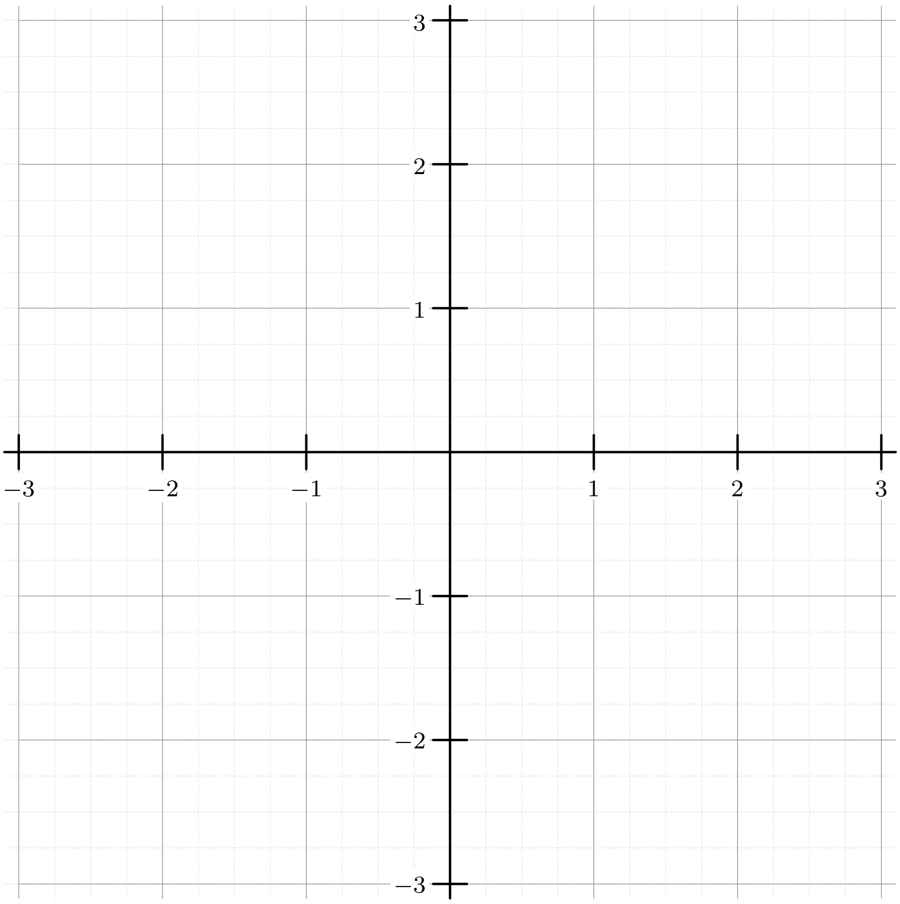


Mark the location of each local max and each min. At each location, calculate the derivative to verify the conclusion of Fermat's Theorem.

Note that the converse of Fermat's Theorem is false. It is true that a differentiable function always has derivative zero at a min or max, but if the derivative is zero at a point, a function may or may not have a max or min at that point. The next exercise illustrates this point.

Exercise 2.5.14. Counterexample to the Converse of Fermat's Theorem

Graph the function $f(x) = x^3$. Verify that the derivative at $x = 0$ is zero by computing $f'(x)$ at that point, but that the function does not have a max or min there.



A point of the above form (derivative zero but neither max nor min) is often called a *saddle point*. In Calculus III, we will see a three-dimensional version of these which will explain the name choice.

2.6 Mean Value Theorem

The Mean Value Theorem is similar to both the Intermediate Value Theorem and the Extreme Value Theorem in that it provides the existence of a point with a certain property without giving a formula for constructing that point! This is not to say we can never find it, but the theorem itself only provides existence. This is no coincidence, as we are going to use the Extreme Value Theorem to prove the Mean Value Theorem.

Statement and Examples of MVT

Theorem 2.6.1. Mean Value Theorem (MVT)

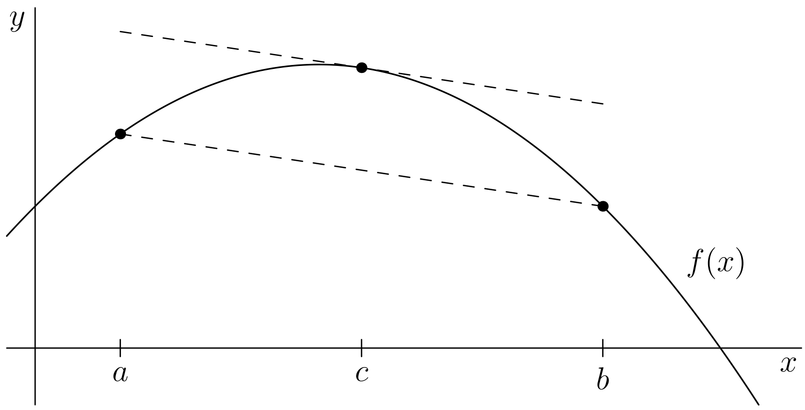
Let $f(x)$ be a differentiable function on an interval $[a, b]$. Then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A nice short wordy restatement of MVT is the following:

Given a differentiable function on an interval, there is a point at which the instantaneous rate of change equals the average rate of change.

Visually, one can think of MVT as drawing a secant line and then finding a tangent line that is parallel to the secant line.



Exercise 2.6.2. Continuity ☕

Should we have also stated that f is continuous in the preconditions of MVT?

Note that the theorem is also true if the function f is only continuous on $[a, b]$ and differentiable on (a, b). Differentiability at the endpoints is not needed, but the cleaner statement above is still true.

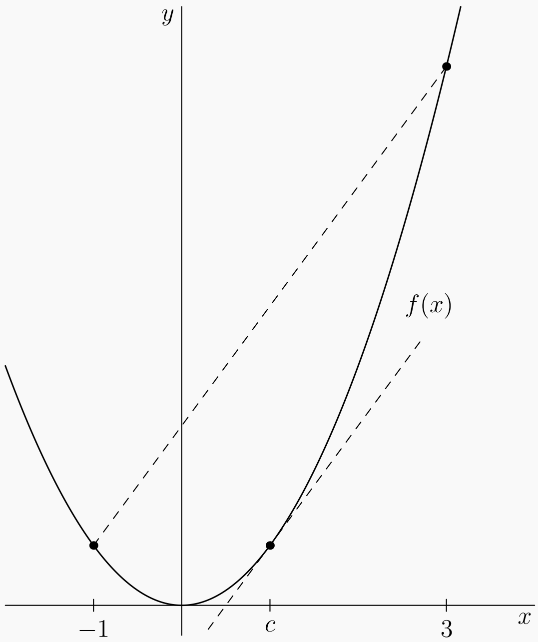
Before proving MVT, we play through some examples to get more intuition for exactly what it says.

Example 2.6.3. A Parabola

Consider our vanilla parabola $f(x) = x^2$ and the interval $[-1, 3]$. The average rate of change is

$$\frac{f(3) - f(-1)}{3 - -1} = \frac{9 - 1}{4} = 2.$$

Then MVT implies that there exists a $c \in (-1, 3)$ such that $f'(c) = 2$.



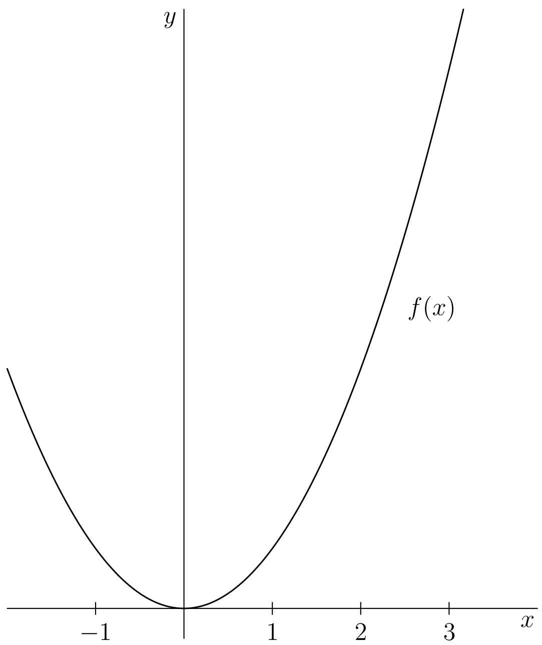
Let's find exactly what that point c is. We simply differentiate f and solve for c. In particular, $f'(x) = 2x$, so that if $f'(c) = 2$ then $2c = 2$ so $c = 1$.

Exercise 2.6.4. Same Graph, Different Interval ☕☕

Find the average rate of change of the function

$$f(x) = x^2$$

on the interval $[0, 3]$ and label the slope of the corresponding secant line.



Explain why MVT applies in this situation. Calculate $f'(x)$ to find the c that MVT guarantees exists! Draw the corresponding tangent line where the instantaneous rate of change matches the average rate of change. Label all relevant points.

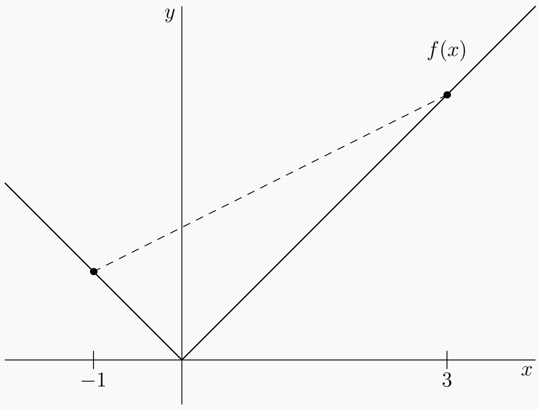
To balance out that example, here is a nonexample!

Example 2.6.5. Absolute Value

Consider the absolute value function $f(x) = |x|$ and the interval $[-1, 3]$. The average rate of change is

$$\frac{f(3) - f(-1)}{3 - -1} = \frac{3 - 1}{4} = 1/2.$$

If MVT applied, it would say there exists a $c \in (-1, 3)$ such that $f'(c) = 1/2$.



However, from the graph one can see that no such c exists, as the tangent line slope is always $1, -1$, or DNE.

Exercise 2.6.6. Nonapplication of MVT

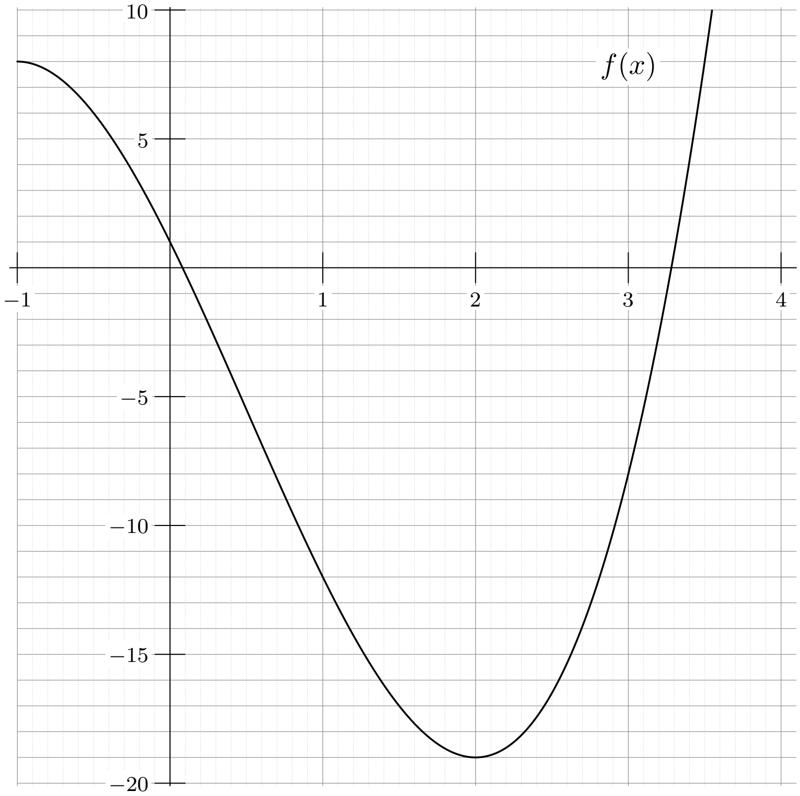
Why did MVT not apply in the above example?

Exercise 2.6.7. A Cubic

Find the average rate of change of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

on the interval $\left[0, \frac{3+\sqrt{105}}{4}\right]$ and label the slope of the corresponding secant line.



Explain why MVT applies in this situation. Calculate $f'(x)$ to find the c that MVT guarantees exists! Draw the corresponding tangent line where the instantaneous rate of change matches the average rate of change. Label all relevant points.

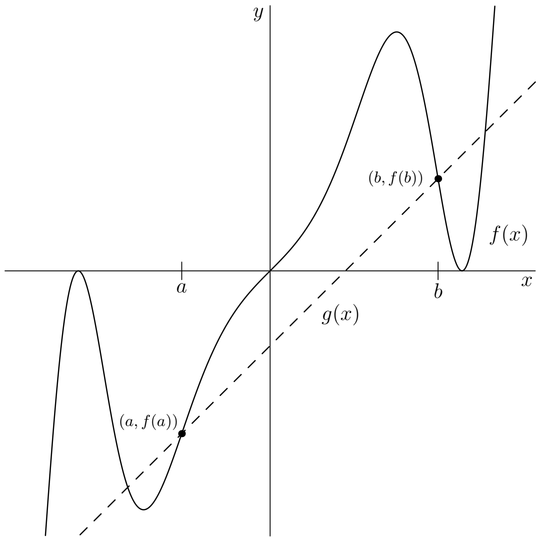
Proof of MVT

Now that we have played through some examples, we see how MVT follows from EVT and Fermat's Theorem.

Exercise 2.6.8. Proof of MVT ☕☕☕

Fill in the blanks in the proof below.

Proof. Define the function $g(x)$ to be the linear function that connects the endpoints of the graph of f on the interval $[a, b]$. That is, it is the line containing the points $(a, f(a))$ and $(b, f(b))$.



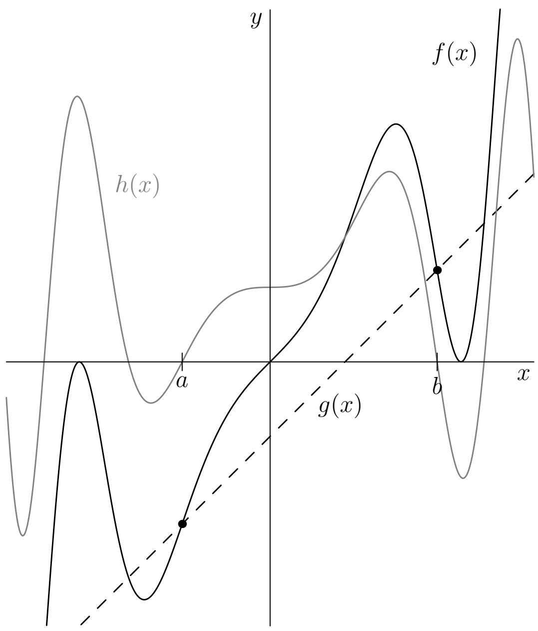
Thus, the formula for g is

$$g(x) = \underline{\hspace{8cm}}.$$

We now construct the function $h(x)$ to represent the difference between $f(x)$ and $g(x)$. Specifically, let

$$h(x) = f(x) - g(x)$$

so that $h(a) = h(b) =$ _______.



We now wish to show that there exists a point $c \in (a, b)$ such that $h'(c) = 0$. (This slope zero case of MVT is sometimes referred to as *Rolle's Theorem.*) We claim that this will be the point at which the instantaneous rate of change of f will equal the average rate of change on the interval. We split this into two cases.

- **Case 1:** Suppose $h(x) = 0$ on the entire interval $[a, b]$. In this case, we can pick any c between a and b to be the point at which $h'(c) = 0$. For sake of being concrete, say we choose $c =$ ______________, the midpoint of the interval.
- **Case 2:** Suppose $h(x)$ is not just the zero function. In this case, it achieves a nonzero absolute maximum or minimum value at some point $c \in (a, b)$ by the ____________________ Theorem. By ___________ Theorem, we know that $h'(c) = 0$.

In either case, we have a point c strictly between a and b where $h'(c) = 0$. If we differentiate both sides of the equation $h(x) = f(x) - g(x)$, we produce

$$h'(x) = \rule{10cm}{0.4pt}.$$

We can then plug in $x = c$ to find that

$$h'(c) = \rule{10cm}{0.4pt}.$$

Since $h'(c) = 0$, we conclude $f'(c) = g'(c)$ at that point. However, since $g(x)$ is just a line, the derivative is equal to its slope regardless of what x value is plugged in. Thus, we have

$$f'(c) = \rule{10cm}{0.4pt}$$

and the proof is complete! $\square$

Exercise 2.6.9. Followups ☕☕

In the above proof, where did we use...

- ...the fact that f was differentiable?
- ...the fact that f was continuous?
- ...linearity of the derivative?

A Corollary to MVT

MVT has many nice implications! Here is a particularly useful one that we want to highlight. A short wordy statement of the theorem is as follows:

If two functions have the same derivative, they can only differ by a constant.

Let us state this more carefully and then prove it below.

Theorem 2.6.10. Same Derivative Implies Off by a Constant

Let $f(x)$ and $g(x)$ be differentiable functions on (a, b) and assume

$$f'(x) = g'(x)$$

for all x in that interval. Then there exists some constant C for which

$$f(x) = g(x) + C$$

for all x in (a, b).

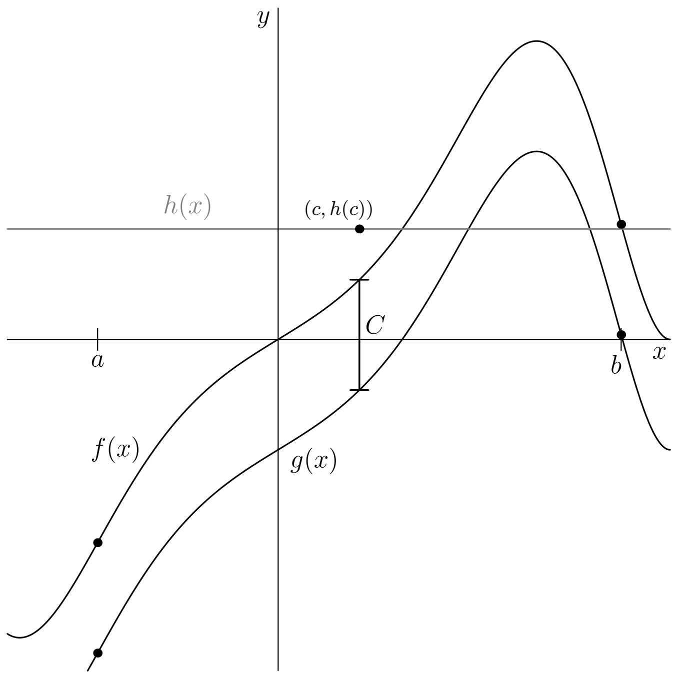
Exercise 2.6.11. Proving the Corollary

Fill in the blanks in the proof of our result!

Proof. Let c be the midpoint of the interval (a, b), in particular

$$c = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

(Note the value of c itself is not important, but we needed a number guaranteed to be inside the interval and constructing one is a nice way to show such a number exists.)



Define the function $h(x) = f(x) - g(x)$. Define the constant C to be

$$C = h(c) = f(c) - g(c).$$

By the definition of h,

$$\begin{aligned} h'(x) &= (\underline{\hspace{2em}})' \\ &= f'(x) - g'(x) \\ &= \underline{\hspace{6em}}. \end{aligned}$$

Note the last equality holds since $f'(x)$ and $g'(x)$ were assumed to be equal.
We now proceed by contradiction. Assume there exists some $x_0 \in (a,b)$ where $f(x_0) \neq g(x_0) + C$. We know this value x_0 could not be c, since C is defined as the gap between the functions at c. Thus, x_0 and c are distinct so the quantity $x_0 - c$ is not ____. Also, the quantity $f(x_0) - (g(x_0) + C)$ is not ____ since $f(x_0)$ and $g(x_0) + C$ were assumed to be different. Thus, the ratio

$$\frac{f(x_0) - (g(x_0) + C)}{x_0 - c}$$

is a nonzero real number. Though not immediately apparent, this fraction is actually the average rate of change for the function h on the interval ____________. To see this, notice that

$$\frac{h(x_0) - h(c)}{x_0 - c} = \underline{\hspace{9em}} = \frac{f(x_0) - (g(x_0) + C)}{x_0 - c}.$$

By MVT, there exists a number d between x_0 and c such that

$$\underline{\hspace{6em}} = \frac{h(x_0) - h(c)}{x_0 - c}.$$

But, we know the right-hand side is nonzero, so we found an x for which $h'(x)$ is nonzero, namely $x = d$. This contradicts the premise that $h'(x) = 0$ for all $x \in (a,b)$.
Thus our original assumption was incorrect, so $f(x) =$ ________________________ for all $x \in (a,b)$, as desired.

$\square$

2.7 Chapter Summary

This chapter introduced the definition of the **derivative**! Woohoo Calculus rite of passage! Intuitively the derivative is the **instantaneous rate of change at a point**, or the **slope of the tangent line.** We define it more formally below.

Definitions of the Derivative

No matter how you slice it, the derivative is a limit of a difference quotient. However, there are two equivalent ways to slice it. We can define the derivative $f'(x)$ as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ or } f'(x) = \lim_{a \to x} \frac{f(x) - f(a)}{x - a}.$$

Note these are the same via the substitution $h = a - x$. One can compute derivatives directly by evaluating the limits above, however for more complicated functions this becomes very difficult and we instead use the properties below.

Properties of the Derivative

Let n, a, and b represent real numbers $f(x)$ and $g(x)$ be differentiable functions, with $g(x)$ not identically zero and $f(x)$ invertible. The following properties of the derivative hold.

- **Power Rule.**
$$(x^n)' = nx^{n-1}$$
- **Linearity.**
$$(a \cdot f(x) + b \cdot g(x))' = a \cdot f'(x) + b \cdot g'(x)$$
- **Product Rule.**
$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
- **Quotient Rule.**
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$
- **Chain Rule.**
$$(f \circ g(x))' = f'(g(x)) \cdot g'(x)$$
- **Inverse Function Theorem.**
$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Theorems Regarding Derivatives

We proved three theorems regarding derivatives.

- **Differentiable implies continuous.** If a function $f(x)$ is differentiable at $x = a$, then it is also continuous at $x = a$. Note that the converse is not true.
- **Fermat's Theorem.** If a function $f(x)$ is differentiable at a and has a max or min at a, then $f'(a) = 0$. Note that in this section we also stated the...
- ...**Extreme Value Theorem**, which does not in and of itself have anything to do with derivatives, but it states that a continuous function on a domain D will have a well-defined maximum value and a well-defined minimum value on D. So it gives conditions under which we can be certain a max and min exist (which Fermat's Theorem in turn might help us find).

- **Mean Value Theorem.** If a function is differentiable on a closed interval, then the average rate of change on that interval is equal to the instantaneous rate of change at some x-value in that interval.

2.8 Mixed Practice

Exercise 2.8.1. ☕

What is the slope of the line $f(x) = x$? Evaluate $f'(x)$ using the power rule. Do the two results confirm each other?

Exercise 2.8.2. ☕☕☕

- Does the Extreme Value Theorem apply to the function $f(x) = x^2 - x - 1$ on the interval $[0, 1]$? Why or why not? If so, what is the max? What is the min?
- Does the Extreme Value Theorem apply to the function $f(x) = \frac{1}{x^2-x-1}$ on the interval $[0, 1]$? Why or why not? If so, what is the max? What is the min?
- Does the Extreme Value Theorem apply to the function $f(x) = \frac{1}{x^2-x-1}$ on the interval $[-2, 2]$? Why or why not? If so, what is the max? What is the min?

Exercise 2.8.3. ☕☕

- Explain in words the relationship between the Extreme Value Theorem and Fermat's Theorem.
- Provide an example of a function with a max at a point $x = c$ that does not satisfy $f'(c) = 0$.

Exercise 2.8.4. ☕☕☕☕

Consider the function

$$f(x) = \sqrt[3]{x} + 1.$$

- Find $f'(x)$ by the limit definition of the derivative.
- Find $f'(x)$ by using the Power Rule. Confirm your answers match!
- Find $f'(x)$ yet one more time by first calculating the inverse function and then using the IFT.

Exercise 2.8.5. ☕☕☕

Consider the function

$$f(x) = \sin(2x).$$

- Differentiate $f(x)$ by using the Chain Rule.
- Differentiate $f(x)$ first applying the double-angle identity $\sin(2x) = 2\sin(x)\cos(x)$ and then using the Product Rule instead!
- Your answers will appear very different from each other. Verify they are in fact the same!

Exercise 2.8.6. ☕☕☕

Consider the function

$$f(x) = \sec(\pi x).$$

- Sketch the graph.
- Does the Mean Value Theorem apply to $f(x)$ on the interval $[0, 1]$? Why or why not? If so, find the point c where $f'(c)$ is equal to the average rate of change on that interval.
- Does the Mean Value Theorem apply to $f(x)$ on the interval $[-1/4, 1/4]$? If so, find the point c where $f'(c)$ is equal to the average rate of change on that interval.

Chapter 3

Applications of the Derivative

3.1 Linearization

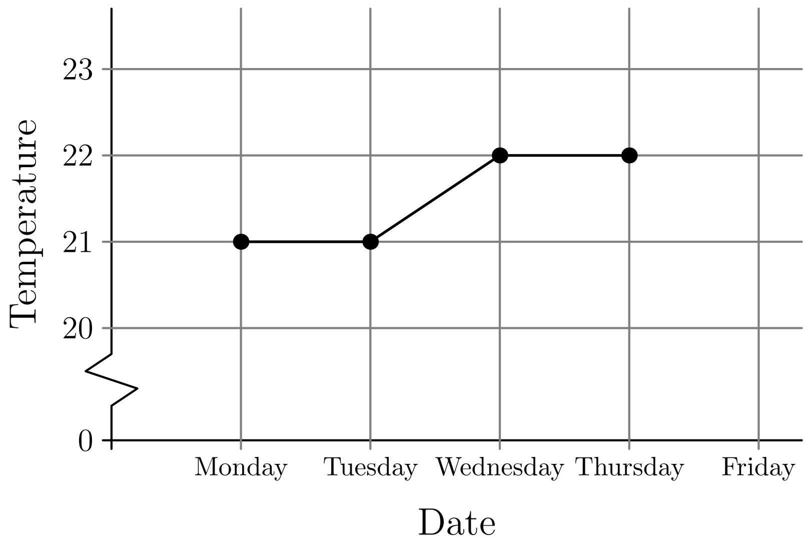
One of the most common uses of derivatives is that of *linearization*, the act of approximating a given (typically nonlinear) function by a degree one polynomial, also known as a linear function. In many models from physics, chemistry, and elsewhere, the model itself is far too difficult to solve. In this case, replacing the functions being used in the model with their linearizations can be an effective way to get an approximate solution to the model. To get a sense for the manner in which the linearization is built, we begin with a classic "too difficult to solve the model" situation: predicting the weather!

One principle that can be applied to modeling weather is a very simple yet effective one, as follows:

The best predictor of tomorrow's weather is today's weather.

Exercise 3.1.1. Predicting the Weather Or Not ☕

Consider the plot below of daily high temperatures from Monday to Thursday.



Use our simple principle stated above to predict Friday's temperature. Draw it on the graph.

Here is a more mathematical restatement of the above principle.

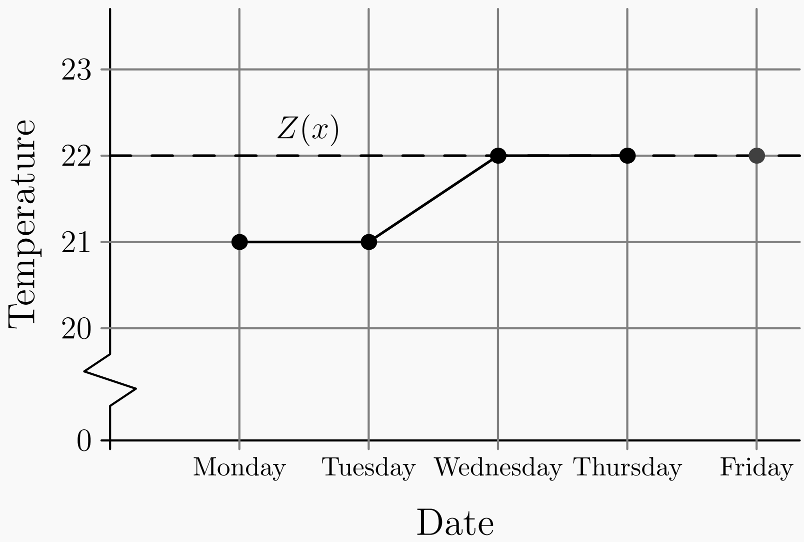
A function $f(x)$ can be approximated near the location $x = a$ by the degree zero polynomial function $Z(x) = f(a)$.

Example 3.1.2. Tomorrow's Weather Predicted by Today's Weather

Call $T(x)$ the temperature function from the above exercise, where x represents the number of days elapsed since Monday and $T(x)$ represents the high temperature on the corresponding day. The function $T(x)$ could be represented as the data table shown below.

x	0	1	2	3
$T(x)$	21	21	22	22

The degree zero polynomial approximation of $T(x)$ at $x = 4$ is the constant function $Z(x) = 22$. Graphically, this could be seen as a horizontal line running through the graph, matching the height of $T(x)$ at $x = 4$.

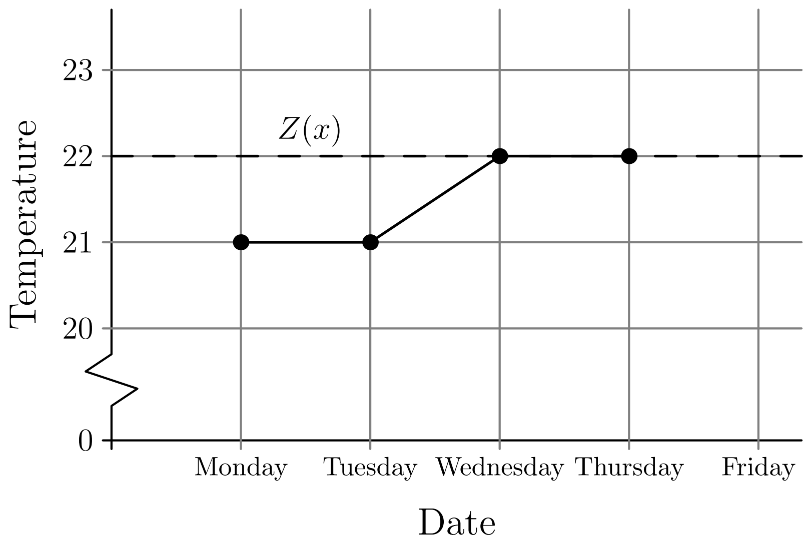


This broken record modeling approach of course does not get one very far. A groovier thing to do is to consider not just a single data point, but a single data point *and* the rate of change nearby. The better weather model represented by this idea could be described as follows:

The best predictor of tomorrow's weather is today's weather, adjusted up or down a bit according to recent trends of warming or cooling.

Exercise 3.1.3. Same Data, New Model

Use this fancier idea to predict Friday's weather given the same four data points discussed above.



This idea, stated a bit more mathematically, is all that linearization is. Instead of limiting ourself to degree zero polynomial approximation (constant functions, or horizontal lines if you are thinking graphically), we allow our approximation to be a degree one polynomial approximation (linear functions, any line that is not vertical if thinking graphically). Linearization can be described as follows:

A function $f(x)$ can be approximated near the location $x = a$ by the degree one polynomial function $L(x) = f(a) + f'(a)(x - a)$.

If the formula $L(x) = f(a) + f'(a)(x - a)$ looks a bit out of the blue, notice that it is just the slightest restatement of point-slope form of a line. We explain this in the following exercise.

Exercise 3.1.4. Noticing the Restatement ☕

- Write down the point-slope form of a line for a line that has slope m and passes through a point (a, b).

- Add b to both sides to isolate y. Rename y as $L(x)$ to get the equation written in function form.

- Replace the generic unknown slope m with the derivative of the function $f(x)$ at the point $x = a$, namely $f'(a)$. Replace the generic y coordinate b with the height of the function $f(x)$ at the point $x = a$, namely $f(a)$. Conclude our formula for the linearization is correct.

Because this comes up so often, we state it in its own nice little box.

Formula 3.1.5. Linearization

The linearization $L(x)$ of a function $f(x)$ at $x = a$ is given by $L(x) = f(a) + f'(a)(x - a)$.

Sometimes the "adjusted according to recent trends of cooling and warming" term, namely $f'(a)(x-a)$, is referred to as the *differential* of f. If one writes Δf to represent change in f near a, then one way to calculate this across small intervals is to take the change in the x coordinate times the rate of change of f with respect to x at a. So the linearization formula can also be thought of as

$$\Delta f \approx \frac{\mathtt{d}f}{\mathtt{d}x}\Delta x.$$

We include this comment because this notation is used quite frequently in other sources, but for our purposes we will stick to just writing it as a function $L(x)$ where $L(x) \approx f(x)$ for x near a.

In our weather example above, our derivative was computed by just eyeballing rates of change off of a graph. If we have an actual formula for $f(x)$, we can use our tricks for computing derivatives (power rule, product rule, chain rule, etc) to compute it more precisely.

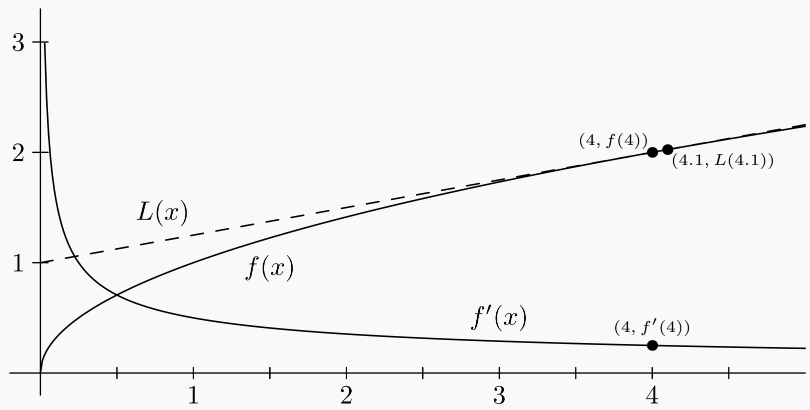
Example 3.1.6. A Square Root

Suppose we wish to obtain an approximate value for the number $\sqrt{4.1}$. Here is one strategy, using linearization. First, define the function $f(x) = \sqrt{x}$. Second, notice that nearby there is the very easy to compute location $x = 4$. Thus, we decide to take the linearization $L(x)$ to the function $f(x)$ at $x = 4$, and the value of $L(4.1)$ will be our approximation to the value of $f(4.1)$. Proceeding, we have the following calculations:

- $f(4) = 2$
- $f'(x) = \frac{1}{2\sqrt{x}}$
- $f'(4) = \frac{1}{4}$
- $L(x) = f'(4)(x - 4) + f(4) = \frac{1}{4}(x - 4) + 2$
- $L(4.1) = \frac{1}{4}(4.1 - 4) + 2 = \frac{1}{4}(0.1) + 2 = 0.025 + 2 = 2.025$

Thus, our approximation is

$$\sqrt{4.1} \approx 2.025.$$



Exercise 3.1.7. Verifying Our Square Root Approximation ☕

Find the square of our approximation 2.025. If our approximation were perfect, what would the square be? How does the square compare to this value?

Exercise 3.1.8. Your Turn! ☕☕

- Use a linearization at $x = 8$ of the function $f(x) = \sqrt[3]{x}$ to calculate an approximate value of $\sqrt[3]{8.1}$. Illustrate your work with a graph of f, a graph of the linearization L, the exact y value trying to be computed on the graph of f, and the exact y value being used to approximate it on the graph of L.

- Cube your answer to verify it is a reasonable approximation to the cubed root of 8.1.

Exercise 3.1.9. Your Turn! ☕☕

- Use a linearization at $x = \pi/3$ of the function $f(x) = \cos(x)$ to calculate an approximate value of $\cos(1)$. Use the approximations $\pi \approx 3.14$ and $\sqrt{3} \approx 1.73$ in your approximation. Illustrate your work with a graph of f, a graph of the linearization L, the exact y value trying to be computed on the graph of f, and the exact y value being used to approximate it on the graph of L.

- Calculate $\cos(1)$ on a calculator or computer algebra system to verify your approximation is close to the true value.

Exercise 3.1.10. Your Turn! ☕☕

- Use a linearization at $x = 1$ of the function $f(x) = \arctan(x)$ to calculate an approximate value of $\arctan(4/5)$. Illustrate your work with a graph of f, a graph of the linearization L, the exact y value trying to be computed on the graph of f, and the exact y value being used to approximate it on the graph of L. Use $\pi \approx 3.14$ in your approximation.

- Calculate $\arctan(4/5)$ on a calculator or computer algebra system to verify your approximation is close to the true value.

It is worth noting that you will revisit this topic in Calculus 2! Here we saw how you can approximate a nonpolynomial function by a degree zero polynomial, or by a degree one polynomial. Why stop there? In the Calculus 2 development of *power series*, you will see how to extend this process to approximation by degree two polynomials, degree three polynomials, and so on, as high as you like! So, keep this section in mind for when you reach the power series chapter in your Calculus 2 course.

3.2 Finding Extrema Using Fermat's Theorem and the Second Derivative

In Section 2.5, we stated the Extreme Value Theorem, which guaranteed that every continuous function on a closed interval attains an absolute max and absolute min. The slightly unsatisfying aspect of that theorem is that it does not give a method for finding the max or min; it just promises that one exists somewhere. Similarly, in your college algebra or precalculus courses, you would often graph polynomial functions and see turning points on the graph between the roots. However, there was little if any way to find the coordinates of the peaks of the mountains and lows of the valleys.

Fortunately, derivatives shine a bit more light on these situations. If the function is differentiable, then by Fermat's Theorem we know that the derivative is zero at the location of a max or min. The second derivative (the derivative of the derivative) will let us then distinguish max from min!

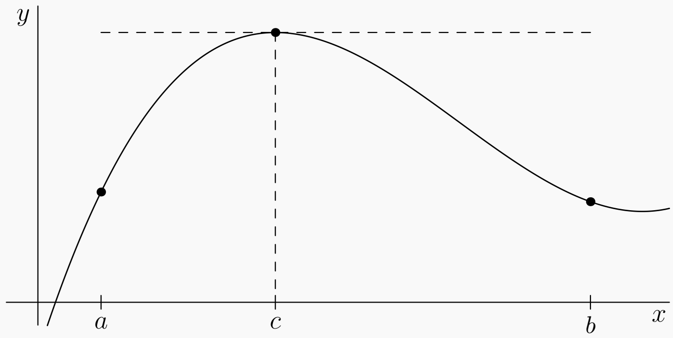
Using the First Derivative to Locate Maxima and Minima

By Fermat's Theorem, the derivative of a function (if it exists) is always zero at a max or min. Thus, if c is a location at which f has a max or min, either the derivative is zero, or does not exist. It might not exist because the limit $f'(c)$ does not converge to a number, or maybe we are simply at the endpoint of the domain over which the function is being considered. This knowledge gives us a good process for finding max and mins of functions, stated below.

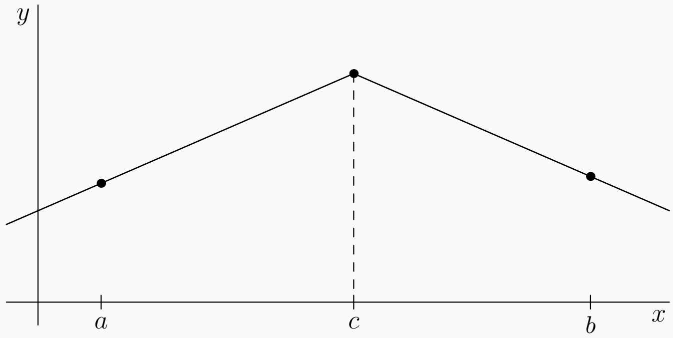
Theorem 3.2.1. Three Cases for Extrema

Let $f(x)$ be continuous on the interval $[a, b]$. Then if f attains an absolute minimum or absolute maximum at a real number $c \in [a, b]$, one of the three following statements is true:

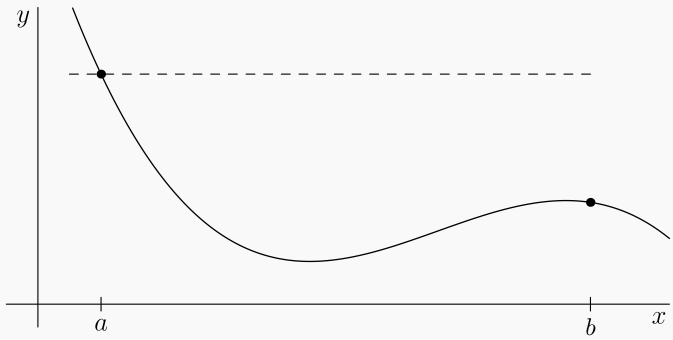
1. The derivative is zero at c. That is, $f'(c) = 0$.



2. The function is not differentiable at c. That is, $f'(c)$ does not exist.



3. The number c is in fact one of a or b (where the function may or may not be differentiable).



A point c in the form of Case 1 or 2 is called a *critical point*. That is, if the derivative of f is zero or does not exist at c, we say c is a critical point of f. This gives the following convenient rephrasing of the above theorem:

Absolute extrema of a continuous function on a closed interval always occur at critical points or endpoints.

If we examine a function on the entire real number line and not on just some closed interval, then we do not have to worry about checking endpoints. However, then there is also no guarantee the function will actually attain an absolute max or min! However, the points where the derivative is zero or undefined can still uncover local max or local min values, places where the function is larger or smaller than all nearby values.

Example 3.2.2. Revisiting a Cubic

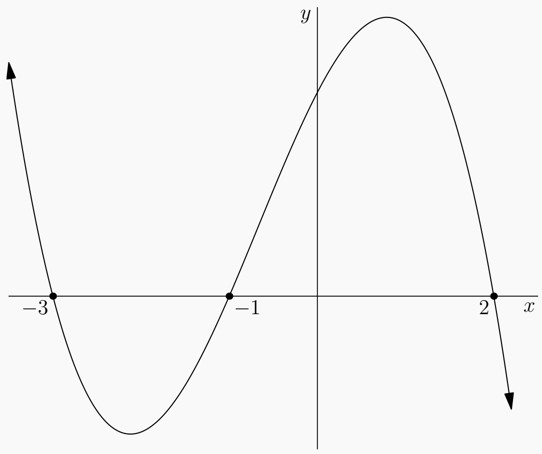
Let us begin by graphing the function

$$f(x) = 6 + 5x - 2x^2 - x^3$$

as we would in a college algebra or precalculus course. Since it is a polynomial, we apply the usual process.

- **Degree/Leading Coefficient:** Since it is odd degree with a negative leading coefficient, we know it has up/down end behavior. To say this in the notation of this course, we would say that
$$\lim_{x\to-\infty} f(x) = \infty$$
and
$$\lim_{x\to\infty} f(x) = -\infty.$$
- **Roots:** Since it is a polynomial with integer coefficients, we try to guess some rational roots whose numerators divide the constant term and denominators divide the leading coefficient (applying the Rational Root Theorem). This leads to $\pm 1, 2, 3, 6$ as the list of possible rational roots. Plugging these in, we find that $x = -3, -1$, and 2 are the roots. Since it is a degree three polynomial, each must have multiplicity of only one, so the graph will cross through the x axis at each root.

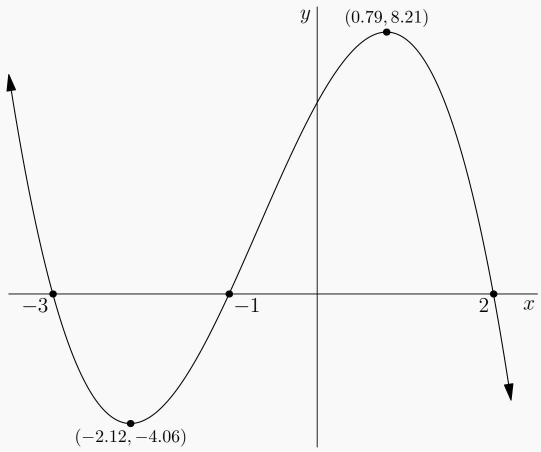
We assemble this information to obtain a graph, exactly as we would in college algebra.



However, at the end of such an example in your college algebra course, there was always an unsatisfying missing point (or two); how high or low does the graph go between those roots? Previously, we did not have a method to answer this question. Now however, we can find those max and min values on the intervals bounded by the roots! From the graph, we can see those are clearly critical points, which we find by solving the equation $f'(c) = 0$. By the quadratic formula, $f'(c) = 5 - 4c - 3c^2$ has zeros at

$$c = -\frac{4 \pm 2\sqrt{19}}{6} \approx -2.12 \text{ or } 0.79.$$

This lets us find locations of those local max and local min values! In particular, we have $f(-2.12) \approx 4.06$ and $f(0.79) \approx 8.21$. We now can plot the graph once again with labels for these max and min values.

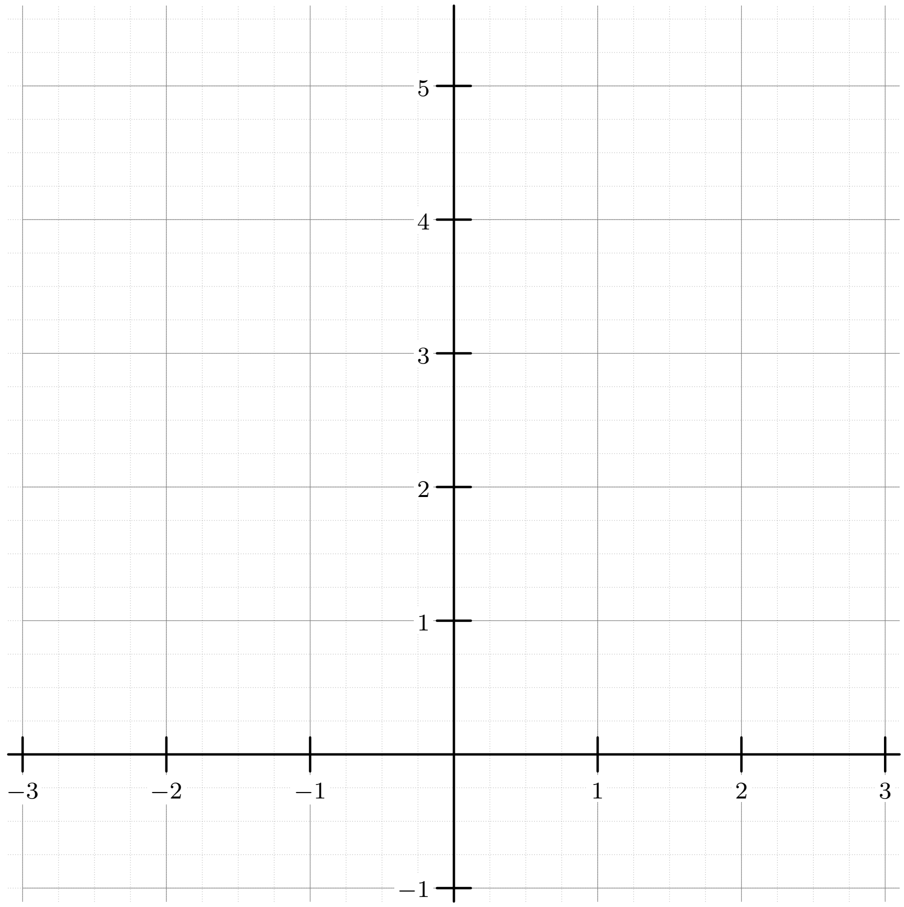


Exercise 3.2.3. A Quartic!

Consider the polynomial function

$$f(x) = 4 + 4x - 3x^2 - 2x^3 + x^4.$$

Follow the method of the previous example to graph the function. That is, first plot it using end behavior and multiplicities of roots. Then enhance your graph by finding and labeling the critical points.

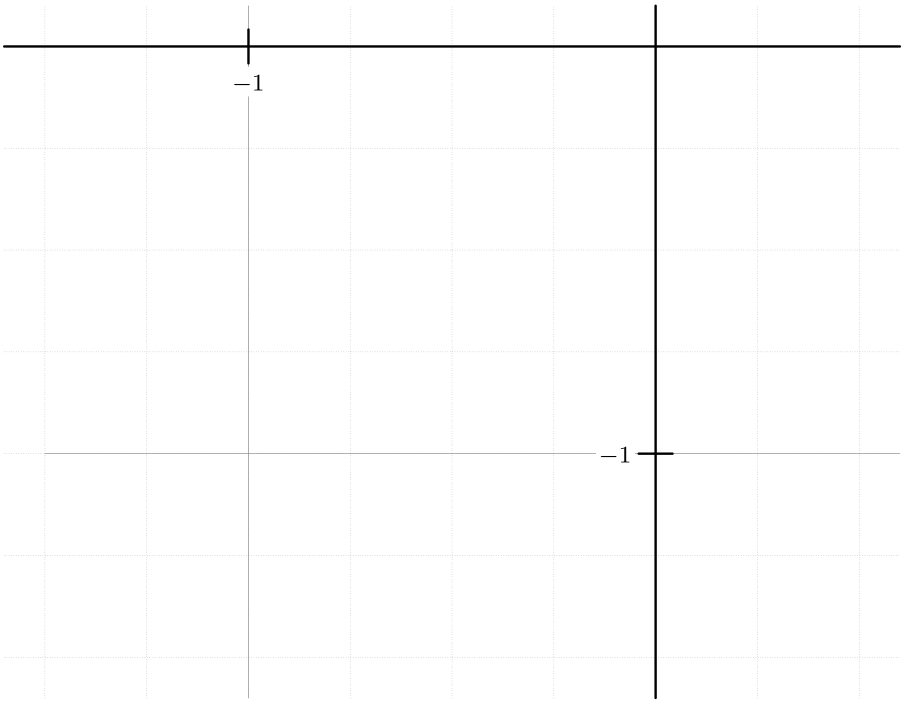


Exercise 3.2.4. Yet Another Cubic

- Explain why the function $f(x) = x^3 + 2x^2 + x - 1$ must have an absolute max value and an absolute min value on the interval $[-1.2, 0]$.

- Find the max and min value of $f(x) = x^3 + 2x^2 + x - 1$ on $[-1.2, 0]$ and state where they occur.

- Use the information above to draw a rough graph of the function $f(x)$ on the interval $[-1.2, 0]$.



Exercise 3.2.5. The Vertex Formula

Recall the *vertex formula* from college algebra. It says that the vertex of a parabola with formula

$$f(x) = ax^2 + bx + c$$

will always occur at

$$x = -\frac{b}{2a}.$$

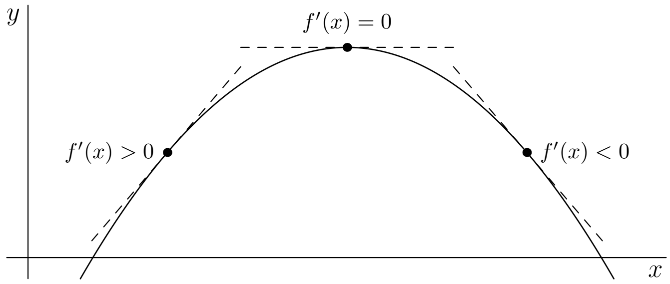
Show that you get the same x coordinate if you try to find the max/min using critical points!

Using the Second Derivative to Classify Max vs Min

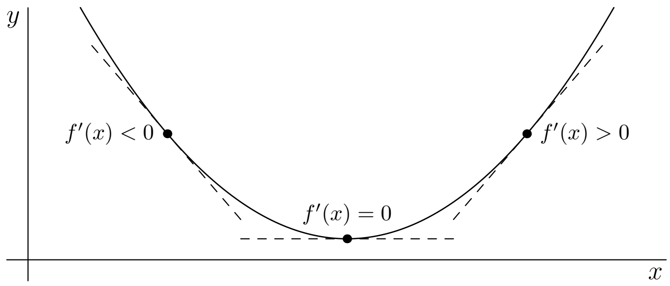
To distinguish a max from a min, we use the *second derivative* of the function, the derivative of the derivative. We often write a double-prime for this expression, so

$$f''(x) = (f')'(x).$$

The second derivative represents the rate of change of the first derivative and thus is the rate of change of the slope of the tangent line. At a maximum, we can see the tangent line slope go from positive to zero to negative.



At a minimum, we can see the tangent line slope go from negative to zero to positive. This leads us to the Second Derivative Test.



Theorem 3.2.6. Second Derivative Test

Let f be a function with $f'(c) = 0$ for some c in the domain of f.

- If $f''(c) > 0$, then the function has a minimum at c.
- If $f''(c) < 0$, then the function has a maximum at c.
- If $f''(c) = 0$, the test provides no information.

Exercise 3.2.7. Observing Each Case ☕

- To see the first case described above, try the function $f(x) = x^2$. Verify that $c = 0$ is a critical point with second derivative positive, and that f has a min.

- To see the second case described above, try the function $f(x) = -x^2$. Verify that $c = 0$ is a critical point with second derivative negative, and that f has a max.

- To see the third case described above, consider the following three functions:

$$f(x) = x^3$$
$$g(x) = x^4$$
$$h(x) = -x^4$$

Verify that each has $c = 0$ as a critical point with second derivative equal to zero, but that f has neither max nor min, g has a min, and h has a max. (Thus in the third case, anything is possible, which is why we say "no information".)

Exercise 3.2.8. Revisiting the Graphs

Revisit each of the graphs in the previous subsection. Compute the second derivatives and verify that they are negative for the maxima and positive for the minima.

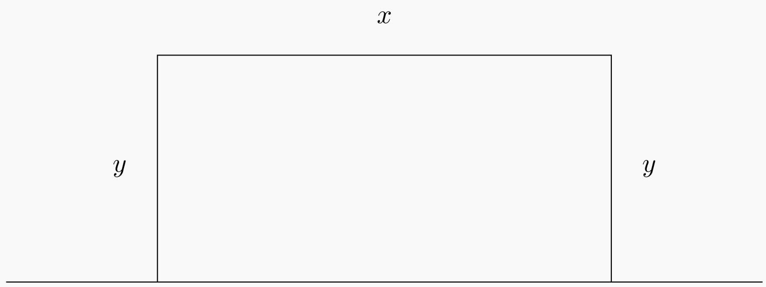
3.3 Applied Optimization

One place where we can apply the ability to find maximums and minimums, locally or globally, is in applied optimization problems. These problems ask you to maximize or minimize some quantity that is subject to a given constraint. These are often problems that ask about geometric objects, though not always. Each optimization problem is slightly different. You should be as careful as possible to make sure that you're carefully reading and understanding what's being asked of you before you start throwing numbers around! Lets start with an example.

Example 3.3.1. Optimizing an Area

A farmer needs to fence in a new rectangular field. He has 200 feet of fencing material. Furthermore, the field will border an already fenced in field on one side. What are the dimensions of the largest field he can enclose?

We'll start by drawing a picture of the field. We'll assume the south side is the side that already has fencing. We'll also need to assign variables to each unknown value. Lets call the width (horizontal or east-west length) of the field x and the height (vertical or north-south length) of the field y.



Then we have two equations that we can find here. First, our objective equation, the actual quantity we want to optimize. We are looking for the largest possible field (by area), so our objective equation is:

$$A = xy.$$

Next, we have our constraint equation. We have to fence in two sides with length y and one side with length x using our total fencing of length 200 feet. This gives us our constraint equation:

$$200 = 2y + x.$$

We want to find maximums of the objective, but we have a problem– there are two variables present! We need to only have one variable in order to find a maximum! So we need to use our constraint to substitute. We can solve for y in our constraint:

$$\begin{aligned} 200 &= 2y + x \\ 200 - x &= 2y \\ 100 - \frac{x}{2} &= y. \end{aligned}$$

Then we can substitute:

$$\begin{aligned} A &= xy \\ &= x\left(100 - \frac{x}{2}\right) \\ &= 100x - \frac{x^2}{2}. \end{aligned}$$

Now we have an equation for our area in terms of just one variable. So we can optimize. We can use the first derivative test, the second derivative test, or we might even be able to identify a closed interval for x. Here, we know that x must be more than 0, since the length has to be positive. Furthermore, x must be less than 200, since we only have 200 feet of fence to begin with. This gives us a closed interval $x \in [0, 200]$. So we just need to identify our critical points, then test our endpoints and critical points! Lets start by taking the derivative of our area function:

$$\begin{aligned} A =& 100x - \frac{x^2}{2} \\ A' =& 100 - x \end{aligned}$$

Then we'll set our derivative equal to 0:

$$\begin{aligned} 0 =& 100 - x \\ x = 100. \end{aligned}$$

So we have a critical point at $x = 100$. We have no singular points, because $100 - x$ is continuous everywhere. Next, we'll test our critical points and endpoints:

x	A
0	0
100	5,000
200	0

So our area function has a global maximum of an area of 5,000 feet when $x = 100$. Looking back towards our original prompt, we wanted to find the dimensions that gave us the largest area. We know that our width should be 100 feet, but we dont have a value for y. Luckily, we know that $100 - \frac{x}{2} = y$, so:

$$\begin{aligned} y =& 100 - \frac{x}{2} \\ =& 100 - \frac{100}{2} \\ =& 100 - 50 \\ y =& 50. \end{aligned}$$

Then the dimensions of the field that has the maximal possible area is $100' \times 50'$.

Phew! That was a lot, for a pretty simple problem. Most optimization problems will be like this–they'll present some information about a problem, then you'll be expected to pull that information out and set up the problem yourself. However, there is generally a pattern.

Claim 3.3.2. Steps for Optimization

1. **Gather Information**: Read the problem carefully. Write down any numbers given. If the problem concerns some kind of geometric object or length, draw a diagram! State and assign the variables you're going to use.

2. **Objective Equation**: Identify and write down your objective equation. This is the equation that relates the thing you want to optimize to the variables that you stated in step 1.

3. **Constraint Equation(s)**: Identify and write down your constraint equation(s). Sometimes you may have more than one. This is the equation that relates the variables you assigned to actual quantities given in the problem. This is usually related to a number given in the prompt. There is sometimes more than one of these equations.

4. **Rewrite Objective**: Your objective equation will almost always have more than one variable in it. You'll use your constraint(s) to substitute variables until you just have one variable.

5. **Optimize**: Now that your objective has a single variable, we can find it's maximum and minimums. Here you have 3 basic options:

   (a) **First Derivative Test**

   (b) **Second Derivative Test**

   (c) **Global Extrema**

   If you can't identify a closed interval to use the global extrema test, you'll use either the first or second derivative test. Any way you do this, you'll be taking at least one derivative and finding critical points. **You must always verify your extrema**.

6. **Answer The Question**: This seems like an obvious step, but after everything you've done it can be easy to forget! Go back to the prompt and see what quantity you were supposed to give for the answer, then find that quantity and give your answer.

This 6(ish) step process gives you a basic outline of how to tackle each optimization problem. Lets go back and see how we applied this process in the example!

Exercise 3.3.3. Processing ☕

For the following questions, refer to the example above.

- What information did we gather? What did we do to enhance our understanding of the prompt before starting to write down equations?

- What did we decide our objective function was? What key words or phrases in the prompt gave us clues?

- What did we decide our constraint was? What key words or phrases in the prompt gave us clues?

- How did we decide that the domain for our objective function was $[0, 200]$? What could we have done if we could not have identified a closed interval domain of feasibility?

- How did we decide that our maximum occured at $x = 100$?

- What key words or phrases in the prompt told us that we should give our answer as $50' \times 100'$?

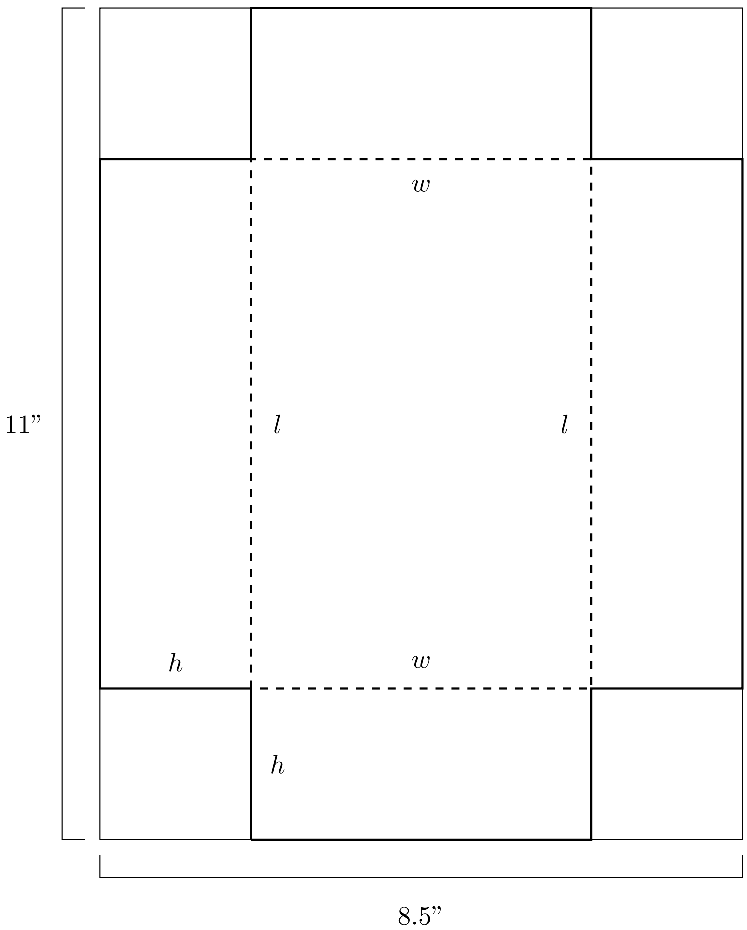
There many different types of problems. Many of them tackle geometric objects.

Exercise 3.3.4. A Different Field ☕☕

We are going to fence in a rectangular field. One end does not need fence, as it is against a building. The sides of the field coming off the building need to be made of chain link fence that costs $4 per foot of fencing. The remaining side should be made of wood, which costs $7 per foot of fence. The area to be enclosed must be 121 square feet. How much does it cost to build the fence that costs the least?
Hint: Your objective function should be a cost function!

Exercise 3.3.5. A box!

We want to construct a box of maximal volume from a 8.5” × 11” sheet of paper by cutting squares out of the corner. Find the dimensions of the squares that should be cut out of the corners.



Exercise 3.3.6. Soda Cans ☕☕

A can of soda is approximately cylindrical with a volume of about 355ml (or 355 cm^3). Because the top and bottom need to be shaped differently and reinforced slightly, the cost for the top and bottom of the can cost twice as much as the label of the cylinder. How tall should the optimal (least expensive) can be?
Hint: The volume of a cylinder is $V = \pi r^2 h$ and the surface area of a cylinder is made up of 3 parts: a top and a bottom that are both circles with area πr^2 and a "label" that is a rolled up rectangle with area $A = 2\pi rh$. Again, your objective should be a cost!

Exercise 3.3.7. Friction ☕☕

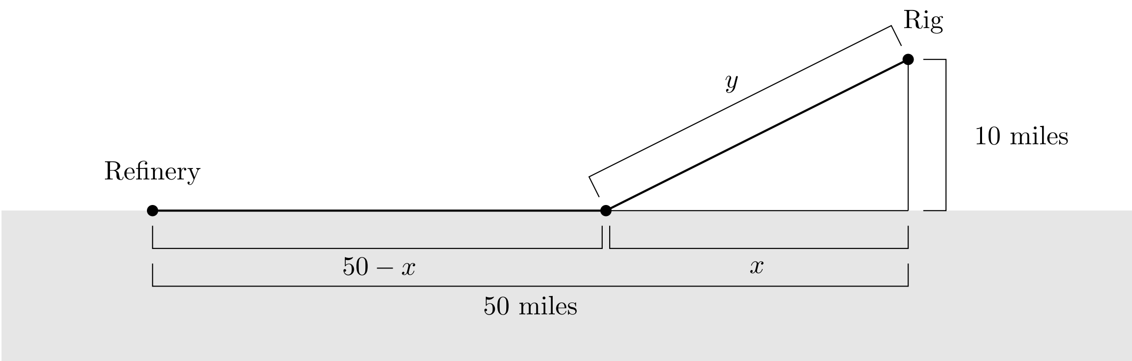
Imagine you're pulling a sled across a flat surface. The sled has mass m, and the surface has a coefficient of friction μ. When you pull the sled with a force of F at an angle of θ, the horizontal component of that force actually pulls the sled, but the vertical component of that force serves to reduce the normal force on the sled, and therefore reduce the amount of friction that resists your motion. What is the optimal angle to pull the sled at?
Hint: The force that "pulls the sled" is

$$f = f_x - f_f.$$

Draw a free-body diagram and use that to figure out what the magnitude of f_x, the x component of your pulling force and f_f, the frictional force are in terms of θ.

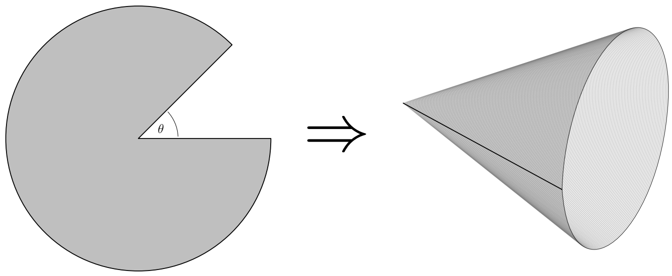
Exercise 3.3.8. The Lifeguard/Pipeline Problem ☕☕☕

Suppose you need to build a pipeline for an offshore oil rig. It costs about \$10 million per mile to build the pipeline underwater, and it costs about \$7 million per mile to build the pipeline over land. The oil rig is 10 miles offshore, and the nearest refinery is 50 miles down the shoreline from the location of the oil rig. How should you lay the pipeline to minimize the cost?



Exercise 3.3.9. Turning a Pizzelle into a Waffle Cone ☕☕☕☕

Suppose we have a disc of radius 1. We wish to leave a sector of angle θ and then identify the edges on either side of the sector to create a right circular cone. We of course would like this to then hold as much ice cream as possible. This leads us to our optimization problem!



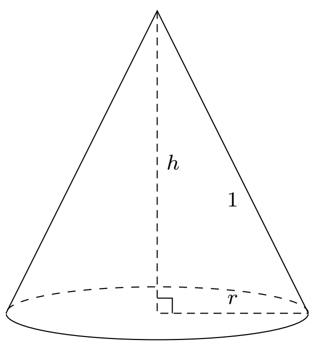
What size sector should be cut out of a disc to fold it into the largest possible cone?

- If θ is too small, why does that not lead to a very good cone?

- If θ is too large, why does that not lead to a very good cone?

- Now, we use calculus to find that sweet spot in the middle! Express the volume of the cone formed by the sector of angle θ and then folding as $V(\theta) = \frac{1}{3}h \cdot A$. This is our objective function. We need to identify constraints.
  - The area of the base should be a circular area, where $A = \pi r^2$. But what's r? The circumference of the base should be the arc length of our pizelle, which is conveniently just θ. Use this fact to write the area of our base in terms of θ.

  - The height of the cone is also related to the radius of the base. Note that we can form a right triangle as follows:



Use the Pythagorean theorem and the value for r in terms of θ you found for the previous part to give an expression for h in terms of θ.

- Substitute your expressions for A and h into the volume function

$$V(\theta) = \frac{1}{3}h \cdot A.$$

Then, using optimization, find the ideal angle θ that maximizes the volume of the cone.

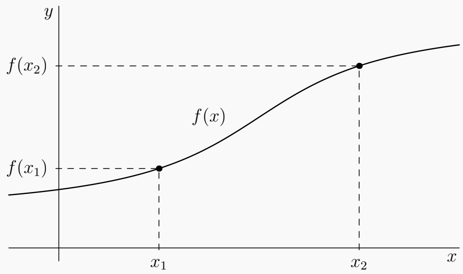
3.4 Curve Sketching with Derivatives

In the previous section, we focused on finding max or min values of functions using the first and second derivative. Here we extend that analysis just a bit by continuing our discussion of how derivatives help us find the shape of a graph. We begin with a few definitions.

Definition 3.4.1. Increasing/Decreasing and Convex/Concave

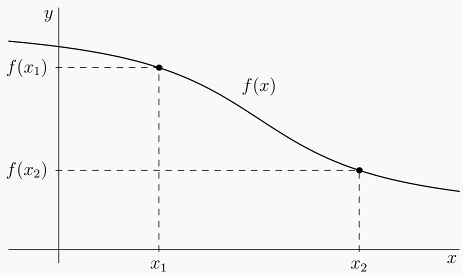
- A function f is *increasing* on an interval $[a, b]$ if and only if for all x_1 and x_2 in the interval,

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

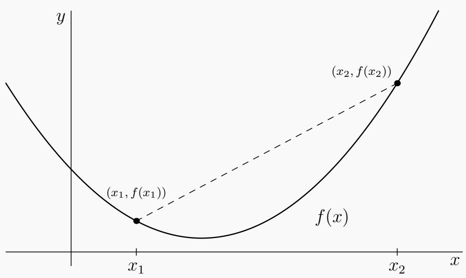


- A function f is *decreasing* on an interval $[a, b]$ if and only if for all x_1 and x_2 in the interval,

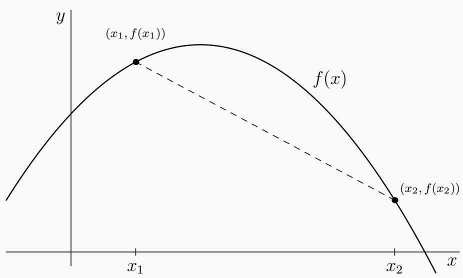
$$x_1 < x_2 \implies f(x_1) > f(x_2).$$



- A function f is *convex* on an interval $[a, b]$ if and only if for all x_1 and x_2 in the interval, the line segment connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies entirely above the graph.

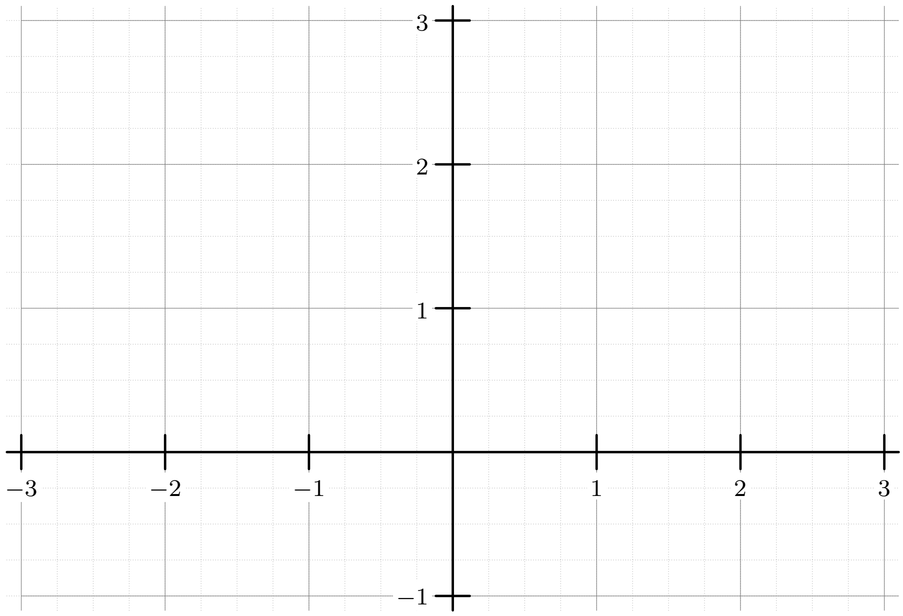


- A function f is *concave* (also known as *concave down*) on an interval $[a, b]$ if and only if for all x_1 and x_2 in the interval, the line segment connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies entirely below the graph.



Exercise 3.4.2. Absolute Value ☕

Graph the function $f(x) = |x|$. Where is this function increasing? Where is it decreasing? On what intervals is it convex? On what intervals is it concave?



Note that in other sources, you may see the phrase *concave up* to describe the graph of a convex function on an interval and *concave down* to describe the graph of a concave function on an interval. As far as easily remembering which shape goes which with word, one can think of the following phonetic and visual resemblances:

Convex sounds like flex, which is how a flexing arm looks. Concave has cave as a subword, which is where a bear would live.

Notice that max/mins can be characterized as locations where a function changes from increasing to decreasing or vice versa. We make an analogous name for places where a function switches from convex to concave or vice versa.

Definition 3.4.3. Point of Inflection

A *point of inflection* is the location where a function changes from convex to concave, or vice versa.

It turns out that if a function is differentiable, we can use derivatives to determine the intervals on which the function is increasing, decreasing, convex, and concave.

Theorem 3.4.4. Increasing/Decreasing and Convex/Concave via Derivatives

Let $f(x)$ be differentiable on (a, b).

- If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).
- If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).
- If $f''(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is convex on (a, b).
- If $f''(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is concave on (a, b).

Proof. We handle each statement one at a time.

- To prove the first statement,

  If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).

  it is easiest to instead prove its contrapositive, namely

  If $f(x)$ is not increasing on (a, b), then $f'(x) \leq 0$ for some $x \in (a, b)$.

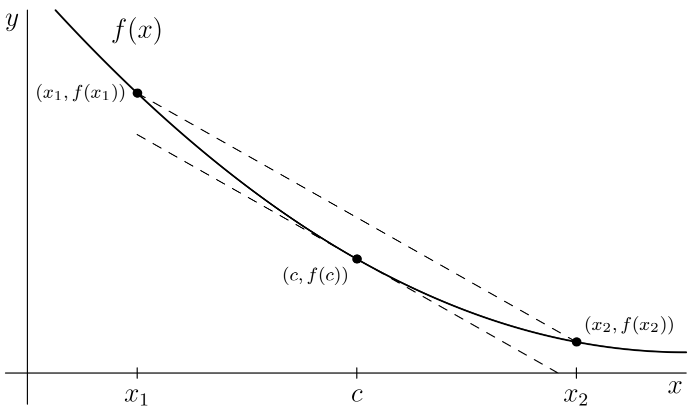
  Assume $f(x)$ is not increasing on (a, b). Then there exists a pair of values x_1 and x_2 in (a, b) where $x_1 < x_2$ but $f(x_1) \geq f(x_2)$. Now consider the average rate of change of f on the interval $[x_1, x_2]$. The secant line has slope

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

  Since $0 \geq f(x_2) - f(x_1)$ while $x_2 - x_1 > 0$, their ratio is nonpositive. By MVT, there exists a point $c \in (x_1, x_2)$ with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0.$$

  Since $(x_1, x_2) \subseteq (a, b)$, our value $c \in (a, b)$ as well.



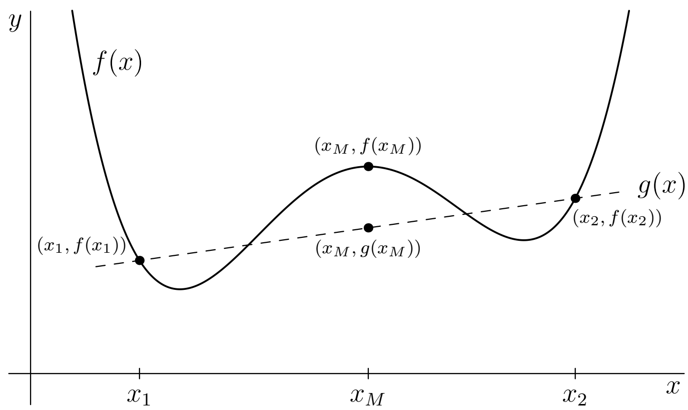
- The second statement is left as an exercise to the reader.
- For the third statement, we again proceed by contrapositive, showing the following:

If $f(x)$ is not convex on (a,b), then $f''(x) \le 0$ for some $x \in (a,b)$.

Proceeding, we assume that f is not convex on (a,b). Then there exists some line segment connecting two points on the graph of f that has a point on the line segment below the graph of f. To say this more precisely, we say that there exist three x-values x_1, x_2, x_M with $a \le x_1 < x_M < x_2 \le b$ such that

$$g(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1)$$

and $g(x_M) < f(x_M)$. Let m represent the slope of this line segment.



We now apply MVT twice, once to the subinterval from x_1 to x_M, and once from the subinterval from x_M to x_2.

– **First MVT Application:** By MVT, there exists $c_L \in [x_1, x_M]$ such that

$$f'(c_L) = \frac{f(x_M) - f(x_1)}{x_M - x_1}.$$

Recall that $f(x_M) > g(x_M)$ and subtract $f(x_1)$ from both sides to see that $f(x_M) - f(x_1) > g(x_M) - f(x_1)$. Since $x_M - x_1$ is positive, we can divide both sides without flipping the sign of the inequality. Thus,

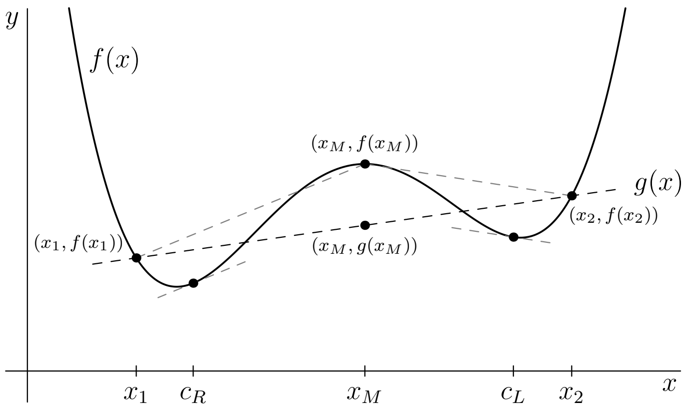
$$f'(c_L) = \frac{f(x_M) - f(x_1)}{x_M - x_1} > \frac{g(x_M) - f(x_1)}{x_M - x_1} = m.$$

– **Second MVT Application:** By MVT, there exists $c_R \in [x_M, x_2]$ such that

$$f'(c_R) = \frac{f(x_M) - f(x_2)}{x_M - x_2}.$$

Recall again that $f(x_M) > g(x_M)$ and subtract $f(x_1)$ from both sides once again to see that $f(x_M) - f(x_1) > g(x_M) - f(x_1)$. Since $x_M - x_2$ is negative, we can divide both sides if we flip the sign of the inequality. Thus,

$$f'(c_R) = \frac{f(x_M) - f(x_2)}{x_M - x_2} < \frac{g(x_M) - f(x_2)}{x_M - x_2} = m.$$



We now put our two applications of MVT together to see that

$$f'(c_R) < m < f'(c_L),$$

but meanwhile

$$c_L < x_M < c_R.$$

Thus, we have that $c_L < c_R$ but $f'(c_L) \not< f'(c_R)$. This shows us that the function $f'(x)$ is not increasing on (a, b). Thus, by the first statement (in contrapositive form, using f' itself as the new "f"), we have that $f''(x) \leq 0$ for some $x \in (a, b)$.

- The fourth statement is again left as an exercise to the reader.

□

Exercise 3.4.5. Statements Two and Four ☕☕☕

Modify or use the arguments above to fill in the proofs of the second and fourth statements.

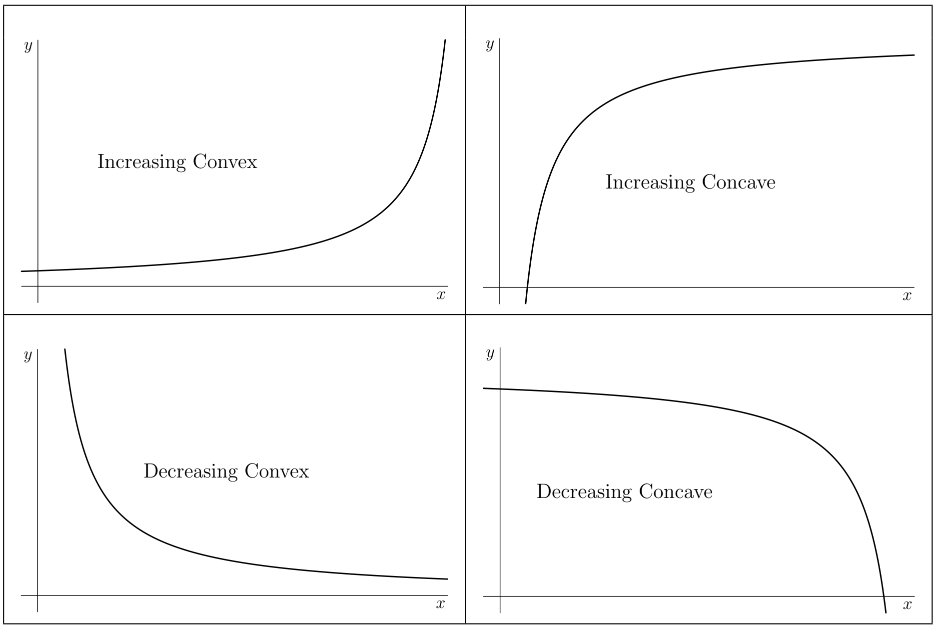
The theorems above motivate the definitions of what it means to be increasing/decreasing or convex/concave at a single point. Since on an interval these can be determined based on the signs of the first and second derivatives, we define those properties at a point in a similar manner.

Definition 3.4.6. Increasing/Decreasing and Concave/Convex at a Point

Let f be a function and let c be a point where $f'(c)$ and $f''(c)$ both exist. Then we define the following terms:

- We say f is *increasing* at c if and only if $f'(c) > 0$.
- We say f is *decreasing* at c if and only if $f'(c) < 0$.
- We say f is *convex* at c if and only if $f''(c) > 0$.
- We say f is *concave* at c if and only if $f''(c) < 0$.

Notice that increasing/decreasing and convex/concave can be paired with each other in any of four ways. Keep the approximate shapes below in mind as you sketch curves.



Now that we have the theory above laid out, we play with some specific functions. We have a nice bit of machinery developed for uncovering the shape of a graph. In particular, to find the shape of the graph of a function $f(x)$ using derivatives, do the following:

- **Find Domain:** Think about what real numbers x are valid inputs to your function f. Watch out for trouble spots like division by zero, square roots of negative numbers, logs of negative numbers, and so on. Call this set D.
- **Use First Derivative:** Calculate the first derivative, $f'(x)$. On the domain D, figure out where $f'(x)$ exists (that is, where f is differentiable). Recall that
  - Wherever $f'(x) > 0$, the original function f is increasing.
  - Wherever $f'(x) < 0$, the original function f is decreasing.
  - Wherever $f'(x) = 0$ or DNE, the original function f has a critical point (and may have a max or min).

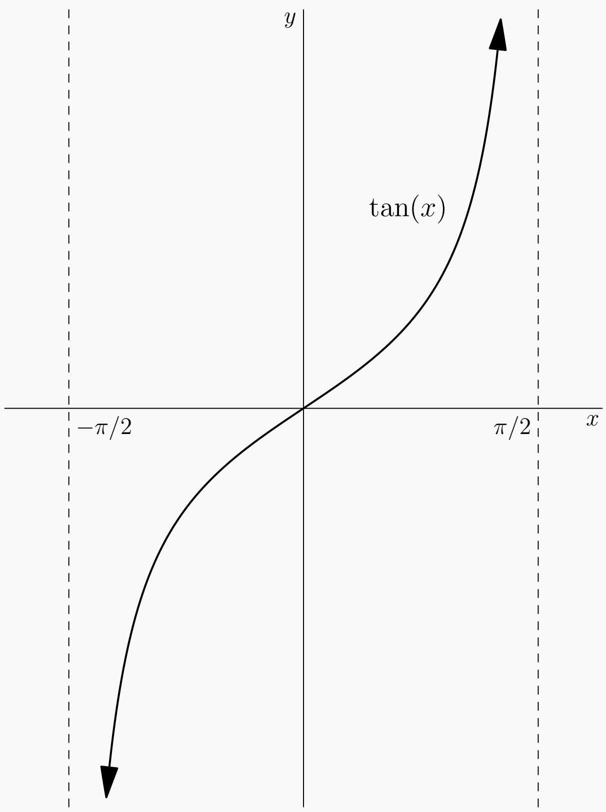
- **Use Second Derivative:** Calculate the second derivative, $f''(x)$. On the domain D, figure out where $f''(x)$ exists (that is, where f is twice differentiable). Recall that
  - Wherever $f''(x) > 0$, the original function f is convex.
  - Wherever $f''(x) < 0$, the original function f is concave.
  - Wherever $f''(x) = 0$ or DNE, the original function f has a possible point of inflection.
- **Classify Critical Points as Max/Min and Identify Points of Inflection:** Fermat's Theorem along with the Second Derivative Test can be used to identify locations of max or min values. A change in sign on the second derivative will identify a point of inflection.

This information can then be combined with any previous graphing techniques that are relevant in that situation. For example, finding intercepts and testing for symmetry is still very useful! If you have a trig function, you still want to find the period, if you have a rational function, you still want to find the asymptotes, and so on. Lastly, don't forget that despite all this fancy machinery, it is still just graphing a function, so you can plug in as many points as you like to help yourself out! Select a few x values from the domain that are easy to work with and plot the corresponding points to anchor your graph.

Example 3.4.7. Arctangent

Let us begin with a fairly familiar graph, just to see all the moving parts in action. Observe the graph of $f(x) = \arctan(x)$ below.

- **Find Domain:** In this case, the domain is the entire set of real numbers. That is, $D = \mathbb{R}$. One way to see this is that the range of tangent (restricted to $(\pi/2, \pi/2)$) is the set of all real numbers.



Thus, the domain of the inverse function, arctangent, is the set of all real numbers, since taking an inverse function interchanges the domain and range. In short,

$$D = \mathbb{R}.$$

- **Use First Derivative:** We use $f'(x) = \frac{1}{1+x^2}$ to find the intervals of increasing or decreasing. Since $1 + x^2$ is always positive, $\frac{1}{1+x^2}$ is as well. Thus, $f'(x) > 0$ for all $x \in \mathbb{R}$, which tells us that f increases forever. In short,

  $$\textit{the function } f \textit{ is increasing on } (-\infty, \infty).$$

- **Use Second Derivative:** We calculate the second derivative to be

  $$f''(x) = \frac{-2x}{(1+x^2)^2}.$$

  The denominator is always positive, so the sign of the second derivative is the same as the sign of the numerator. The numerator is positive when x is negative, negative when x is positive, and zero when x is zero. We conclude that $f''(x) > 0$ on $(-\infty, 0)$ and $f''(x) < 0$ on $(0, \infty)$. In short,

  $$\textit{the function } f \textit{ is convex on } (-\infty, 0) \textit{ and concave on } (0, \infty).$$

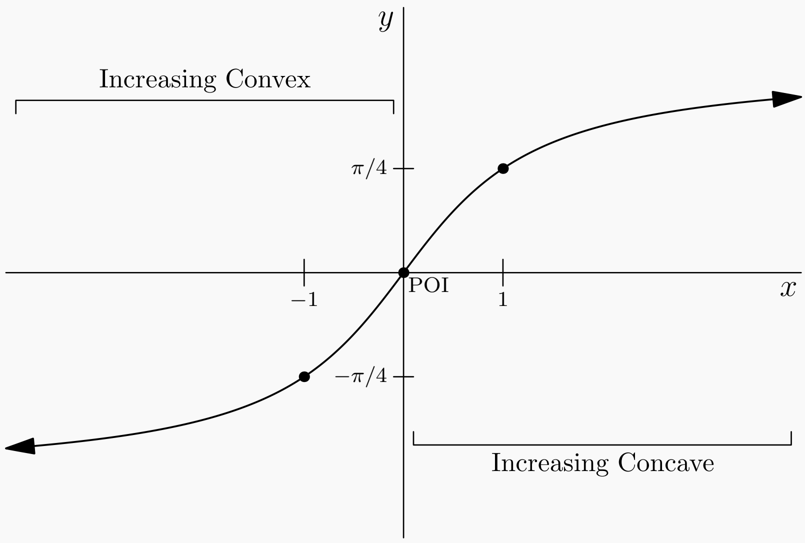
- **Classify Critical Points as Max/Min and Identify Points of Inflection:** The function has no critical points since the first derivative is always a positive real number. Since the second derivative flips from positive to negative at $x = 0$, we do have a point of inflection at that location. In short,

  the function has no local max or min, but it has a point of inflection at $x = 0$.

Finally, we plot just a few random points to get ourselves going. In particular, we strategically select a few values that are easy to compute as shown below:

x	-1	0	1
$\arctan(x)$	$-\pi/4$	0	$\pi/4$

At last, we assemble all of this info to build the graph of our function.



Exercise 3.4.8. Filling in Details ☕

In the above example. . .

- . . . show the details of the second derivative calculation using the quotient rule.

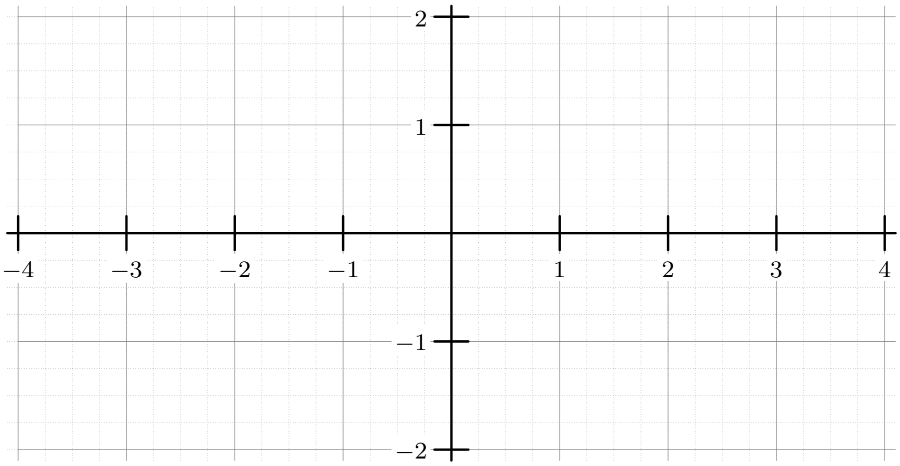
- . . . show the details of the second derivative calculation using the power and chain rule (treating the reciprocal as a negative one power rather than a quotient).

- . . . show the details of where those values in the data table came from. Specifically, identify triangles or locations on the unit circle that verify $\tan(y) = x$, allowing you to conclude $\arctan(x) = y$ for each such pair in the table.

In addition to looking at the first and second derivative of $f(x)$ symbolically as we did in the above example, sometimes it is nice to graph $f'(x)$ and $f''(x)$ as separate functions to see the relationship with $f(x)$ in a more visual manner.

Exercise 3.4.9. Followup to Arctangent ☕☕

On the graph paper below, again graph $f(x) = \arctan(x)$, but this time also draw the graphs of $f'(x) = \frac{1}{1+x^2}$ and $f''(x) = \frac{-2x}{(1+x^2)^2}$ on the same axes. Verify that the graph of $f'(x)$ is positive/negative wherever the graph of $f(x)$ is increasing/decreasing, and that the graph of $f''(x)$ is positive/negative wherever the graph of $f(x)$ is convex/concave.



One of the nicest contexts for using this is in graphing polynomials and rational functions. You can begin just as you would in college algebra, but the first and second derivative allow you to identify much more interesting detail on the graph!

Exercise 3.4.10. A Guided Rational Function

Let us graph the rational function

$$f(x) = \frac{x^3 - 1}{x^3 + 8}$$

together, one step at a time.

- Factor the numerator and denominator. Explain why this determines $x = -2$ as the only vertical asymptote and $x = 1$ as the only x intercept.

- Based on the factorizations you found, does the sign of the function stay the same or switch at $x = -2$? What about at $x = 1$?

- Use polynomial long division to identify any horizontal asymptotes. In calculus speak, what are the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$?

- Show that the first derivative is
$$f'(x) = \frac{27x^2}{(x^3+8)^2}.$$
Notice that it is much more difficult to differentiate $f(x)$ in the original form in which it is given than the form you create using polynomial long division!

- Use that information to conclude that the function is increasing on the set $(-\infty,-2) \cup (-2,\infty)$ and is never decreasing. Furthermore, explain why there are no local extrema.

- Show that the second derivative is
$$f''(x) = \frac{432x - 108x^4}{(x^3+8)^3}.$$

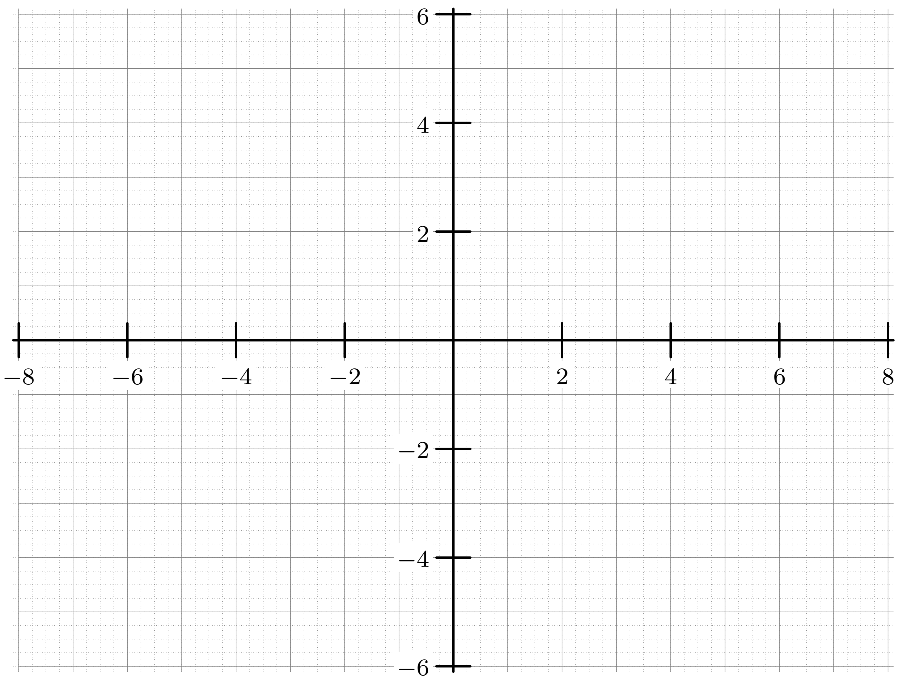
- Use the formula for the second derivative to conclude that $f''(x) = 0$ only for $x = 0$ and $x = \sqrt[3]{4}$, and does not exist at $x = -2$.

- Use the information above to conclude that f is convex on the set $(-\infty,-2) \cup \left(0, \sqrt[3]{4}\right)$ and concave on the set $(-2,0) \cup \left(\sqrt[3]{4}, \infty\right)$. Furthermore, explain why f has a point of inflection at $x = \sqrt[3]{4}$ but not at $x = -2$.

- Plot a few random points to get yourself going. In particular, fill out the table below and use them!

x	-1	0	1
$f(x)$			

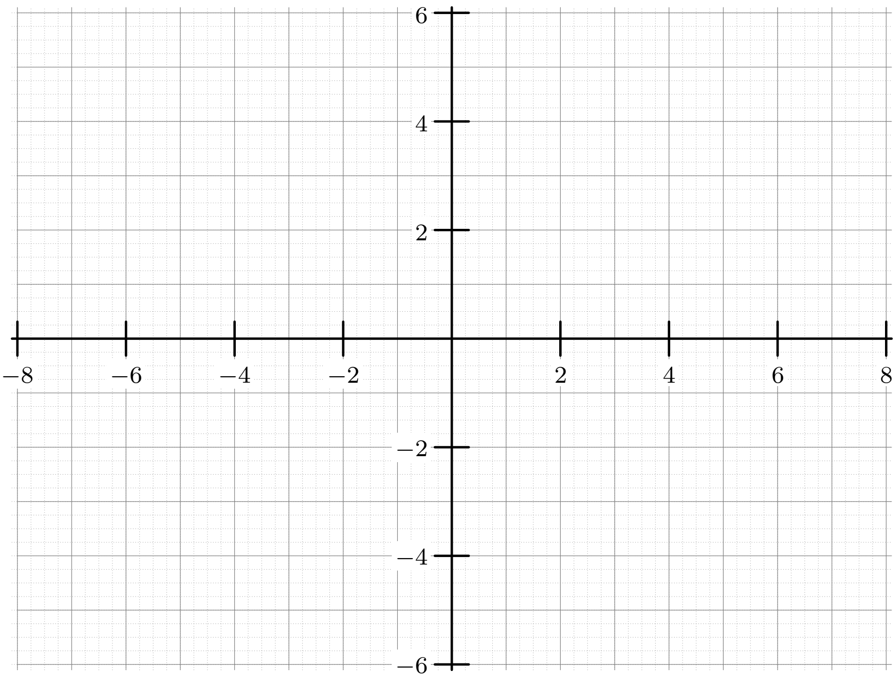
- At last, assemble all of this information to build the graph of the function. Label all intervals of increasing/decreasing, intervals of concavity/convexity, extrema, points of inflection, and intercepts.



Alright, now try a few completely on your own!

Exercise 3.4.11. A Rational Function

Graph the function $f(x) = \frac{x+1}{x^2-4}$. Include any local max/min, intervals of increasing, decreasing, concavity, and points of inflection.

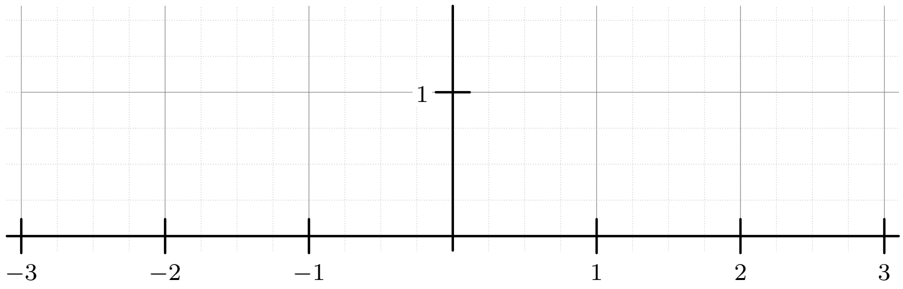


Exercise 3.4.12. A Radical Rational Function ☕☕

Graph the function $f(x) = \sqrt{\frac{1}{1-x-x^2}}$. Include any asymptotes, max/min, intervals of increasing, decreasing, concavity, and points of inflection.

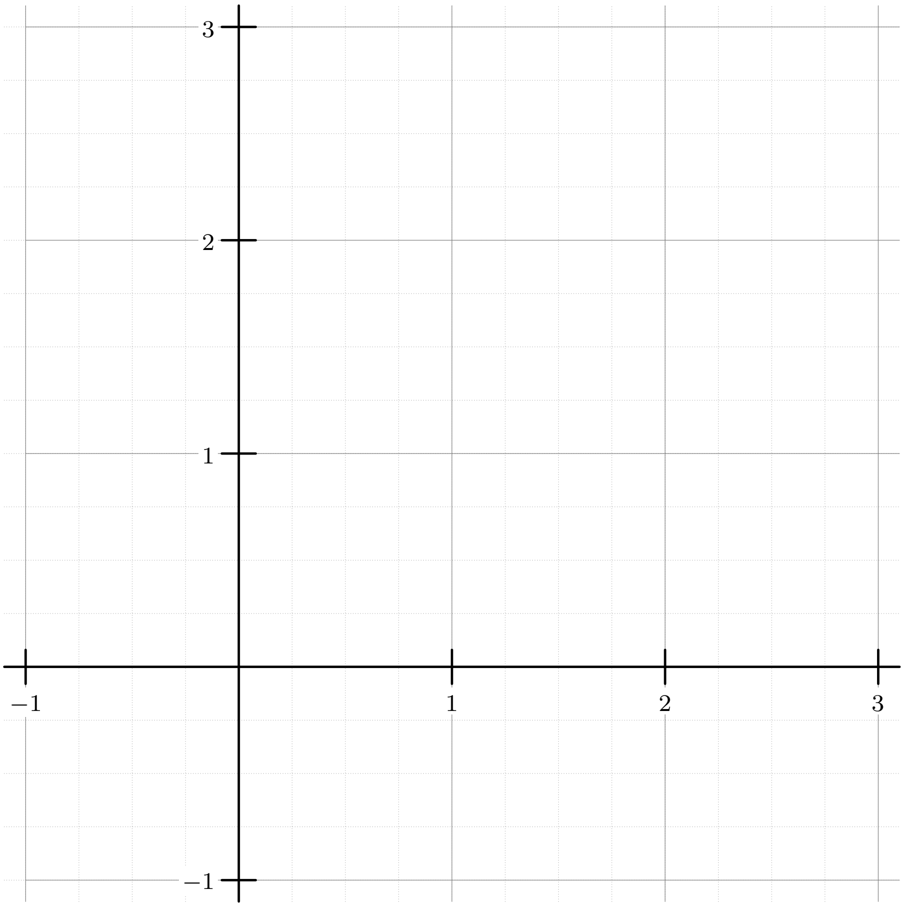
Exercise 3.4.13. An Exponential Function ☕☕

Graph the function $f(x) = e^{-x^2}$. Include any asymptotes, max/min, intervals of increasing, decreasing, concavity, and points of inflection.



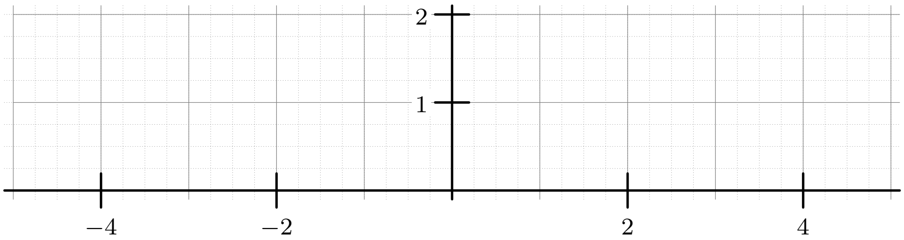
Exercise 3.4.14. A Logarithmic Function

Graph the function $f(x) = x\ln(x)$. Include any asymptotes, max/min, intervals of increasing, decreasing, concavity, and points of inflection.



Exercise 3.4.15. An Inverse Trig Function ☕☕

Graph the function $f(x) = \arctan\left(x^2\right)$. Include any asymptotes, max/min, intervals of increasing, decreasing, concavity, and points of inflection.



3.5 Related Rates

Often an equation has several dependent variables that are all (perhaps unknown) functions of some independent variable (time, for example, comes up often). In this case, even though the functions may be unknown, we can use implicit differentiation to relate the rates of change of those dependent variables by differentiating with respect to the dependent variable.

Example 3.5.1. Several Dependent Variables

Suppose we have quantities a, b, c, and d that are all changing throughout time, t. Suppose k is a constant that is not changing with time. Suppose also that the quantities a, b, c, and d are related by the equation

$$\sin(a) + b + e^{kcd} = 1.$$

Then we can differentiate both sides with respect to t to obtain

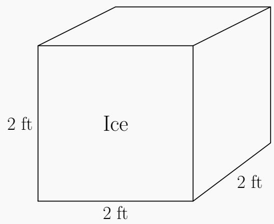
$$\cos(a)\frac{\mathtt{d}a}{\mathtt{d}t} + \frac{\mathtt{d}b}{\mathtt{d}t} + e^{kcd}\left(kc\frac{\mathtt{d}d}{\mathtt{d}t} + k\frac{\mathtt{d}c}{\mathtt{d}t}d\right) = 0.$$

Notice that the above differentiation doesn't provide a formula in terms of t for any of the derivatives, but it does actually relate them to each other! So, if you for some reason knew particular values of a, b, c and their rates of change, you could then use the original equation to solve for d and the implicit differentiation to solve for $\frac{\mathtt{d}d}{\mathtt{d}t}$. This is exactly the technique of *related rates*; taking related quantities, and then relating their rates of change through implicit differentiation.

Example 3.5.2. A Melting Ice Cube

Let us answer the following question:

Suppose a large block of ice measures 2ft×2ft×2ft and is melting at a consistent rate of 2in³ per minute. At what rate is the side length decreasing?



Here we have two dependent variables, which we can call V for volume and s for side length. These are our quantities of interest. They both in turn depend on the independent variable t. What we need is some relationship between V and s that can be implicitly differentiated with respect to t. That relationship is the formula for the volume of a cube, namely

$$V = s^3.$$

Differentiating with respect to t produces

$$\frac{\mathtt{d}V}{\mathtt{d}t} = 3s^2\frac{\mathtt{d}s}{\mathtt{d}t}.$$

We know the current side length $s = 2\text{ft} = 24\text{in}$ and the current rate of change of the volume, namely $\frac{\mathtt{d}V}{\mathtt{d}t} = -2\text{in}^3$. Substituting these values into our equation will allow us to solve for $\frac{\mathtt{d}s}{\mathtt{d}t}$.

Proceeding thusly, we have

$$-2 = 3 \cdot 24^2 \frac{\mathrm{d}s}{\mathrm{d}t}$$

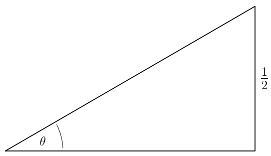
which shows that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{1}{864} \text{in/min.}$$

Note that it makes sense that the side length was decreasing very slowly, since the volume was decreasing at a very small rate compared to the very large size of the ice block. Nice. nIce.

Exercise 3.5.3. Broscience Except Real ☕

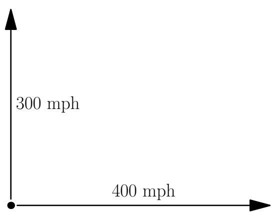
A leg press machine involves a person pressing weights upwards at an angle of 30 degrees from the horizontal. A squat happens straight up and down. Thus, a lifter's one rep maximum (ORM) weight on squats will always be roughly sin(30) = 1/2 times their ORM on leg press.



If the lifter gains 10 pounds per month on their squats ORM, at what rate will their leg press ORM change? Show how the answer can be obtained by setting up an equation relating the squat ORM and the leg press ORM, and then using implicit differentiation with respect to time t.

Exercise 3.5.4. Parting Ways ☕☕☕

Two planes take off from Denver International Airport at the same time. One travels due north at 300mph while the other travels due east at 400mph. Assume an infinite flat earth for this problem!



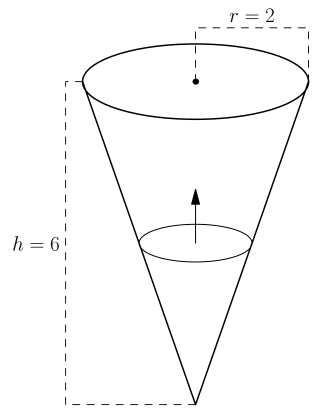
- After 1 hour, how quickly are they moving apart from each other?

- After 2 hours, how quickly are they moving apart from each other?

- Is the rate at which they move apart from one another dependent on how long they have been flying? Or is it the same rate regardless of air time?

Exercise 3.5.5. Filling a Soda Glass ☕☕

Consider the following situation: we have a conical soda glass with height 6in and a circular top of radius 2in. Recall the formula for the volume of such a cone: $V = \frac{1}{3}\pi r^2 \cdot h$ where r represents the radius and h represents the height.



Suppose at time 0 the glass is empty. We begin filling it with soda from a tap that pours at a rate of 2 in^3 per second.

- Intuitively, where is the rate of change of height the fastest? Where is the rate of change of height the slowest? Write a sentence to explain your answer.

- Compute $\frac{dh}{dt}$ after one second of pouring.

- Compute $\frac{dh}{dt}$ after two seconds of pouring.

- Do these computations verify or contradict your intuition? Explain.

Let us try an example that is less geometric in nature.

Exercise 3.5.6. Ideal Gas Law ☕☕

The *Ideal Gas Law* states that

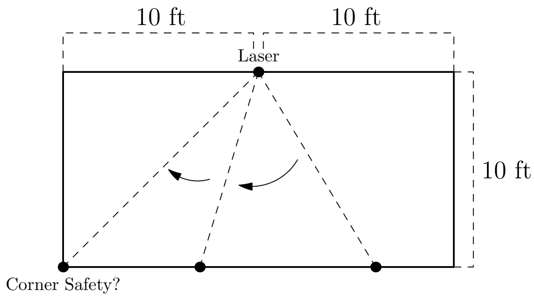
$$PV = nRT$$

for an ideal gas, where P represents the pressure, V represents the volume, n is the number of moles, and T represents the temperature of the gas. The symbol R is just a placeholder for the unsightly constant $8.31\mathrm{J} \cdot \mathrm{mol}^{-1} \cdot \mathrm{K}^{-1}$.
Suppose we have a closed container of gas (like a sealed balloon for example). Thus, n is constant with respect to time as no gas is entering or leaving the balloon. But, suppose pressure, volume, and temperature can all vary with time. Differentiate both sides with respect to time t in order to find a relationship between the rates of change of pressure, volume, and temperature.

Ok, back to geometry.

Exercise 3.5.7. Laser Pointers ☕☕☕

Wanting to create a cat heaven at home for when they are at work, a cat lover installs a rotating motor on the ceiling to which a laser pointer is attached. The motor rotates at a constant angular speed, completing thirty revolutions per minute. The room is 10ft high and 20ft long and the laser is attached to the very center of the ceiling (assume the length of the pointer is negligible).



A cat can sustain a 20mph impact with no adverse health effects whatsoever. However, anything beyond that will hurt. Assuming the cat will blindly chase the red dot and slam into the wall each time (because he/she will), is the setup safe for the cat? If not, what should the angular speed of the motor be reduced to?

3.6 Chapter Summary

Here we looked at a wide spread of derivative applications.

1. **Linearization.** Derivatives let us approximate a function $f(x)$ for x-values near $x = a$ with a linear function $L(x)$ by calculating the tangent line at a. Specifically,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

   for x-values near a.

2. **Optimization.** Given a function $f(x)$ and an interval $[a, b]$ on which we wish to find the maximum/minimum value of $f(x)$, we can compare the values of

$$f(a), f(b) \text{ and } f(c)$$

   where c takes on each value for which $f'(c) = 0$ or DNE. The biggest of these values will be the max, and the smallest will be the min.

3. **Graphing with derivatives.** First and second derivatives add a wealth of information to graphing. Most importantly, they enable you to determine where you function is increasing/decreasing and convex/concave as follows:
   - If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).
   - If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).
   - If $f''(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is convex on (a, b).
   - If $f''(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is concave on (a, b).

4. **Implicit differentiation.** We usually work with explicit formulas of the form $y = f(x)$ when we differentiate. However, if we have only an implicit formula for y in terms of x, that is, an equation with x and y tangled up in some silly manner, then we can still find $\frac{dy}{dx}$. To accomplish this, we perform the following steps:
   - Differentiate both sides of the equation. Treat y as an unknown function of x, and apply product rule/chain rule/etc as appropriate.
   - Solve for $\frac{dy}{dx}$ using algebra.

5. **Related rates.** Often geometric formulas (area formulas, volume formulas, Pythagorean Theorem, etc) or physical laws (Ideal Gas Law, Kirchhoff's Law, etc) can be written as an equation to describe some situation that is changing with time. Performing implicit differentiation on such an equation with respect to time enables one to solve for one rate of change given another.

3.7 Mixed Practice

Exercise 3.7.1.

Consider the function $f(x) = x^3$. Use the linearization at $x = 1$ to approximate the value of 1.1^3. How does this compare to the true value?

Exercise 3.7.2.

Identify each of the following statements as true or false.

- If a function $f(x)$ is convex and increasing at a point $x = c$, then $f(x)$ must have a local minimum at that point.
- If a function $f(x)$ is increasing on an interval $[a, b]$, then any line segment drawn between points $(c_1, f(c_1))$ and $(c_2, f(c_2))$ for $c_1, c_2 \in [a, b]$ must lie entirely below the graph of $f(x)$.
- If a function $f(x)$ is twice-differentiable on an interval $[a, b]$ and for some $c \in [a, b]$, $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ must be the absolute minimum value of $f(x)$ on $[a, b]$.

Exercise 3.7.3.

- Suppose we were to use the linearization of $f(x) = \tan(x)$ at $x = \pi/4$ to approximate the values of $\tan(0.8)$ and $\tan(1)$. Which of those two values would you expect to be approximated more accurately and why?
- Calculate the linearization and use it to approximate those given values, $\tan(0.8)$ and $\tan(1)$. Compare to calculator/CAS values of these numbers to see if your prediction regarding the accuracy was correct.

Exercise 3.7.4.

Consider a linear function $f(x) = mx + b$.

- From the graph, what would be the linearization of $f(x)$ at any point $x = a$?
- Use the linearization formula to confirm your answer from the previous part.

Exercise 3.7.5. ☕☕☕

Graph the function

$$f(x) = \frac{x}{x^2 - 4}.$$

Be sure to include the following in your process:

- Find the domain.
- Use the first derivative to determine intervals on which the graph is increasing/decreasing.
- Use the second derivative to determine intervals on which the graph is concave/convex.
- Plug in a few nice points to get started and sketch the graph!

Exercise 3.7.6. ☕☕

A martini bar owner currently sells 300 martinis each week at \$9 per martini. She notices that any time she lowers the price 50 cents, that week's sales go up by 25 martinis.

- Let x represent the price of a martini in dollars. Let $Q(x)$ be the quantity of martinis sold. For example, $Q(9) = 300$, $Q(8) = 350$, $Q(7) = 400$, and so on. Find a formula for $Q(x)$.
- Let $R(x) = x \cdot Q(x)$ be the total weekly revenue function. Find a formula for $R(x)$.
- Use the first and second derivative to find the price that will maximize revenue for the martini bar owner.

Exercise 3.7.7. ☕☕☕☕

Let us study the graph of the equation

$$x^2 + xy + y^2 = 1.$$

- First, explain why the graph must have symmetry across the line $y = x$.
- Find the x and y intercepts.
- Use implicit differentiation to calculate $\frac{dy}{dx}$.
- Use the derivative to determine where the curve has horizontal tangents and where the curve has vertical tangents.
- Assemble the above information to draw the graph.

Exercise 3.7.8. ☕☕☕

Suppose soft-serve ice cream is pouring out of a high spout into a large flat dish. Upon landing, it always takes the shape of a right circular cone with height equal to its radius. The machine pours at $1\frac{\text{cm}^3}{s}$. (Recall the formula for the volume of a cone: $V = \frac{1}{3}h\pi r^2$.)

- Would you predict that the radius grows more quickly when the cone is small or when the cone is large? Explain.
- What is the rate of change of the radius when the cone has a radius of 5cm?
- What is the rate of change of the radius when the cone has a radius of 10cm?
- Compare your two answers above. Are they consistent with your prediction?

Part III

Integrals

Chapter 4

Riemann Sum Definition of Integrals

This chapter may initially appear to be a bit of a non sequitur considering what just happened in Part II. Derivatives will not show up for quite some time, but like so many good stories, that beloved character will reappear soon, specifically in Chapter 5.

Here is a motivating question that will lead us to the idea of an integral.

Notice that in trying to answer the question above, we are always adding up big long lists of numbers. The following section sets up good compact notation for this! Also notice that the as we are computing probabilities above, we are also computing areas. So one can think of this as a probability question or as a geometry question!

4.1 Summation Notation and Properties

A *series* or a *summation* is a sum of a list of numbers. We often use the very compact *sigma notation* to represent series.

Definition 4.1.1. Sigma Notation for Series

If a_n is a sequence and j, k are both natural numbers, then we define the series:

$$\sum_{n=j}^{k} a_n = a_j + a_{j+1} + a_{j+2} + \cdots + a_k$$

That is, we add up all consecutive terms of the sequence a_n, starting at index j and stopping at index k.

Notice that the above summation has $k - j + 1$ terms in it, not $k - j$ as one might quickly guess. One way to see this is to rewrite the sum slightly as

$$a_j + a_{j+1} + a_{j+2} + \cdots + a_k = \underbrace{a_{j+0}}_{\text{One extra for term } 0\ldots} + \underbrace{a_{j+1} + a_{j+2} + \cdots + a_k}_{\ldots\text{then count terms } 1,2,\ldots k}.$$

This subtlety is often called an *off by one error* or *fencepost problem*, since one can view it as if the plus signs were sections of fence, and the terms in the sequence were posts holding up those sections. To support $k - j$ sections of fence, we need $k - j + 1$ posts, since each section has a post to the right of it, but the very first section of fence also has a post to the left which is not to the right of any section.

If the starting index is greater than the stopping index, we consider the sum to be empty. Since it has no terms, we define the total to be zero. Thus, an *empty sum* is the additive identity zero, just like an empty product is the multiplicative identity one.

The sequence a_n that is being totaled is called the *summand*, much as the function $f(x)$ is referred to as the radicand in the expression $\sqrt{f(x)}$.

Example 4.1.2. Evaluating a Summation

Consider the sum of all even numbers between six and fourteen. Of course we don't need sigma notation to evaluate such a sum, but just for proof of concept, let's write this sum in sigma notation, expand, and evaluate.

- First we discuss the summand. The sequence of all even numbers has the explicit formula $a_n = 2n$, so $2n$ will be our summand.
- We want the first term to be six, so we set $n = 3$ as the starting index.
- We want the last term to be fourteen, so we set $n = 7$ as the stopping index.

Thus our summation is

$$\begin{aligned}\sum_{n=3}^{7} 2n &= \underbrace{6}_{n=3} + \underbrace{8}_{n=4} + \underbrace{10}_{n=5} + \underbrace{12}_{n=6} + \underbrace{14}_{n=7} \\ &= 60.\end{aligned}$$

Exercise 4.1.3. Not Crashing Into That Extra Fencepost ☕

The summation in the above example has starting index 3 and stopping index 7. So, does the sum have $7 - 3 = 4$ terms, or does it have $7 - 3 + 1 = 5$ terms?

Exercise 4.1.4. Sigma Notation ☕

Evaluate the following sums:

- $\sum_{n=0}^{3} 2n$
- $\sum_{n=0}^{3} (-1)^n n^2$
- $\sum_{n=0}^{3} 2^n$

Exercise 4.1.5. Properties of Summations ☕☕☕

Let c be an arbitrary real number, j and k natural numbers with $j < k$, and a_n and b_n be arbitrary sequences. For each of the following properties, explain why it is true, or come up with a counterexample that shows it is not.

- $\sum_{n=j}^{k} c \cdot a_n = c \sum_{n=j}^{k} a_n$

- $\sum_{n=j}^{k} (a_n + b_n) = \left(\sum_{n=j}^{k} a_n \right) + \left(\sum_{n=j}^{k} b_n \right)$

- $\sum_{n=j}^{k} (a_n \cdot b_n) = \left(\sum_{n=j}^{k} a_n \right) \cdot \left(\sum_{n=j}^{k} b_n \right)$

- $\sum_{n=0}^{k} a_n = \sum_{n=1}^{k+1} a_{n-1}$

- $\sum_{n=0}^{k} c = ck$

- $\sum_{n=1}^{k} c = ck$

Exercise 4.1.6. Rewriting the Dartboard Probabilities in Sigma Notation ☕☕

- Let us rewrite the Exercise 4.0.0.2 probability calculations as summations in sigma notation. In particular, explain why the summation
$$\sum_{n=1}^{N} \frac{1}{N} \left(\frac{n}{N} \right)^2$$
represents the approximation corresponding to n rectangles.

- Evaluate this sum for the N values in the table below. You may use a CAS to make the computation more efficient!

N	5	10	100	1000
$\sum_{n=1}^{N} \frac{1}{N}\left(\frac{n}{N}\right)^2$				

- As N gets larger, what does it appear the area is converging to? What might you guess the true probability of the dart landing under that parabola is?

Rather than just looking at little tables of values and guessing what number it converges to, we would be able to get the exact value if we could just evaluate that summation to a closed form. That is, if we could get a formula that involves just N and not n, we could then take the limit as N goes to infinity and get the true area. We present a few such useful formulas for evaluating summations in the following section.

4.2 Summation Formulas

Sum of Consecutive Natural Numbers

We begin with *Gauss's Formula*, the formula for the sum of consecutive natural numbers.

Theorem 4.2.1. Gauss's Formula

For $N \in \mathbb{N}$, the sum of all positive integers is

$$\sum_{n=1}^{N} n = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}.$$

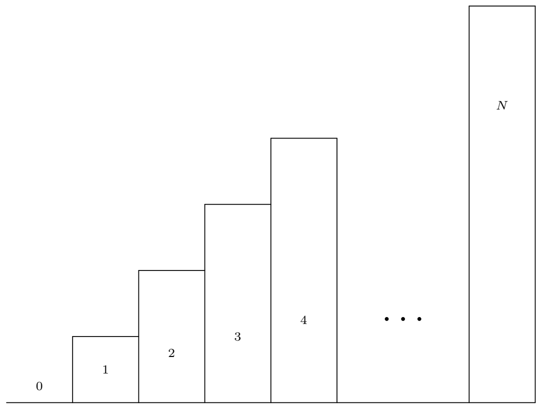
The next exercise provides one of many possible proofs of this formula.

Exercise 4.2.2. A Visual Argument for the Arithmetic Series Formula ☕☕☕

Here we draw a diagram to show why the Arithmetic Series Formula works. Consider the sum

$$1 + 2 + 3 + \cdots + N.$$

- For each term in the sum, we draw a corresponding rectangle. Specifically, a 1×1 rectangle represents the first term, a 1×2 rectangle represents the second term, and so on. These rectangles are stacked in order in the first quadrant, next to each other on the x-axis, with sides of width 1 all on the x-axis. Explain why the area of the region is equal to the sum.



- Duplicate the entire region in the opposite order to build one giant rectangle. Draw a $1 \times N$ rectangle on top of the leftmost, then a $1 \times (N-1)$ rectangle on top of the second, and so on until the last rectangle gets topped with a 1×1 rectangle. In this new giant rectangle that is formed...
  - ...what is the width?
  - ...what is the height?
  - ...what is the total area?
- Explain why the total area of that rectangle must be exactly double the value of the summation.

- Divide the total area by two to arrive at the Gauss's Formula!

Just to get a more concrete sense for what it says, let's use the formula a bit.

Example 4.2.3. Using Gauss's Formula

Suppose we wish to find the sum of all multiples of six between 1000 and 2000. We notice that neither 1000 nor 2000 are divisible by six. However, multiples of six can never be too far away. In particular, $1002 = 6 \cdot 167$ and $1998 = 6 \cdot 333$. Thus, the summation we wish to evaluate is

$$1002 + 1008 + 1014 + \cdots + 1998$$

which can also be written as

$$6 \cdot 167 + 6 \cdot 168 + 6 \cdot 169 + \cdots + 6 \cdot 333.$$

Placing this into the notation used in Gauss's Formula, we have

$$6 \cdot (166 + 1) + 6 \cdot (166 + 2) + 6 \cdot (166 + 3) + \cdots + 6 \cdot (166 + 167).$$

From the above form, we now have all the information we need to apply the Gauss's Formula, along with the properties from Exercise 4.1.0.5. We rewrite the sum in sigma notation and then break it apart using those properties as follows:

$$\begin{aligned} 6 \cdot (166 + 1) + 6 \cdot (166 + 2) + 6 \cdot (166 + 3) + \cdots + 6 \cdot (166 + 167) &= \sum_{n=1}^{167} 6 \cdot (166 + n) \\ &= 6 \sum_{n=1}^{167} (166 + n) \\ &= 6 \left(\sum_{n=1}^{167} (166) + \sum_{n=1}^{167} n \right) \\ &= 6 \left(166 \cdot 167 + \frac{167 \cdot 168}{2} \right) \\ &= 250{,}500. \end{aligned}$$

Exercise 4.2.4. Recognizing Properties ☕

Annotate the above exercise. Specifically, mark where the properties from Exercise 4.1.0.5 were used and where Gauss's Formula was used.

Now try to replicate the techniques of the above example on your own!

Exercise 4.2.5. Practice with Gauss's Formula ☕☕

- Add up all the whole numbers from 1 to 1000 inclusive.

- Add up all the whole numbers from 1000 to 2000 inclusive.

- What is the sum of all multiples of seven between 1000 and 2000?

- Compute the following summation using the Gauss's Formula:

$$\sum_{n=4}^{13}(3n-1).$$

Sums of Consecutive Squares and Cubes

There are similar formulas for sums of consecutive squares and consecutive cubes of natural numbers, starting at 1. We state these below.

Theorem 4.2.6. Sums of Squares and Cubes

For $N \in \mathbb{N}$, sums of consecutive squares and cubes evaluate to the following closed forms:

$$\sum_{n=1}^{N} n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$$
$$\sum_{n=1}^{N} n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = \left(\frac{N(N+1)}{2}\right)^2$$

We won't prove these formulas in this course, but they can be proved by similar methods to what we did for Gauss's Formula.

Exercise 4.2.7. Checking the Formulas ☕

To verify the above formulas in an example, try the following:

- For the sum of consecutive squares formula, write out the summation for $N = 5$. Then proceed to evaluate the left-hand side but simply adding the numbers. Evaluate the right-hand side using our formula. Verify these quantities match.

- Do the same for the sum of consecutive cubes for $N = 5$.

Exercise 4.2.8. Polynomial Degrees ☕

- Considering N as the variable, what is the polynomial degree of $\sum_{n=1}^{N} n = 1+2+3+\cdots+N = \frac{N(N+1)}{2}$?
- What is the polynomial degree of $\sum_{n=1}^{N} n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$?
- What is the polynomial degree of $\sum_{n=1}^{N} n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = \left(\frac{N(N+1)}{2}\right)^2$?
- Even though we haven't specified a formula for it, what would you expect the degree to be for the closed form of the summation $\sum_{n=1}^{N} n^4 = 1^4 + 2^4 + 3^4 + \cdots + N^4$?

It turns out that the sum of consecutive kth powers is always a degree $k+1$ polynomial. How to come up with these polynomials is something you will see later in your mathematical career!

Geometric Series

If the summand is an exponential function, the summation is called an *geometric series.* In this case, we again have a nice formula for the sum!

Theorem 4.2.9. Finite Geometric Series Formula

Let $a_n = a \cdot r^n$ be a geometric sequence. Thus, a_n is an exponential function with y-intercept a and base r. Then the following sum has closed form

$$\sum_{n=0}^{N} a \cdot r^n = a + ar + ar^2 + ar^3 + \cdots + ar^N = a \cdot \frac{1 - r^{N+1}}{1 - r}.$$

Sometimes the base of the exponential is referred to as the *common ratio* since it is the ratio between any two consecutive terms in the summation. In words, you can state the geometric series formula as follows:

The sum of a geometric series is equal to the first term times one minus the common ratio raised to the number of terms, divided by one minus the common ratio.

Exercise 4.2.10. An Algebraic Argument for the Geometric Series Formula ☕☕☕

Here we use algebra to demonstrate why the Geometric Series Formula is valid. Consider the following geometric series and call it S for sum:

$$S = (a) + (ar) + \left(ar^2\right) + \left(ar^3\right) + \cdots + \left(ar^N\right).$$

- Explain why the following equality holds:

$$rS = (ar) + \left(ar^2\right) + \left(ar^3\right) + \left(ar^4\right) + \cdots + \left(ar^{N+1}\right).$$

- Subtract the two above equations. Fill in the right hand side below.

$$S - rS =$$

- Solve for S in the equation above to construct the Geometric Series Formula!

Exercise 4.2.11. Trying Out the Geometric Series Formula ☕

Consider the summation $1 + 10 + 10^2 + 10^3 + 10^4 + 10^5$.

- Find the total by just doing the arithmetic. Evaluate the powers of ten and then add them up.

- Find the total by using the Geometric Series Formula. Verify that your answers match!

Exercise 4.2.12. Powers of Two ☕☕

Consider the summation $1 + 2 + 2^2 + 2^3 + 2^4 + 2^5$.

- Find the total by just doing the arithmetic. Evaluate the powers of two and then add them up.

- Find the total by using the Geometric Series Formula. Verify that your answers match!

- Use the Geometric Series Formula to evaluate

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^N.$$

- Write in words the answer to the following: "*A finite sum of consecutive powers of two, starting at one, is equal to...* "

Example 4.2.13. Difference of Two Quartics Formula

Here we show how the geometric series formula can be used to obtain a factorization formula! In particular, let us evaluate the summation

$$A^3 + A^2B + AB^2 + B^3.$$

We follow the little remark above, noting that the first term is $\left(A^3\right)$ and the common ratio is (B/A). We now evaluate the sum and then clean up the resulting compound fraction:

$$\begin{aligned} A^3 + A^2B + AB^2 + B^3 &= A^3 \frac{1-\left(\frac{B}{A}\right)^4}{1-\left(\frac{B}{A}\right)} \\ &= A^3 \frac{A - B^4/A^3}{A-B} \\ &= \frac{A^4 - B^4}{A-B}. \end{aligned}$$

Multiplying both sides by $A - B$, we have the difference of two quartics factorization as follows:

$$A^4 - B^4 = (A-B)\cdot\left(A^3 + A^2B + AB^2 + B^3\right).$$

Exercise 4.2.14. Difference of Two Cubes

- In the same manner, use the geometric series formula to build the more familiar difference of two cubes formula:

$$A^3 - B^3 = (A-B)\cdot\left(A^2 + AB + B^2\right).$$

- Again using the same technique, figure out a formula for factoring $A^5 - B^5$.

Exercise 4.2.15. Alternate Factorization of a Difference of Quartics

Notice that we could also factor a difference of two quartics by using the difference of two squares formula. In particular,

$$A^4 - B^4 = \left(A^2\right)^2 - \left(B^2\right)^2 = \left(A^2 - B^2\right)\left(A^2 + B^2\right).$$

Is this factorization compatible with the one we found via the geometric series formula? Explain.

Summary of Formulas

Just to have them all in one place, here they are again.

- Gauss's Formula:
$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$
- Sum of Consecutive Squares:
$$1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$$
- Sum of Consecutive Cubes:
$$1^3 + 2^3 + 3^3 + \cdots + N^3 = \left(\frac{N(N+1)}{2}\right)^2$$
- Finite Geometric Series:
$$1 + r + r^2 + r^3 + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r}$$

4.3 The Riemann Integral

Armed with better notation and some sum formulas, we can return and answer Exercise 4.0.0.2 our little parabola dartboard question!

Example 4.3.1. Revisiting Our Parabola! ☕☕☕

Let us take the summation from Exercise 4.1.0.6 and evaluate it using our summation properties and formulas as follows:

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{N}\left(\frac{n}{N}\right)^2 &= \frac{1}{N}\sum_{n=1}^{N}\frac{n^2}{N^2}\\ &= \frac{1}{N^3}\sum_{n=1}^{N} n^2\\ &= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6}\\ &= \frac{N(N+1)(2N+1)}{6N^3}\\ &= \frac{(N+1)(2N+1)}{6N^2}.\end{aligned}$$

We now have a closed formula, namely $\frac{(N+1)(2N+1)}{6N^2}$, that tells us the area under an N-rectangle approximation of the parabola on our dartboard.

Exercise 4.3.2. Checking the Numerics ☕

Verify that plugging $N = 5, 10, 100$, and 1000 into the formula $\frac{(N+1)(2N+1)}{6N^2}$ produces the same area values as we found in the table computed in Exercise 4.1.0.6!

The advantage of having this formula is of course not just having a cleaner way to compute values that we already found, but rather that we can take the limit as the number of rectangles goes to infinity!

Example 4.3.3. Taking the Limit

Since the area estimates get more and more accurate as we take more rectangles, we now take the limit as N goes to infinity to get the exact area under the curve. Proceeding, we have

$$\begin{aligned}\lim_{N\to\infty}\frac{(N+1)(2N+1)}{6N^2} &= \lim_{N\to\infty}\frac{2N^2+3N+1}{6N^2}\\ &= \frac{2}{6}\\ &= \frac{1}{3}.\end{aligned}$$

Thus, the area under the parabola is one-third. Another way to think of this is to say that one in three darts randomly thrown at that board should land under the parabola, and two out of three will land above it.

One note about the above example; we treated N as if it were a real number and $\frac{2N^2+3N+1}{6N^2}$ as if it were a function on the positive reals. Technically the expression only makes sense for whole numbers N since it is counting the number of rectangles being used in an approximation. However, if we just pretend it is a real number and take the limit, the sequence of values for natural number values N will have to converge to the same number that the real-valued reinterpretation of the function converges to. In the future, we will make this identification automatically and take the limit without worrying about it. This enables us to use all of our familiar tricks for finding limits of real-valued functions (like we did in the above example, where it was a rational function with tied degree in the numerator and denominator).

Definition of the Riemann Integral

We now take the process from the parabola dartboard example and generalize it to finding the area under any function. We call this process *integration*. There are many ways to do it, but this particular way of building an integral is closest to Bernhard Riemann's development and thus is named after him. Recall the steps involved in determining that parabolic area as one-third:

1. **Split the Region into Rectangles:** The width of the rectangles came from splitting the interval $[0, 1]$ into N equal parts. The height of the rectangles came from the height of the function $y = x^2$.

2. **Evaluate the Sum of Areas of N Rectangles:** We wrote the sum of areas in sigma notation and then used summation properties and relevant summation formulas to get a closed form of the summation. This sum of rectangles is often called a *Riemann sum*.

3. **Take the Limit as the Number of Rectangles Goes to Infinity:** The closed form was a function of N. Taking the limit as N approached infinity produced the exact area.

We replicate this same process but with a more generic interval $[a, b]$ (instead of just $[0, 1]$) and a more generic function $f(x)$ (instead of specifically $y = x^2$). This generic process will be the definition of the *Riemann Integral*, also sometimes called a *definite integral*.

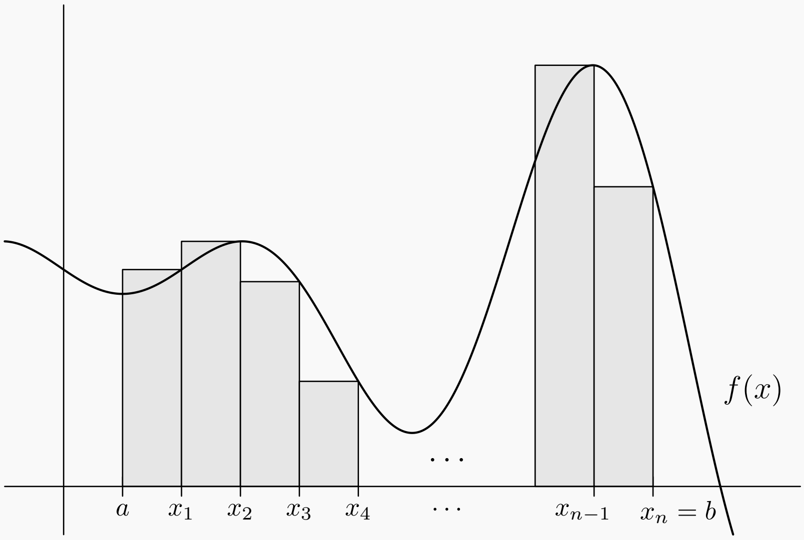
Definition 4.3.4. Integration via Riemann Sums

Let $f(x)$ be a continuous function on an interval $[a, b]$. Then the Riemann Integral of $f(x)$ on $[a, b]$, written $\int_a^b f(x)\,\mathtt{d}x$, is defined to be

$$\int_a^b f(x)\,\mathtt{d}x = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\,\Delta x$$

where

$$\Delta x = \frac{b-a}{N} \text{ and } x_n = a + n\Delta x.$$



Let us look through each symbol above and interpret them geometrically.

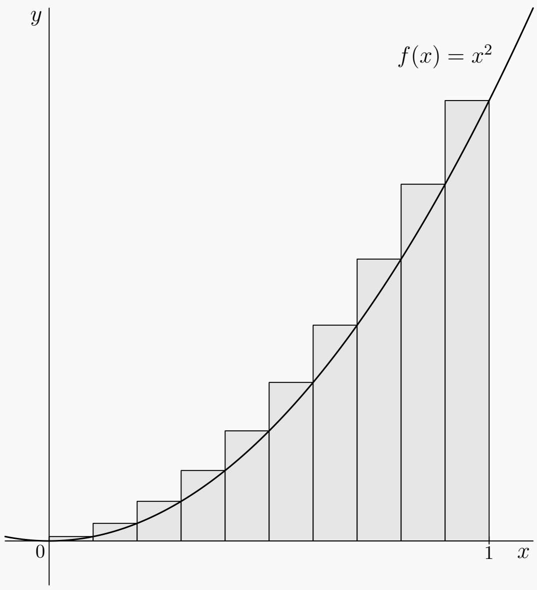
Exercise 4.3.5. Unpacking the Notation

- What length is measured by $b - a$?
- What length is measured by $\frac{b-a}{n}$?
- What is x_0 the same as?
- What is x_N the same as?
- Think of the expression $a + n\Delta x$ as the instruction set that says "start at a and then take n steps to the right of size Δx". What lengths then do the values $f(x_n)$ correspond to?

- What areas do the products $f(x_n)\,\Delta x$ correspond to?
- What area does the summation $\sum_{n=1}^{N} f(x_n)\,\Delta x$ correspond to?
- Why at the end do we take a limit as N goes to infinity?

Although we had already settled the question regarding the parabola dartboard, let us write our solution all in one place using the new notation.

Example 4.3.6. Quadrature of the Parabola via Riemann Sums



Here we compute the integral of $f(x) = x^2$ on the interval $[0, 1]$ via the Riemann sum definition. First we compute $\Delta x = \frac{1-0}{N} = \frac{1}{N}$. Second, we have that $x_n = 0 + n\Delta x = n\Delta x$. Lastly, we plug these pieces into the definition and evaluate everything! The area under the parabola is

$$
\begin{aligned}
\int_a^b f(x)\,\mathtt{d}x &= \lim_{N\to\infty} \sum_{n=1}^{N} f\left(x_n\right)\Delta x \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} x_n^2 \Delta x \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} (n\Delta x)^2\,\Delta x \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \left(n\frac{1}{N}\right)^2 \frac{1}{N} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{n^2}{N^3} \\
&= \lim_{N\to\infty} \frac{1}{N^3} \sum_{n=1}^{N} n^2 \\
&= \lim_{N\to\infty} \frac{1}{N^3} \frac{N(N+1)(2N+1)}{6} \\
&= \lim_{N\to\infty} \frac{2N^2+3N+1}{6N^2} \\
&= 1/3.
\end{aligned}
$$

Here are a few things to note about our definition of integral.

- **The assumption of continuity is stronger than is needed.** If $f(x)$ has only finitely many discontinuities, we can split the integral into several integrals and add them up. That is, suppose $f(x)$ is continuous on $[a, c)$ and $(c, b]$ but discontinuous at c. Then we can calculate

$$\int_a^b f(x)\,\mathtt{d}x = \int_a^c f(x)\,\mathtt{d}x + \int_c^b f(x)\,\mathtt{d}x.$$

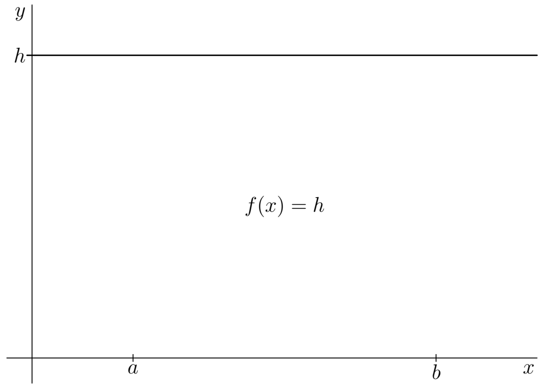
  However, if even one limit of the resulting Riemann sums diverges, we will say the whole integral does not exist.

- **We are restricting our definition to piecewise continuous functions.** Many horribly discontinuous functions (with infinitely many discontinuities) are still integrable, but it requires a more general notion of integration to handle such functions.

- **This integral technically gives "signed area" and not "area".** Usually we think of area as a positive quantity, in the intuitive sense of how much paint would be required to paint a region (which cannot be negative). Our definition of integral though can produce a negative answer, since $f\left(x_n\right)$ may be negative. Thus, when the graph of $f(x)$ is below the x-axis, the area is counted as negative. When it is above, the area is counted as positive.

Let us now try this framework out on some different regions! A good place to start out is a place where you already know the answer.

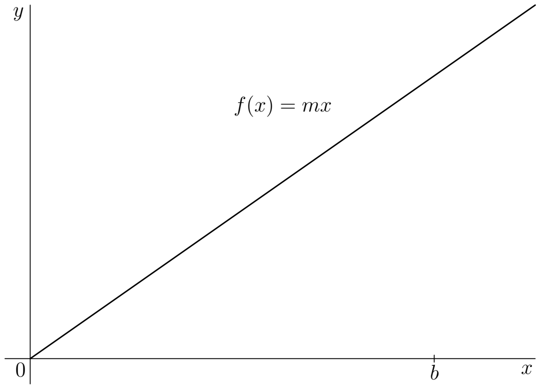
Exercise 4.3.7. Horizontal Lines! ☕☕



- Just using basic geometry, determine the area under the constant function $f(x) = h$ over the interval $[a, b]$.

- Use the Riemann sum definition of the integral to compute $\int_a^b h \, \mathrm{d}x$. Verify that your answers match.

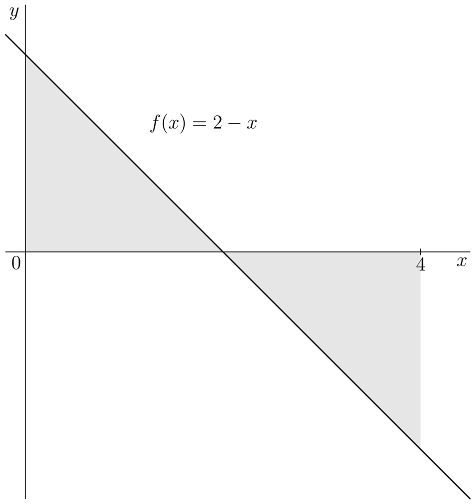
Exercise 4.3.8. Lines! ☕☕



- Just using basic geometry, determine the area under the linear function $f(x) = mx$ over the interval $[0, b]$.

- Use the Riemann sum definition of the integral to compute $\int_a^b mx \, \mathtt{d}x$. Verify that your answers match.

Exercise 4.3.9. Yet Another Line ☕☕

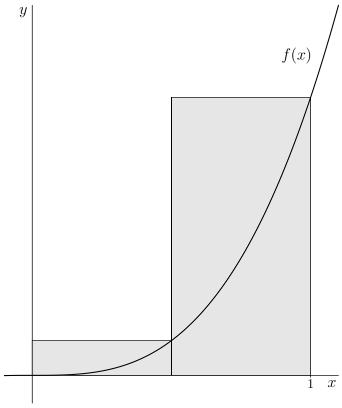


- Just using basic geometry, determine the signed area of $f(x) = 2 - x$ over the interval $[0, 4]$. Remember that when the function dips below the x-axis, that counts as negative area!

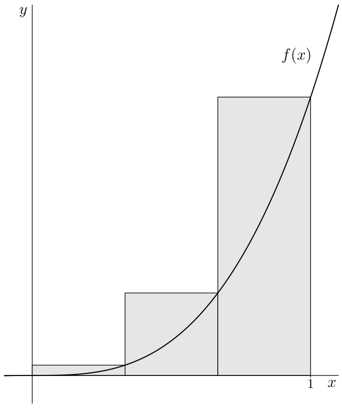
- Use the Riemann sum definition of the integral to compute $\int_0^4 2 - x \, \mathtt{d}x$. Verify that your answers match.

Exercise 4.3.10. Quadrature of a Cubic

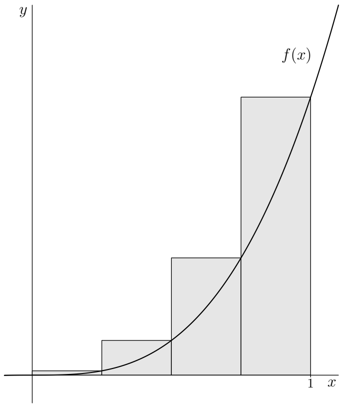
- Estimate the area under the curve $f(x) = x^3$ on the interval $[0, 1]$ using two rectangles.



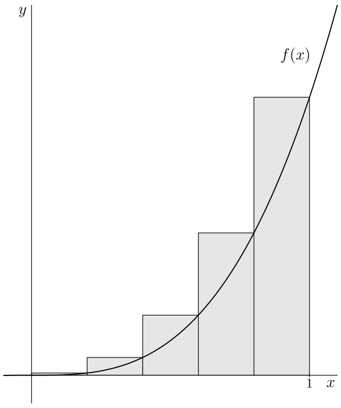
- Estimate the area under the curve $f(x) = x^3$ on the interval $[0, 1]$ using three rectangles.



- Estimate the area under the curve $f(x) = x^3$ on the interval $[0, 1]$ using four rectangles.

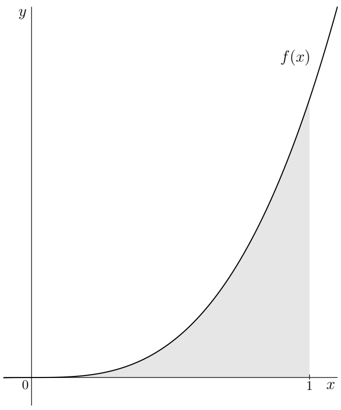


- Estimate the area under the curve $f(x) = x^3$ on the interval $[0, 1]$ using five rectangles.



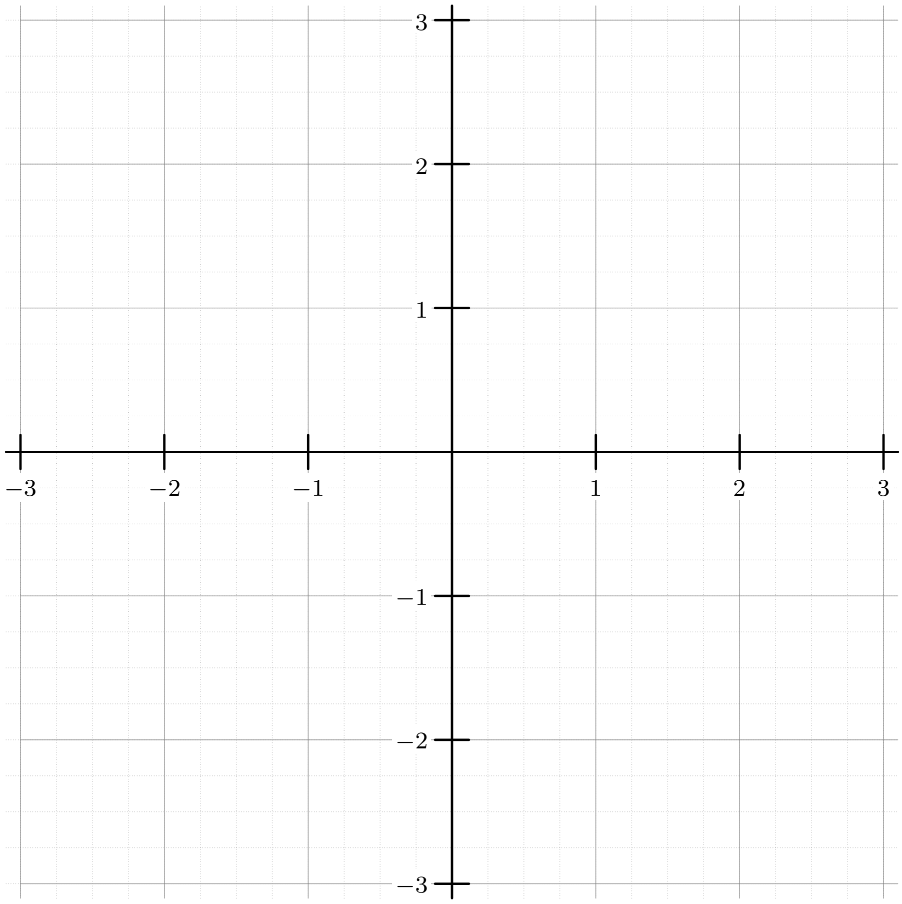
- Are the approximations above too big or too small? What might you guess for the true area?

- Evaluate the integral
$$\int_{x=0}^{x=1} x^3 \, \mathtt{d}x$$
via a limit of Riemann sums. Does it confirm your approximation above?



Exercise 4.3.11. A Bigger Polynomial ☕☕☕

- Carefully graph the function $f(x) = x^3 - x^2 - 2x + 2$. Include labels for all roots and relative extrema.



- Estimate the area under $f(x)$ between the first and second roots using basic geometry.

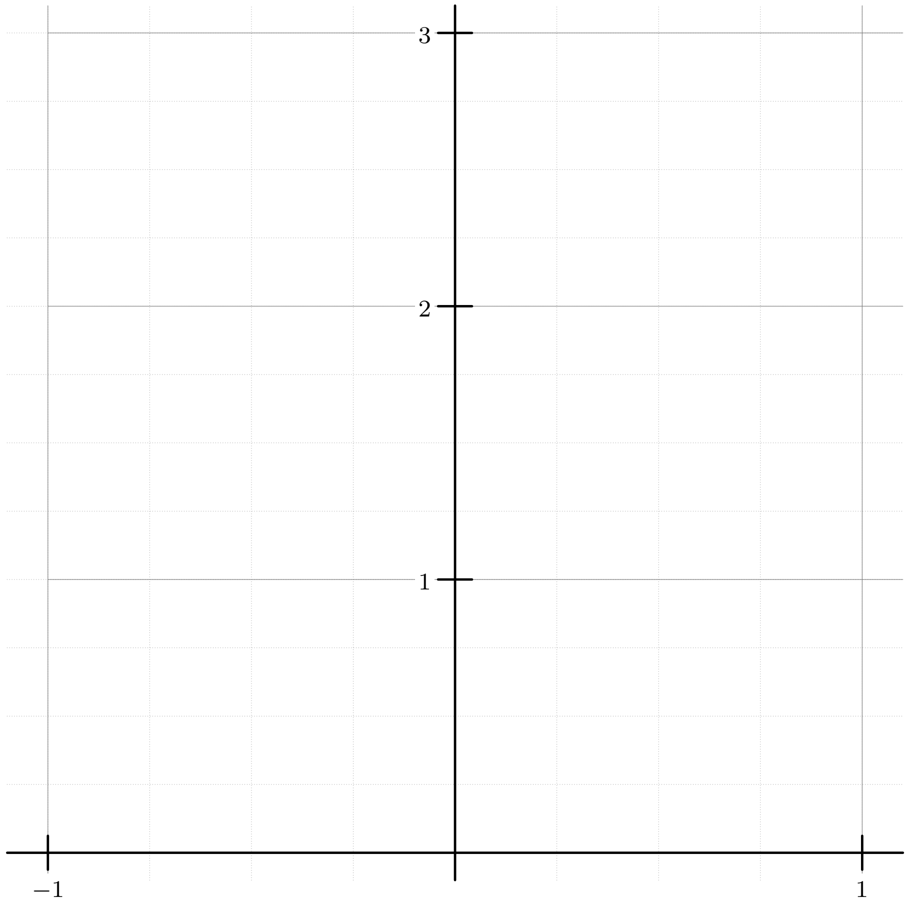
- Find the exact area of that region using the Riemann sum definition of an integral.

- Estimate the area under $f(x)$ between the second and third roots using basic geometry.

- Find the exact area of that region using the Riemann sum definition of an integral.

Exercise 4.3.12. An Exponential ☕☕☕

- Carefully graph the function $f(x) = e^x$ on the interval $[0, 1]$.



- Estimate the area under $f(x)$ between 0 and 1 using basic geometry.

- Set up the Riemann sum for that region. Use the geometric series formula to simplify the summation to

$$\sum_{n=1}^{N} f\left(x_n\right) \Delta x = \frac{e^{1/N}(e-1)}{N\left(e^{1/N}-1\right)}.$$

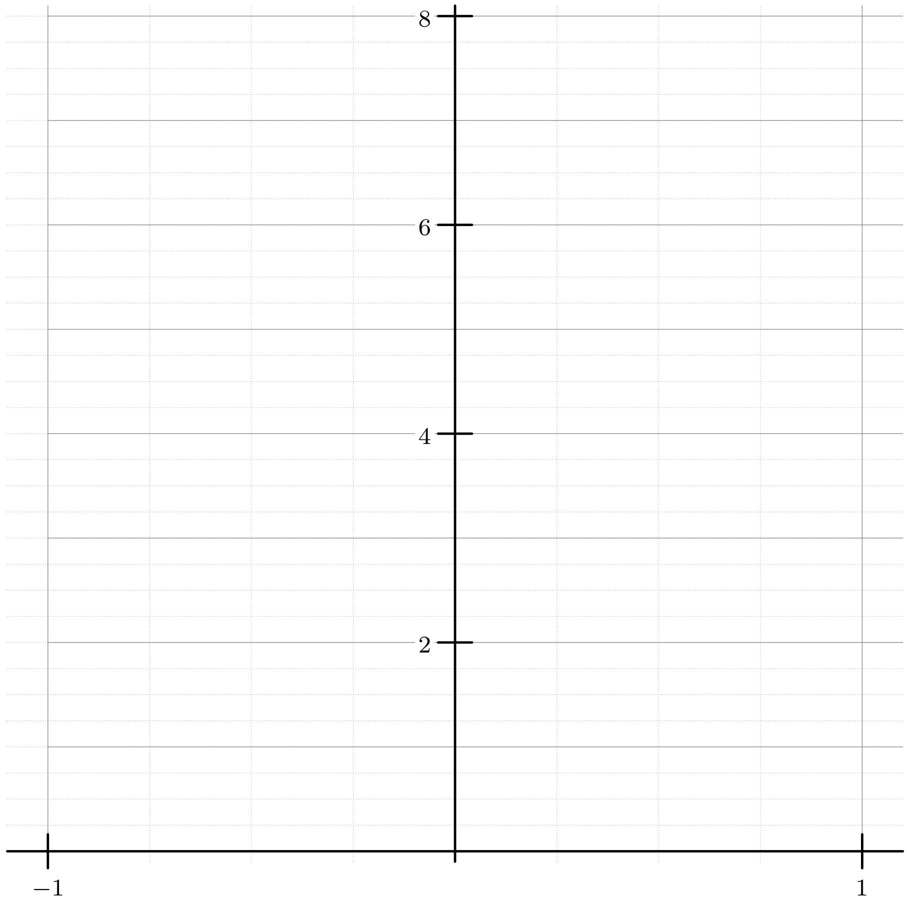
Show the details of this simplification below.

- To evaluate the limit as N goes to infinity, make the substitution $t = 1/N$ and apply the Special Limit for e from Subsection 2.1.

The next problem is exponentially harder than the previous, if your exponential has base 1.

Exercise 4.3.13. Another Exponential ☕☕☕

- Graph the function $f(x) = e^{2x}$ on the interval $[0, 1]$. How should the area under the curve compare to what you computed in the previous exercise?



- Calculate the integral $\int_0^1 e^{2x}\,\mathtt{d}x$ using the Riemann sum definition of integrals. Verify your prediction from the previous part.

Next, we cook up a function that is discontinuous everywhere. This example will explain why our definition of integral was restricted to continuous functions! In later courses, you will see much more general definitions of integral (like the Lebesgue Integral) that can handle integrating functions like the crazy one below.

Exercise 4.3.14. The Devil's Comb ☕☕☕

Recall the definition of *rational numbers* from Chapter 0. We now define a harmless little piecewise function as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

- Compute $f(1/2)$. Compute $f\left(\sqrt{2}/2\right)$.

- If you tried to apply our definition of integral to compute $\int_0^1 f(x)\,\mathtt{d}x$, what do you get?

- The vast majority of real numbers are not rational. One may think of this intuitively as follows: rational numbers are fractions and thus have repeating decimal expansions. If you selected a decimal expansion at random (digit by digit), there is probability zero that it would come out to be repeating. Why does this make the calculation above seem wrong?

4.4 Properties of the Riemann Integral

Just as when we defined limits or derivatives, it is worth going through the new object (integrals) and seeing what properties it satisfies!

Exercise 4.4.1. Prove or Give a Counterexample ☕☕☕

For each of the following properties, use the Riemann sum definition of integrals to verify the property is true. Or, if the property is false, find a counterexample.
Assume $f(x)$ and $g(x)$ are continuous on all domains being considered below. Let a, b, and c be real constants.

- $\int_a^b f(x) + g(x)\,\mathtt{d}x = \int_a^b f(x)\,\mathtt{d}x + \int_a^b g(x)\,\mathtt{d}x$

- $\int_a^b f(x) \cdot g(x)\,\mathtt{d}x = \int_a^b f(x)\,\mathtt{d}x \cdot \int_a^b g(x)\,\mathtt{d}x$

- $\int_a^b c \cdot f(x)\,\mathtt{d}x = c \cdot \int_a^b f(x)\,\mathtt{d}x$

- $\int_a^b f(x)\,\mathtt{d}x = -\int_b^a f(x)\,\mathtt{d}x$

- $\int_a^a f(x)\,\mathtt{d}x = 0$

The value of these properties lies in expressing more complicated integrals in terms of simpler integrals. Often a nastier integral can be broken up into smaller pieces which might be easier to handle.

Exercise 4.4.2. Not Reinventing the Wheel ☕☕

Evaluate the integral

$$\int_1^0 5e^x - x^2 \, \mathtt{d}x$$

without setting up a Riemann sum and doing everything from scratch! Rather, use our work from the previous section and the properties listed above. Indicate clearly which property you use where.

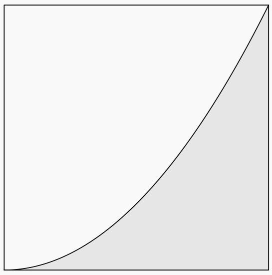
4.5 Numerical Methods

Often an integral is just too hard to actually evaluate. In these cases, it can be desirable to seek a numerical approximation of the integral even when obtaining an exact answer is intractable. One way of course is to just take a Riemann sum using lots of rectangles, as we saw earlier in this chapter. One can fancy up these approximations a bit by allowing the tops of your Riemann sum boxes to be sloped lines rather than horizontal (creating trapezoids) or allowing the tops to be parabolas (a method of numerical integration called Simpson's Rule). Here, we focus on a more modern method known as Monte Carlo Integration.

Example 4.5.1. Darts and Probability

Suppose we throw a dart at the dartboard shown below, adorned with the graph of $y = x^2$. Our aim is good enough to guarantee it lands on the board, but besides that it is equally likely to land anywhere. The natural question of course is...

What is the probability that the dart lands below the parabola?

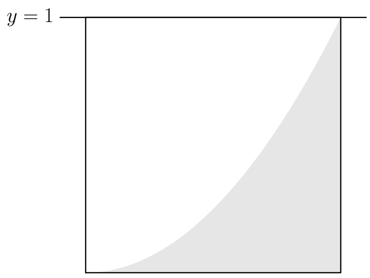


A more precise way to state this is as follows: "Let x and y both be real numbers randomly selected from the interval $[0, 1]$. What is the probability that $y < x^2$?"

Exercise 4.5.2. If a Problem is Too Hard...

...throw it away and solve an easier one!

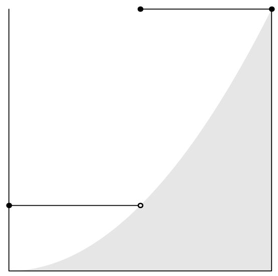
- Suppose instead of the parabola $y = x^2$, it was instead the line $y = 1$. Then what would the probability be that the dart lands below the line?



- Suppose instead of that one horizontal line, it was a piecewise function composed of a few horizontal lines, specifically:

$$y(x) = \begin{cases} 1/4 & \text{if } 0 \le x < 1/2 \\ 1 & \text{if } 1/2 \le x \le 1. \end{cases}$$

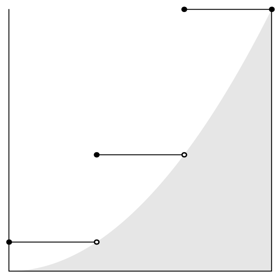
Then what would the probability be that the dart lands below the piecewise function?



- Suppose instead of that one horizontal line, it was a piecewise function composed of three horizontal lines, specifically:

$$y(x) = \begin{cases} 1/9 & \text{if } 0 \le x < 1/3 \\ 4/9 & \text{if } 1/3 \le x < 2/3. \\ 1 & \text{if } 2/3 \le x \le 1. \end{cases}$$

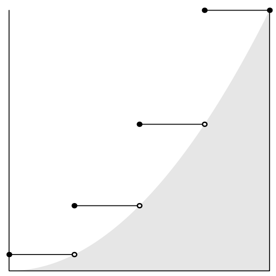
Then what would the probability be that the dart lands below the piecewise function?



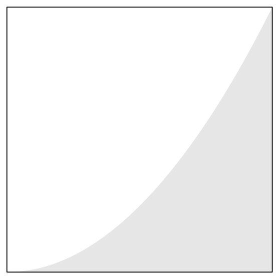
- Suppose instead of that one horizontal line, it was a piecewise function composed of four horizontal lines, specifically:

$$y(x) = \begin{cases} 1/16 & \text{if } 0 \le x < 1/4 \\ 1/4 & \text{if } 1/4 \le x < 1/2. \\ 9/16 & \text{if } 1/2 \le x < 3/4. \\ 1 & \text{if } 3/4 \le x \le 1. \end{cases}$$

Then what would the probability be that the dart lands below the piecewise function?



- Perform the analogous construction for five horizontals instead of four. Show the drawing below.



- How do the probabilities we are computing above relate to the original question with the parabola? Say we wanted the probability accurate to a tenth of a percent. Do you think we could take sufficiently many rectangles to obtain this desired accuracy?

Monte Carlo Integration

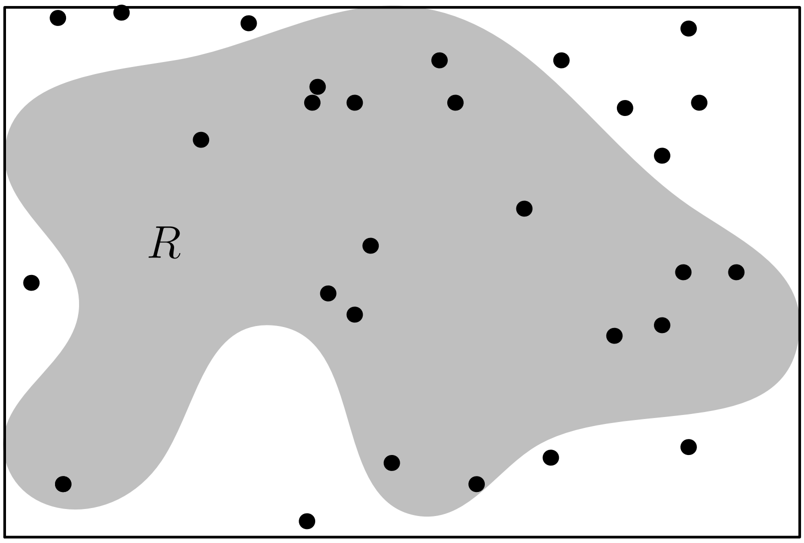
Our dartboard problem back in Exercise 4.0.0.2 is exactly the idea of Monte Carlo Integration! To approximate the area of a region R, perform the following steps:

- **Choose a rectangle B that contains R.** It isn't too important what rectangle B you choose, just that it contains the entirety of R. This B will serve as our rectangular dartboard.
- **Throw a large number of darts at the dartboard B.** To throw a dart, just select a random x coordinate from the width of B and a random y coordinate from the height of B. The point (x, y) you select is where the dart lands. What "large number" means depends entirely on how accurate of an approximation you want.
- **Observe that the proportion of darts that lands inside R should be proportional to the areas!** At last, the key idea is the following proportion:

  $$\frac{\text{Number of darts that land inside } R}{\text{Total number of darts thrown}} = \frac{\text{Area of } R}{\text{Area of } B}.$$

  The area of B is easy to calculate (since it is a rectangle). The ratio on the left-hand side will be calculated from examining our list of randomly chosen values. Thus, the area of R will be the only unknown quantity, so we can solve for it.

- **Calculate the area of R as the proportion of darts that landed inside times the area of B.** Bold sentence said it all.



Again, let us first try this out in a context where we already know the answer!

Exercise 4.5.3. The Parabola

Consider yet again our parabola $y = x^2$ on the region $[0, 1]$. Pick somewhat random numbers for yourself by flipping a coin four times, recording a digit of 1 for heads and 0 for tails to create a binary decimal string (always leading with a zero as the units digit). For example, if we flipped the sequence

Tails, Heads, Heads, Tails

we would build the number

$$0.0110$$

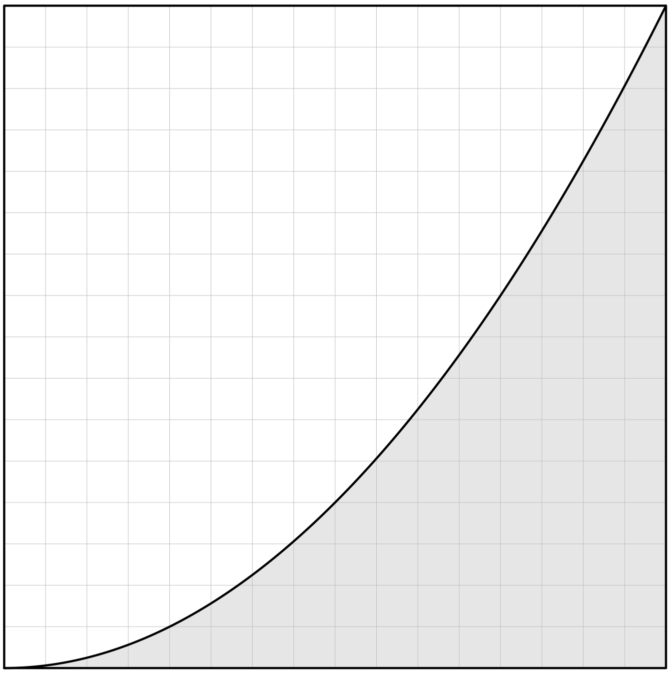
which in binary is

$$0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + 0 \cdot \frac{1}{16} = \frac{3}{8}.$$

This of course is not a totally random real number, because we're really only selecting from rational numbers with denominator sixteen. But, one imagines you could make this as close to random as desired by doing more flips and getting longer decimal sequences.

- In this manner, select twelve random numbers and call them $x_1, y_1, x_2, y_2, \ldots, x_6, y_6$. List your numbers below.

- For each of $i \in \{1, 2, 3, 4, 5, 6\}$, decide if the point (x_i, y_i) is below the parabola. Specifically, if $x_i^2 > y_i$, then the dart landed under the parabola. Graph your six points in the box below.



- Use the proportion of darts under the curve to estimate the area. How does it compare to the true area?

Exercise 4.5.4. The Parabola Again

Consider yet again our parabola $y = x^2$ but now on the interval $[0, 2]$. Choose B to be the rectangle with lower-left corner at $(0, 0)$ and upper-right corner at $(2, 4)$. Repeat the above exercise but instead pick your random x-values by flipping a coin five times, recording a digit of 1 for heads and 0 for tails to create a binary decimal string starting with the units digit. For example, if we flipped the sequence

Heads, Tails, Heads, Heads, Tails

we would build the number

$$1.0110$$

which in binary is

$$1 + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + 0 \cdot \frac{1}{16} = 1 + \frac{3}{8}.$$

Pick your random y values by doing similarly but using six flips (giving yourself two numbers in front of the decimal). For example, if we flipped the sequence

Heads, Heads, Tails, Heads, Heads, Tails we would build the number

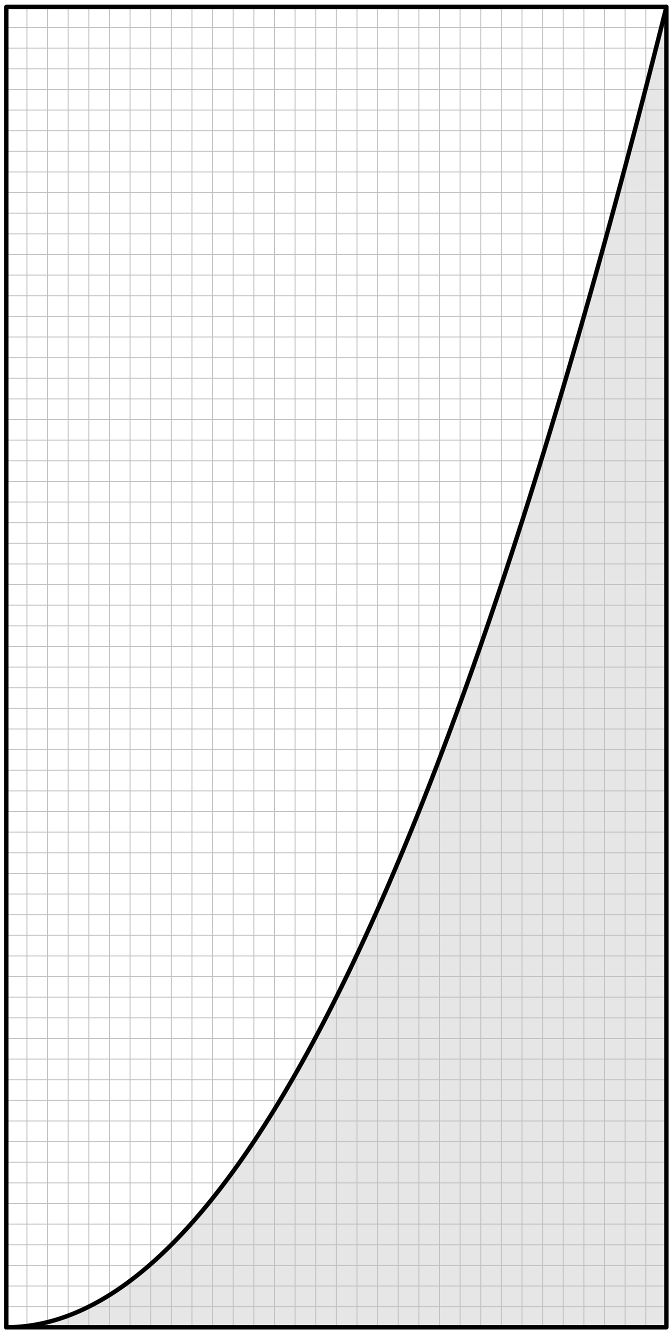
$$11.0110$$

which in binary is

$$2+1+0\cdot\frac{1}{2}+1\cdot\frac{1}{4}+1\cdot\frac{1}{8}+0\cdot\frac{1}{16}=3+\frac{3}{8}.$$

- In this manner, select six x values and six y values. Call them $x_1, y_1, x_2, y_2, \ldots, x_6, y_6$. List your numbers below.

- For each of $i \in \{1,2,3,4,5,6\}$, decide if the point (x_i, y_i) is below the parabola. Specifically, if $x_i^2 > y_i$, then the dart landed under the parabola. Graph your six points in the box below.



- Use the proportion of darts under the curve to estimate the area. How does it compare to the true area?

Unfortunately, our integrals we developed so far did not lead to any summation formula which will allow us to evaluate the integral

$$I = \int_0^1 \sqrt{1-x^2}\,\mathtt{d}x.$$

This is most unfortunate, as this means we cannot use Riemann sums to calculate the area of a circle! However, we can approximate with Monte Carlo methods.

Exercise 4.5.5. Area of a Circle

- Explain why the function $f(x) = \sqrt{1-x^2}$ is in fact the upper boundary of a unit circle as claimed above.
- Explain why four times the integral I should equal π.
- Use Monte Carlo integration to estimate the integral I, thus creating an estimate for the number π (after multiplication by four)! What do you get? Explain what your box B was and how you chose your random numbers.

Note that one big advantage of a Monte Carlo method like what we describe above is that it allows for parallel processing to get more accuracy. If twenty people all compute a Riemann sum approximation with ten rectangles, they all just get exactly the same answer and nothing more has been accomplished than would have been if just one person had performed the approximation. However, if twenty people all throw ten darts, then we can merge their data into one data set and get a two-hundred dart Monte Carlo approximation, which will likely provide a much more accurate answer than any one individual had obtained!

4.6 Chapter Summary

Let $f(x)$ be a continuous function on an interval $[a, b]$. The **Riemann integral** of $f(x)$ on $[a, b]$ is defined as

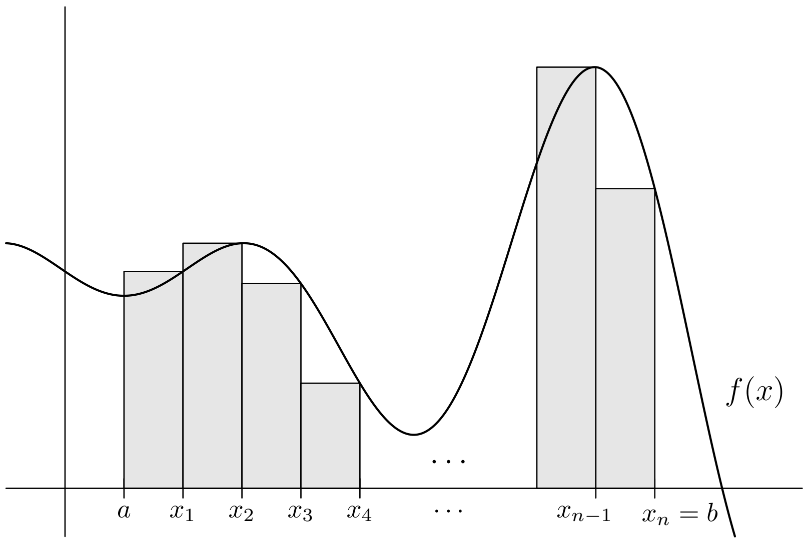
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\,\Delta x$$

where

$$\Delta x = \frac{b-a}{N}$$

and

$$x_n = a + n\Delta x.$$



This calculation returns the **signed area under the curve**. It can be approximated by taking just a finite N, or the exact value can be calculated by taking the limit. In order to take the limit, typically one first needs to evaluate the summation. The summation formulas below are often useful in calculating Riemann Sums.

- **Gauss's Formula.** $\sum_{n=1}^{N} n = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$
- **Sum of Consecutive Squares.** $\sum_{n=1}^{N} n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$
- **Sum of Consecutive Cubes.** $\sum_{n=1}^{N} n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = \left(\frac{N(N+1)}{2}\right)^2$
- **Finite Geometric Series.** For a real number $r \neq 1$, $\sum_{n=1}^{N} r^n = 1 + r + r^2 + r^3 + \cdots + r^N = \frac{r^{N+1}-1}{r-1}$

Besides just taking finitely many rectangles, the Riemann integral can also be approximated probabilistically via a **Monte Carlo method**; choose any box that contains the region of interest. Select random ordered pairs in this box, and multiply the area of the box by the percent of ordered pairs that lie under the curve.

4.7 Mixed Practice

Exercise 4.7.1. ☕

What situations could cause an integral to be zero?

Exercise 4.7.2. ☕☕

True or false!

- Integrals distribute over addition.
- Integrals distribute over multiplication.
- If the integrand is multiplied by a constant, the constant can be pulled out of the integral.

Exercise 4.7.3. ☕☕☕

Let us use a Riemann sum to approximate π. This is admittedly a bit silly, but, well. . . cool!

- Graph the equation $x^2 + y^2 = 1$. Explain why the area inside is equal to π, and as a result the area of the top half is $\pi/2$.
- Solve the equation for y to obtain a function $f(x)$ for the top half of the graph.
- Explain why $\int_{x=-1}^{x=1} f(x)\,\mathtt{d}x = \pi/2$ and thus $\pi = 2\int_{x=-1}^{x=1} f(x)\,\mathtt{d}x$.
- Use a four-rectangle Riemann sum of the integral to approximate π. How close is the approximation?
- Use an eight-rectangle Riemann sum of the integral to get a better approximation of π. How close is the approximation?
- Use a twenty-rectangle Riemann sum of the integral to get an even better approximation of π. How close is it now? (Feel free to use a computer algebra system or spreadsheet to fast-forward the tediousness of evaluating that many rectangles by hand.)

Exercise 4.7.4. ☕☕☕

- Sketch the graph of the function $f(x) = 3x^2$ on the interval $[1, 2]$.
- Find the exact area under the curve using the definition of the integral as a limit of Riemann sums. Show all work.

Exercise 4.7.5. ☕☕☕

Use Riemann sums to calculate the integral $\int_{x=0}^{x=1} x^2 - x\,\mathtt{d}x$. Why is the result negative?

Exercise 4.7.6. ☕☕☕

Use basic geometry to calculate the integral $\int_{x=1}^{x=2} 8 - x\,\mathtt{d}x$. Then, calculate the same integral using Riemann Sums. Confirm your answers match!

Exercise 4.7.7. ☕☕☕

You choose 1000 ordered pairs (x, y) where x and y are both randomly chosen numbers between 0 and 1. How many of those ordered pairs would you expect to satisfy $y > x^3$?

Chapter 5

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is typically split into two parts. To state them briefly in words, one would say:

- **Fundamental Theorem of Calculus Part I:** A derivative will cancel an integral.
- **Fundamental Theorem of Calculus Part II:** An integral will cancel a derivative.

Roughly speaking, the Fundamental Theorem of Calculus is the fact that derivatives and integrals are inverse operations. They cancel each other in either order of application. We state these more formally in the sections that follow.

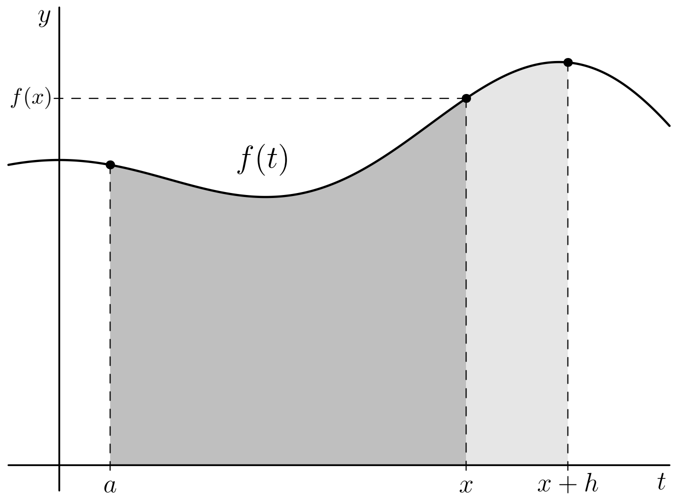
5.1 Fundamental Theorem of Calculus Part I

Theorem 5.1.1. FTC Part I

Let $f(x)$ be a continuous function on the interval $[a, b]$. Let x be any value in (a, b). Then

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\int_a^x f(t)\,\mathtt{d}t\right) = f(x).$$

Intuitively, all this is saying is that at x, the instantaneous rate of change of the area under $f(t)$ should be given by the height of the function $f(x)$ itself, since that is the little "stick" of area that gets added as we adjust x by a tiny amount. Writing up a careful proof requires quite a bit more care, of course.



Exercise 5.1.2. Proof of FTC Part I

Fill in the blanks in the following proof!

Proof. Here we "simply" evaluate the left-hand side using the limit definition of the derivative and the Riemann sum definition of the integral, and then keep bashing until we reach the right-hand side. Deep breath. Ready?

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\int_a^x f(t)\,\mathtt{d}t\right) = \lim_{h\to 0}\frac{\rule{6cm}{0.4pt}}{h}$$
$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,\mathtt{d}t}{h}.$$

We now apply the Riemann sum definition of the integral to the numerator. Specifically, we have

$$\int_x^{x+h} f(t)\,\mathtt{d}t = \lim_{N\to\infty}\sum_{n=1}^{N} f(t_n)\,\Delta t$$

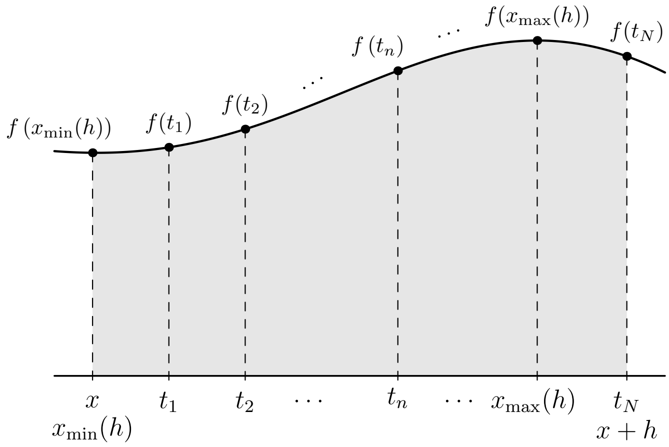
where

$$\Delta t = \frac{\rule{3cm}{0.4pt}}{N} = \frac{h}{N} \text{ and } t_n = x + n\Delta t.$$

This is fortunate, since the h in the denominator of the limit will cancel with the h in the numerator of Δt. Proceeding, we have

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\int_a^x f(t)\,\mathtt{d}t\right) = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,\mathtt{d}t}{h}$$
$$= \lim_{h\to 0}\frac{\lim_{N\to\infty}\sum_{n=1}^{N} f(t_n)\,\Delta t}{h}$$
$$= \lim_{h\to 0}\frac{\lim_{N\to\infty}\sum_{n=1}^{N} f\left(x+\frac{nh}{N}\right)\frac{h}{N}}{h}$$
$$= \lim_{h\to 0}\lim_{N\to\infty}\sum_{n=1}^{N} f\left(x+\frac{nh}{N}\right)\frac{1}{N}.$$

By EVT, f attains ____________________ on the interval $[x, x+h]$. For a given h, call the locations where these occur by $x_{\max}(h)$ and $x_{\min}(h)$, respectively.



Meanwhile, notice that $\sum_{n=1}^{N} f\left(x+\frac{nh}{N}\right)\frac{1}{N}$ is just the average of N values of f with inputs selected from the interval $[x, x+h]$. Thus,

$$f\left(x_{\min}(h)\right) \leq \sum_{n=1}^{N} f\left(x+\frac{nh}{N}\right)\frac{1}{N} \leq f\left(x_{\max}(h)\right),$$

since the average value of any set of values cannot be greater than the maximum value or less than the minimum value. (This statement is often known as the *Pigeonhole Principle* and is one of the most surprising, beautiful, and powerful results in all of mathematics!) If we take the limit of all three sides of the inequality with respect to _______, we get

$$\lim_{N\to\infty} f\left(x_{\min}(h)\right) \leq \lim_{N\to\infty} \sum_{n=1}^{N} f\left(x+\frac{nh}{N}\right)\frac{1}{N} \leq \lim_{N\to\infty} f\left(x_{\max}(h)\right).$$

Notice that the limit in the middle converges and is in fact equal to $\frac{\int_x^{x+h} f(t)\,\mathrm{d}t}{h}$, by the computation above. The functions $f\left(x_{\min}(h)\right)$ and $f\left(x_{\max}(h)\right)$ vary with h but are constant with respect to N, and thus we can discard the limits. Therefore, our inequality becomes

$$f\left(x_{\min}(h)\right) \leq \lim_{N\to\infty} \sum_{n=1}^{N} f\left(x+\frac{nh}{N}\right)\frac{1}{N} \leq f\left(x_{\max}(h)\right).$$

Since we have proven the middle expression is a number, we can think of it as a real-valued function of h. We proceed to use the Sandwich Theorem to show that it goes to $f(x)$ as h approaches 0. Proceeding, we have that

$$\lim_{h\to 0} f\left(x_{\min}(h)\right) = f(x)$$

and

$$\lim_{h\to 0} f\left(x_{\max}(h)\right) = f(x),$$

because h represents the width of the interval, and the max and min values can be forced to be arbitrarily close to $f(x)$ itself by shrinking the interval sufficiently small. (Note this is essentially just a slight restatement of the definition of continuity, choosing $\delta = h$.) Since the upper and lower

bounds both converge to $f(x)$ as h goes to 0, we can apply the Sandwich Theorem and conclude that $\lim_{N\to\infty}\sum_{n=1}^{N} f\left(x+\frac{nh}{N}\right)\frac{1}{N}$ goes to ________ as h approaches 0 as well.
Thus,

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\int_a^x f(t)\,\mathtt{d}t\right) = f(x)$$

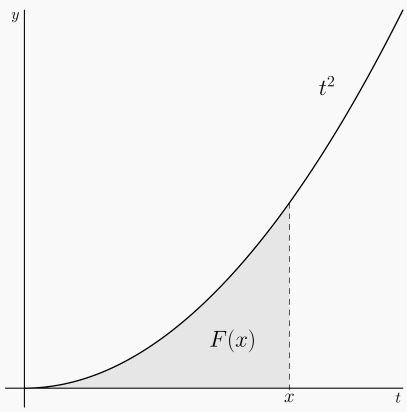
as desired. □

Example 5.1.3. Quadrature of a Parabola Yet Again

Let the function $f(x) = x^2$, our good old parabola once again. Let

$$F(x) = \int_0^x t^2\,\mathtt{d}t,$$

the area under the parabola from 0 to x.



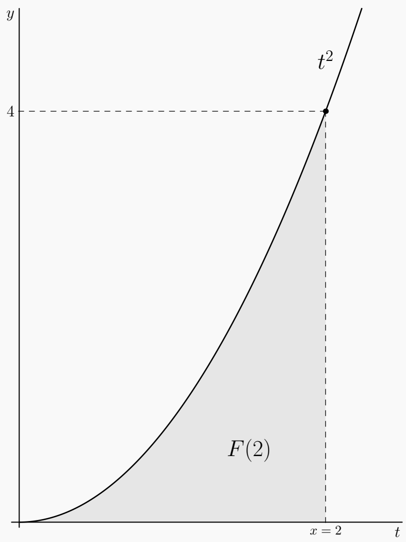
To see FTC I in action a bit, let's look at the average rate of change vs the instantaneous rate of change at $x = 2$. FTC I tells us that at $x = 2$, the instantaneous rate of change of the area under the curve should be equal to the height of the function, namely 4. To say the same thing more mathematically, we have

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\int_0^x t^2\,\mathtt{d}t\right) = x^2,$$

or that

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(F(x)\right) = x^2.$$

Evaluating this at $x = 2$, we have $F'(2) = 4$.



Let us find a few values of the function $F(x)$ and see if we can observe this instantaneous rate of change numerically, approximating it as an average rate of change. Computing some areas under the curve (just with our good old Riemann sum machinery), we have

x	2	2.1
$F(x)$	$2.\overline{6}$	3.087

Thus, the average rate of change is

$$\frac{F(2.1) - F(2)}{2.1 - 2} = \frac{3.087 - 2.\overline{6}}{0.1} = \frac{0.421\overline{6}}{0.1} = 4.216\ldots$$

which indeed is close to the instantaneous rate of change of 4.

Exercise 5.1.4. Filling in Details ☕

Fill in the details of the calculations of $F(2) = 2.\overline{6}$ and $F(2.1) = 3.087$ using Riemann sums. Show your work below!

Exercise 5.1.5. A Cubic!

Repeat the above example but with a cubic. Specifically, do the following:

- Let $F(x) = \int_0^x t^3 \, \mathrm{d}t$, the area under a cubic from 0 to x. Use FTC I to find the instantaneous rate of change at $x = 2$.

- Verify this instantaneous rate of change by comparing to an average rate of change. Specifically, use Riemann sums to compute the values of $F(2)$ and $F(2.1)$, and then compute an average rate of change based on those values.

Exercise 5.1.6. Alternate Definition of Natural Log ☕☕

Often the natural log function is defined in the following manner:

For a positive real number x, the quantity $\ln(x)$ is defined to be the area under the curve $1/t$ from 1 to t.

- Write the same definition using an integral instead of words.

- This definition illuminates a lot of the properties we associate with logarithms. Use the integral definition to explain (with a picture showing area under the curve $1/t$ as well as words) the following properties of the natural logarithm:
  - The x intercept: $\ln(1) = 0$

  - Negativity: $\ln(x) < 0$ for $x \in (0, 1)$

  - Positivity: $\ln(x) > 0$ for $x \in (1, \infty)$

  - Increasing: $\ln(x)$ is increasing on its entire domain

- Use FTC I to calculate $(\ln(x))'$. Does it agree with what we computed in Subsection 2.3?

5.2 Fundamental Theorem of Calculus Part II

Part I showed that taking the derivative of an integral will cancel out the integral, leaving just the integrand. Part II shows essentially the same thing but in the reverse order, that taking an integral of a derivative will hand you back the original function.

Theorem 5.2.1. FTC Part II

Let $F(x)$ be a differentiable function on the interval (a, b). Then

$$\int_a^b F'(x)\,\mathtt{d}x = F(b) - F(a).$$

The trick in proving this theorem is to look at the integral as a function in b, the right-hand endpoint. To emphasize the fact that we will think of b as the independent variable, we will replace the x in the integrand with t (just as a placeholder variable) and replace b with x for the course of the proof. That is, we will prove the statement $\int_a^x F'(t)\,\mathtt{d}t = F(x) - F(a)$ instead.

Exercise 5.2.2. Proof of FTC Part II

Fill in the blanks in the following proof!

Proof. This proof is just an application of Theorem 2.6.10. We build our two functions, the first being

$$F(x)$$

and the second being

$$\int_a^x F'(t)\,\mathtt{d}t.$$

The derivative of the first is just $F'(x)$. The derivative of the second is just $F'(x)$ by ______________________________. Since the two functions have the same derivative, we know they differ by a _______________. In particular,

$$\int_a^x F'(t)\,\mathtt{d}t = F(x) + C$$

for some constant C. Fortunately, C is independent of x, so we can plug in any value at all of x to solve for C. We strategically choose $x =$ ____ to make the left-hand side become 0. Plugging this value in, we have

$$0 = \underline{\qquad\qquad}.$$

Solving for C produces $C =$ ____. Thus,

$$\int_a^x F'(t)\,\mathtt{d}t = F(x) - F(a)$$

as desired. □

Exercise 5.2.3. Verifying FTC Part II in the Linear Case ☕☕

Let $F(x) = mx + c$ where m and c are real constants. (Apologies for the odd choice of c rather than b for the y intercept. We do this to avoid a collision with b in the interval $[a, b]$ below.) Let's run this function through the theorem and see what happens!

- Compute $F'(x)$.

- Compute
$$\int_a^b F'(x)\,\mathtt{d}x$$
using basic geometry. Draw a graph to support your answer, showing the region whose area

is being computed.

- Compute
$$\int_a^b F'(x)\,\mathtt{d}x$$
using FTC II. Verify your answers match!

Exercise 5.2.4. Verifying FTC Part II in the Quadratic Case ☕☕

Let $F(x) = a_0 + a_1x + a_2x^2$ where the a_i are real constants. Let's run this function through the theorem and see what happens!

- Compute $F'(x)$.

- Compute
$$\int_a^b F'(x)\,\mathtt{d}x$$
using basic geometry. Draw a graph to support your answer, showing the region whose area is being computed.

- Compute
$$\int_a^b F'(x)\,\mathtt{d}x$$

using FTC II. Verify your answers match!

Notice that FTC Part II actually gives us an alternative to find an area under a curve that can completely circumvent Riemann sums! To calculate

$$\int_a^b f(x)\,\mathtt{d}x,$$

simply do the following:

- **Find an Antiderivative:** Find a function $F(x)$ such that $F'(x) = f(x)$. Such a function F is called an *antiderivative* of f.
- **Apply FTC Part II:** Calculate the integral as

$$\int_a^b f(x)\,\mathtt{d}x = \int_a^b F'(x)\,\mathtt{d}x = F(b) - F(a).$$

Though this seems much easier than computing a Riemann sum, it turns out that finding an antiderivative is actually a very nontrivial task! We dedicate the next section (along with probably the first month of your Calculus II course) to exactly this task. In the following example, we just provide an antiderivative (without any explanation of where it came from) for sake of seeing how FTC Part II is used.

Exercise 5.2.5. Area Under the Natural Log ☕☕

Let the function $f(x) = \ln(x)$.

- Verify that $F(x) = x\ln(x) - x$ is an antiderivative for $f(x)$.

- Use the above and FTC Part II to evaluate

$$\int_1^e \ln(x)\,\mathtt{d}x.$$

- Confirm that your answer is reasonable by estimating the area with one big right triangle. Support your work with a graph.

5.3 Antiderivatives by Inspection

As mentioned in the previous section, Calculus II typically provides a very detailed treatment of techniques for finding antiderivatives. In this section, we provide a short introduction to this topic. One bit of notation that is commonly used in this context is written below.

Definition 5.3.1. The Indefinite Integral

Given a function $f(x)$, we write the *indefinite integral*

$$\int f(x)\,\mathtt{d}x = F(x) + C$$

where $F(x)$ is an antiderivative for $f(x)$. That is, the string of symbols we are defining above is just another way of writing

$$F'(x) = f(x).$$

It is tradition to put the "$+C$" on the end of the indefinite integral just to remind the reader that a constant can always be added to an antiderivative to produce yet another valid antiderivative. That is, if we differentiate $F(x) + C$, we still get $F'(x) = f(x)$ since the constant C goes away. This C is often called the *constant of integration*.

The integral notation being used above does not directly correspond to an area under the curve, since no interval is specified. Rather, it serves to remind the reader that once bounds are specified, the area under the curve can be computed using FTC Part II and the function $F(x)$.

The easiest way to generate a nice little list of antiderivatives is to start with derivatives and just reverse the order of the functions. For example, we can take the differentiation formula

$$\frac{\mathtt{d}}{\mathtt{d}x}(\sin(x)) = \cos(x)$$

and change it into the antidifferentiation formula

$$\int \cos(x)\,\mathtt{d}x = \sin(x) + C.$$

Exercise 5.3.2. Some Basic Antiderivatives ☕

Find the antiderivatives below by just reversing a differentiation formula that we already have.

- $\int -\sin(x)\,\mathtt{d}x$

- $\int \sec^2(x)\,\mathtt{d}x$

- $\int 2\,\mathtt{d}x$

- $\int 2x\,\mathtt{d}x$

- $\int 1/x \, \mathtt{d}x$
- $\int \frac{1}{2\sqrt{x}} \, \mathtt{d}x$
- $\int e^x \, \mathtt{d}x$
- $\int -\csc(x)\cot(x) \, \mathtt{d}x$
- $\int \frac{1}{1+x^2} \, \mathtt{d}x$

Example 5.3.3. Sliding Around Constants

Suppose we wanted the antiderivative of $\sin(x)$. We do not directly have any function in our list of basic derivatives that differentiates to $\sin(x)$. However, we have one that of course gets us very close, namely cosine. If we use the derivative of cosine as a starting point, we should be able to fix it. Recall that

$$\frac{\mathtt{d}}{\mathtt{d}x}(\cos(x)) = -\sin(x).$$

Since we can pull constants in and out of derivatives as we like, we can negate both sides to produce

$$-\frac{\mathtt{d}}{\mathtt{d}x}(\cos(x)) = \sin(x)$$

and pull the negative into the derivative to get

$$\frac{\mathtt{d}}{\mathtt{d}x}(-\cos(x)) = \sin(x).$$

We now have a differentiation formula that we can reverse to produce an antidifferentiation formula. Thus, the indefinite integral of sine is

$$\int \sin(x) \, \mathtt{d}x = -\cos(x) + C.$$

Exercise 5.3.4. Slightly More Advanced Antiderivatives ☕☕

Use the "get kind of close with a guess and then fix the constants later" trick demoed in the above example to find the antiderivatives below.

- $\int x \, \mathtt{d}x$
- $\int x^2 \, \mathtt{d}x$
- $\int x^3 \, \mathtt{d}x$
- $\int \frac{1}{\sqrt{x}} \, \mathtt{d}x$
- $\int \sqrt{x} \, \mathtt{d}x$
- $\int \sqrt[3]{x} \, \mathtt{d}x$
- $\int 2^x \, \mathtt{d}x$
- $\int \frac{2}{\sqrt{1-x^2}} \, \mathtt{d}x$

Exercise 5.3.5. Power Rule for Antiderivatives

In the previous exercise, many of the integrals were of the form

$$\int x^n \, \mathtt{d}x$$

for some real number n. What general pattern (in terms of n) does the answer have? Are there any n values for which the pattern is not valid? If so, how would you handle that case?

Since derivatives satisfy linearity (split over addition and constants pull out), antiderivatives do too. To say this more mathematically, we remark that for functions $f(x)$ and $g(x)$ and constants c and d,

$$\int cf(x) + dg(x)\,\mathtt{d}x = c\int f(x)\,\mathtt{d}x + d\int g(x)\,\mathtt{d}x.$$

Exercise 5.3.6. Verifying the Linearity of Antidifferentiation ☕

Separately compute the derivative of both the left- and right-hand sides above. Verify they are they same to see that those antiderivatives are in fact equivalent.

Thanks to that property, if we have a sum of many terms, we can just antidifferentiate each separately and then add the results.

Example 5.3.7. Using Linearity of Antidifferentiation

Suppose we wish to find the following antiderivative:

$$\int \frac{1-x}{x}\,\mathtt{d}x.$$

The integrand does not particularly remind us of any known derivative formula. So instead, we try to split it up into simpler terms using linearity as follows:

$$\begin{aligned}\int \frac{1-x}{x}\,\mathtt{d}x &= \int \frac{1}{x} - \frac{x}{x}\,\mathtt{d}x \\ &= \int \frac{1}{x} - 1\,\mathtt{d}x \\ &= \int \frac{1}{x}\,\mathtt{d}x - \int 1\,\mathtt{d}x \\ &= \ln(x) - x + C.\end{aligned}$$

Note that in the above example we did not need a separate "$+C$" for each integral, which might be tempting to put there. Imagine we did put a separate constant for each integral; call the first C_1, the second C_2, and their total C. The sum of two constants is just a constant anyway.

Exercise 5.3.8. Similar Trickery ☕☕

Use polynomial long division and linearity of antiderivatives to find the following integral:

$$\int \frac{3+2x^2}{1+x^2}\,\mathtt{d}x.$$

Exercise 5.3.9. Back to the Dartboard ☕

Answer Exercise 4.0.0.2 once again, but this time using FTC Part II. Verify it matches the answer we came up with previously.

Exercise 5.3.10. Revisiting a Very Messy Riemann Sum ☕☕

Answer Exercise 4.3.11 once again, but this time using FTC Part II. Verify it matches the answer we came up with previously.

Exercise 5.3.11. A Comment on Log ☕☕

Sometimes you will see the antiderivative of $1/x$ written as

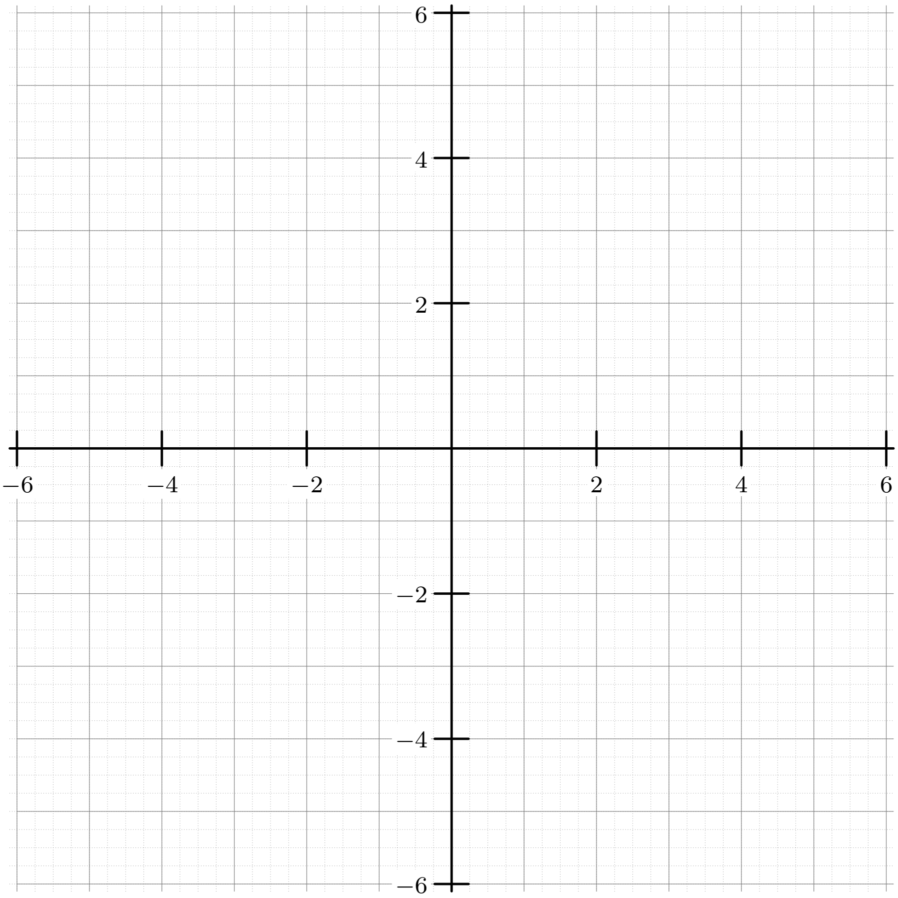
$$\int \frac{1}{x}\,\mathtt{d}x = \ln|x| + C$$

rather than just

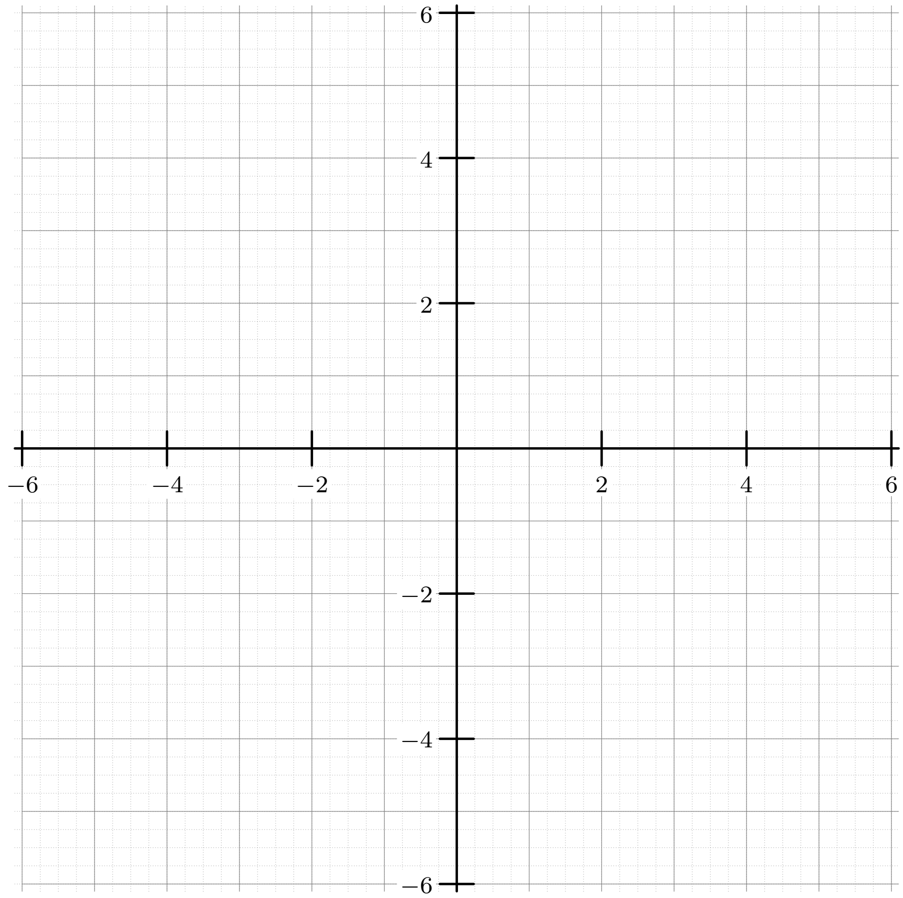
$$\int \frac{1}{x}\,\mathtt{d}x = \ln(x) + C.$$

The second formula is not wrong, but the absolute values extend the domain on which the formula is valid. We analyze this a bit here.

- Graph the function $\ln(x)$ and the function $1/x$ on the same axes. For what x values is the slope of the tangent line to $\ln(x)$ actually equal to $1/x$?



- Graph the function $\ln|x|$ and the function $1/x$ on the same axes. For what x values is the slope of the tangent line to $\ln|x|$ actually equal to $1/x$?



5.4 Antiderivatives by Substitution

Undoing the Chain Rule

The technique of u-substitution (affectionately known as "u-sub" from here on) can be seen as the reverse of the chain rule for antiderivatives.

Exercise 5.4.1. What Was the Chain Rule Again?

- First, write down the chain rule.

$$(f\,(g(x)))' =$$

- Take the antiderivative of both sides of that equation.

$$\int \qquad\qquad \mathtt{d}x = f\,(g(x)) + C$$

In practice, we often make the substitution $u = g(x)$ to condense the notation. This will take a nastier integral with respect to x and replace it by a hopefully friendlier integral with respect to u. This process of transforming from x to u involves the following three steps:

1. **Choose u:** Pick u to be equal to some expression involving x. Frequently, it is helpful to pick u to be some "inner function" in a composition of functions that appears in the integrand. However, there is a *lot* of freedom regarding what substitution you make. Some choices of u will be helpful, and others will not be! It is important to be brave and just try some.
2. **Differentiate u:** Once you have a formula for u, differentiate with respect to x to get a formula for $\frac{\mathtt{d}u}{\mathtt{d}x}$. This will tell us what the conversion factor is between x units and u units.
3. **Solve for $\mathtt{d}x$:** Use your derivative to solve for $\mathtt{d}x$. Substitute that expression for the $\mathtt{d}x$ in the integral to replace it with $\mathtt{d}u$.

For the sake of having this process written in one nice little line, here is the above paragraph rewritten concisely and precisely.

$$\int f'\left(g(x)\right) g'(x)\,\mathtt{d}x = \int f'(u)\,\mathtt{d}u = f(u) + C = f\left(g(x)\right) + C.$$

Example 5.4.2. An Example of Integration via u-sub

To evaluate $\int x\cos\left(x^2\right)\mathtt{d}x$, we identify $u = x^2$ as a plausible choice based on our recollection of chain rule. This gives the following change of variables:

Three Steps of u-Substitution		
Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u = x^2$	$\frac{\mathtt{d}u}{\mathtt{d}x} = 2x$	$\mathtt{d}x = \frac{1}{2x}\,\mathtt{d}u$

We now replace x^2 by u and replace $\mathtt{d}x$ by $\frac{1}{2x}\,\mathtt{d}u$ in our integral.

$$\int x\cos\left(x^2\right)\mathtt{d}x = \int x\cdot\cos(u)\frac{1}{2x}\,\mathtt{d}u = \frac{1}{2}\int\cos(u)\,\mathtt{d}u = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin\left(x^2\right) + C$$

Exercise 5.4.3. Checking Our Work ☕

As a follow up to the previous example, differentiate the answer to verify that you end up with the original integrand!

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\frac{1}{2}\sin\left(x^2\right) + C\right) =$$

Example 5.4.4. A Trickier u-sub

Suppose we wish to evaluate the following integral:

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\,\mathtt{d}x$$

One possible approach is to let u be the denominator. The denominator can be thought of as the "inner function" inside a reciprocal function and thus often makes a good choice for u.

Three Steps of u-Substitution		
Choice of u	Differentiate u	Solve for $\mathtt{d}x$
$u = \sqrt[3]{x}+1$	$\frac{\mathtt{d}u}{\mathtt{d}x} = \frac{1}{3}x^{-2/3}$	$\mathtt{d}x = 3x^{2/3}\,\mathtt{d}u$

We now perform the substitutions on the denominator and the $\mathtt{d}x$.

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\,\mathtt{d}x = \int \frac{\sqrt[3]{x}}{u}3x^{2/3}\,\mathtt{d}u = 3\int \frac{x}{u}\,\mathtt{d}u$$

At the moment, it seems like things are going very poorly! We hoped that x in the numerator would nicely cancel out, like it did back in the more civilized age of Exercise 5.4.2. To fix this, we solve for x in the equation $u = \sqrt[3]{x}+1$ to obtain $x = (u-1)^3$. We now substitute that expression for x in the integral.

$$\begin{aligned}
3\int \frac{x}{u}\,\mathtt{d}u &= 3\int \frac{(u-1)^3}{u}\,\mathtt{d}u \\
&= 3\int \frac{u^3-3u^2+3u-1}{u}\,\mathtt{d}u \\
&= 3\int u^2-3u+3-\frac{1}{u}\,\mathtt{d}u \\
&= u^3-\frac{9}{2}u^2+9u-3\ln|u|+C \\
&= \left(\sqrt[3]{x}+1\right)^3-\frac{9}{2}\left(\sqrt[3]{x}+1\right)^2+9\left(\sqrt[3]{x}+1\right)-3\ln\left|\sqrt[3]{x}+1\right|+C \\
&= x-\frac{3}{2}\sqrt[3]{x}^2+3\sqrt[3]{x}-3\ln\left|\sqrt[3]{x}+1\right|+C
\end{aligned}$$

Exercise 5.4.5. Missing Constants ☕

In the above example, all of the constant terms disappeared on the final step! Was that ok?

Exercise 5.4.6. Practice with u-sub ☕☕

- Evaluate $\int \frac{6x+3}{x^2+x+8}\,\mathtt{d}x$.

- Evaluate $\int \frac{(\ln(x))^2}{x}\,\mathtt{d}x$.

- Evaluate $\int xe^{-x^2}\,\mathtt{d}x$.

- Consider the integral
$$\int e^{\left(x^2\right)}\,\mathtt{d}x$$
Explain in words why the substitution $u = x^2$ will not work in this case. Where do you get stuck?

Antiderivatives of the Six Trig Functions

It turns out u-substitution is the right technique for finding antiderivatives of all six trig functions! We already obtained the antiderivatives of sine and cosine. But for the other four, we need u-sub.

Example 5.4.7. Antiderivative of Tangent

We compute the antiderivative of tangent by rewriting as $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and then using the substitution $u = \cos(x)$. Differentiating both sides produces $\mathtt{d}x = \frac{\mathtt{d}u}{-\sin(x)}$. We now apply these substitutions:

$$\begin{aligned}
\int \tan(x)\,\mathtt{d}x &= \int \frac{\sin(x)}{\cos(x)}\,\mathtt{d}x \\
&= \int \frac{\sin(x)}{u}\frac{\mathtt{d}u}{-\sin(x)} \\
&= -\int \frac{1}{u}\,\mathtt{d}u \\
&= -\ln|u| + C \\
&= -\ln|\cos(x)| + C
\end{aligned}$$

The method used to antidifferentiate tangent can be adapted to also antidifferntiate cotangent.

Exercise 5.4.8. Integral of Cotangent ☕☕

Find the antiderivative of cotangent.

$$\int \cot(x)\,\mathtt{d}x =$$

The antiderivative of secant is much trickier! The process is not intuitive and requires a rabbit out of a hat.

Example 5.4.9. Integral of Secant

Since multiplication by 1 does not change the integrand, we are free to multiply by 1 whenever it is helpful. Here, it turns out to be helpful to multiply by $\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$. This is the rabbit.

$$\begin{aligned}
\int \sec(x)\, \mathtt{d}x &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\, \mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\, \mathtt{d}x \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{u} \frac{1}{\sec(x)\tan(x)+\sec^2(x)}\, \mathtt{d}u \\
&= \int \frac{1}{u}\, \mathtt{d}u \\
&= \ln|u| + C \\
&= \ln|\sec(x)+\tan(x)| + C
\end{aligned}$$

The above method can be adapted to antidifferentiate cosecant.

Exercise 5.4.10. Integral of Cosecant ☕☕

Find the antiderivative of cosecant.

$$\int \csc(x)\, \mathtt{d}x =$$

Antiderivatives and Completing the Square

Completing the square can often be used in combination with u-sub, where the result of CTS creates an inner function that can be used as u.

Example 5.4.11. Arctangent in Disguise

Suppose we want to find the antiderivative

$$\int \frac{1}{16x^2+8x+17}\,\mathtt{d}x.$$

The initial instinct could be to aim for the derivative of $\arctan(x)$. This derivative, $\frac{1}{x^2+1}$, gets us kind of close. However, that $8x$ term in the denominator is a bit troublesome, so we CTS to remove it as follows:

$$16x^2+8x+17 = 16\left(x^2+\frac{1}{2}x\right)+17 = 16\left(x^2+\frac{1}{2}x+\frac{1}{16}-\frac{1}{16}\right)+17 = 16\left(x+\frac{1}{4}\right)^2+16.$$

This motivates the choice of inner function $u = x+\frac{1}{4}$. Conveniently, this implies that $\mathtt{d}u = \mathtt{d}x$ since $\frac{\mathtt{d}u}{\mathtt{d}x} = 1$. We now evaluate the integral!

$$\begin{aligned}
\int \frac{1}{16x^2+8x+17}\,\mathtt{d}x &= \int \frac{1}{16\left(x+\frac{1}{4}\right)^2+16}\,\mathtt{d}x \\
&= \frac{1}{16}\int \frac{1}{(u)^2+1}\,\mathtt{d}u \\
&= \frac{1}{16}\arctan(u) \\
&= \frac{1}{16}\arctan\left(x+\frac{1}{4}\right).
\end{aligned}$$

Exercise 5.4.12. Checking Our Work ☕

Verify our antiderivative above is correct by differentiating.

Example 5.4.13. Again But an Inverse Hyperbolic This Time

Consider the antiderivative
$$\int \frac{1}{\sqrt{2x^2-2x+1}}\,\mathtt{d}x.$$
This formula vaguely reminds us of the fact that
$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\operatorname{arcsinh}(x)\right)=\frac{1}{\sqrt{x^2+1}}.$$
Again, we CTS to figure out our choice of u.
$$2x^2-2x+1=2\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)+1=2\left(x-\frac{1}{2}\right)^2-\frac{1}{2}+1=2\left(x-\frac{1}{2}\right)^2+\frac{1}{2}$$
As in the previous example, this motivates the substitution $u=x-\frac{1}{2}$, which again has $\mathtt{d}u=\mathtt{d}x$. This transforms the integral as follows:
$$\begin{aligned}\int \frac{1}{\sqrt{2x^2-2x+1}}\,\mathtt{d}x &= \int \frac{1}{\sqrt{2\left(x-\frac{1}{2}\right)^2+\frac{1}{2}}}\,\mathtt{d}x\\ &= \int \frac{1}{\sqrt{2\left(u\right)^2+\frac{1}{2}}}\,\mathtt{d}u\end{aligned}$$
Aiming for that hyperbolic arcsine derivative, we next work on turning that $\frac{1}{2}$ into a 1, which we accomplish by factoring out a constant as follows:
$$\int \frac{1}{\sqrt{2u^2+\frac{1}{2}}}\,\mathtt{d}u=\int \frac{1}{\sqrt{\frac{1}{2}4u^2+\frac{1}{2}\cdot 1}}\,\mathtt{d}u=\frac{1}{\sqrt{\frac{1}{2}}}\int \frac{1}{\sqrt{4u^2+1}}\,\mathtt{d}u=\sqrt{2}\int \frac{1}{\sqrt{\left(2u\right)^2+1}}\,\mathtt{d}u.$$
Notice the last step motivates yet another substitution! We choose $w=2u$, picking the letter w of course because this is in fact a double u substitution. In this case, $\frac{\mathtt{d}w}{\mathtt{d}u}=2$, so $\mathtt{d}u=\frac{1}{2}\,\mathtt{d}w$. Implementing this second substitution, we finally land at something we can integrate:
$$\sqrt{2}\int \frac{1}{\sqrt{\left(2u\right)^2+1}}\,\mathtt{d}u=\sqrt{2}\int \frac{1}{\sqrt{\left(w\right)^2+1}}\frac{1}{2}\,\mathtt{d}w=\frac{\sqrt{2}}{2}\int \frac{1}{\sqrt{w^2+1}}\,\mathtt{d}w=\frac{\sqrt{2}}{2}\operatorname{arcsinh}(w)+C.$$
We now finish the calculation by undoing our two substitutions as follows:
$$\begin{aligned}\int \frac{1}{\sqrt{2x^2-2x+1}}\,\mathtt{d}x &= \frac{\sqrt{2}}{2}\operatorname{arcsinh}(w)+C\\ &= \frac{\sqrt{2}}{2}\operatorname{arcsinh}(2u)+C\\ &= \frac{\sqrt{2}}{2}\operatorname{arcsinh}\left(2\left(x-\frac{1}{2}\right)\right)+C\\ &= \frac{\sqrt{2}}{2}\operatorname{arcsinh}\left(2x-1\right)+C.\end{aligned}$$

Exercise 5.4.14. Now You Try! ☕☕☕

Find the following antiderivative:

$$\int \frac{1}{1-4x+3x^2}\,\mathtt{d}x$$

Exercise 5.4.15. Now Two Try! ☕☕☕

Find the following antiderivative:

$$\int \frac{1}{(x+1)\sqrt{x^2+2x}}\,\mathtt{d}x$$

5.5 Chapter Summary

We began by proving **Fundamental Theorem of Calculus Part I.** This is roughly the statement that **a derivative will cancel an integral** and is more formally stated below:

Let $f(x)$ be a continuous function on the interval $[a, b]$. Let x be any value in (a, b). Then

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(\int_a^x f(t)\,\mathtt{d}t\right) = f(x).$$

We proceeded to prove **Fundamental Theorem of Calculus Part II.** This is roughly the statement that **an integral will cancel a derivative** and is more formally stated below:

Let $F(x)$ be a differentiable function on the interval (a, b). Then

$$\int_a^b F'(x)\,\mathtt{d}x = F(b) - F(a).$$

Because of this relationship, we often use the word **integral** interchangeably with the word **antiderivative**, especially when written without bounds like this: $\int f(x)\,\mathtt{d}x$. This provides us with an easier way to evaluate integrals instead of always doing Riemann sums. In particular, to calculate

$$\int_a^b f(x)\,\mathtt{d}x,$$

take the following two steps:

1. **Find an Antiderivative.** Find a function $F(x)$ such that $F'(x) = f(x)$. That is, find the antiderivative of $f(x)$.
2. **Apply FTC Part II.** Calculate the integral as
$$\int_a^b f(x)\,\mathtt{d}x = \int_a^b F'(x)\,\mathtt{d}x = F(b) - F(a).$$

Step 2 is always easy; unfortunately Step 1 can be incredibly difficult. Techniques for antidifferentiation form a significant part of any Calculus II course. Here, we saw just a few introductory methods for antidifferentiation:

1. **Recognize the function as a known derivative.** Having a list of derivative formulas memorized will make many functions easy to antidifferentiate. For example, calculating $\int \frac{1}{1+x^2}\,\mathtt{d}x$ is quite easy if you remember that $(\arctan(x))' = \frac{1}{1+x^2}$, but a bit tricky otherwise.
2. **Perform a u-substitution.** If the integral is a bit messier, often we can clean it up by making a change of variables from x to u, where u is defined in terms x. Specifically, take the following steps:
   (a) **Choose u.** Pick u to be equal to some expression involving x. Frequently, it is helpful to pick u to be some "inner function" in a composition of functions that appears in the integrand.
   (b) **Differentiate u.** Once you have a formula for u, differentiate with respect to x to get a formula for $\frac{\mathtt{d}u}{\mathtt{d}x}$.
   (c) **Solve for $\mathtt{d}x$.** Use your derivative to solve for $\mathtt{d}x$. Substitute that expression for the $\mathtt{d}x$ in the integral to replace it with $\mathtt{d}u$.

   For example, the integral $\int \frac{1}{1+4x^2}\,\mathtt{d}x$ suggests the subsitution $u = 2x$ since it can be written as $\int \frac{1}{1+(2x)^2}\,\mathtt{d}x$. If we differentiate u, we find $\frac{\mathtt{d}u}{\mathtt{d}x} = 2$ so $\mathtt{d}x = \frac{1}{2}\,\mathtt{d}u$. Thus, $\int \frac{1}{1+(2x)^2}\,\mathtt{d}x = \int \frac{1}{1+(u)^2}\frac{1}{2}\,\mathtt{d}u$. Since constants factor out of integrals, we have that the integral is $\frac{1}{2}\int \frac{1}{1+u^2}\,\mathtt{d}u = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(2x) + C$.

5.6 Mixed Practice

Exercise 5.6.1. ☕

- After computing an antiderivative, how can you check your answer?
- Carry out this "checking process" on the following formula:

$$\int \ln(x)\,\mathtt{d}x = x\ln(x) - x + C$$

Exercise 5.6.2. ☕☕

Why do we put a "$+C$" on the end of an antiderivative?

Exercise 5.6.3. ☕☕☕

Here we attempt to use the Fundamental Theorem of Calculus Part I to prove the log property $\ln\left(x^2\right) = 2\ln(x)$.

- Recall the integral definition of the natural logarithm, and call this function $f(x)$. Specifically,

$$f(x) = \ln(x) = \int_{t=1}^{t=x} \frac{1}{t}\,\mathtt{d}t.$$

  What does this tell us the derivative $f'(x)$ is?
- Consider the functions $f(x) = \ln(x)$ and $g(x) = x^2$. Write an integral formula for the function $f \circ g(x) = \ln\left(x^2\right)$.
- Calculate the derivative of $f \circ g$ using the integral formula, chain rule, and Fundamental Theorem of Calculus Part I.
- Calculate the derivative of the function $2\ln(x)$ by any means.
- Why does your work above let you conclude that the functions $\ln\left(x^2\right)$ and $2\ln(x)$ must only differ by a constant? That is, there exists a real number C such that

$$\ln\left(x^2\right) = 2\ln(x) + C.$$

- Plug in $x = 1$ to solve for C and conclude that $\ln\left(x^2\right) = 2\ln(x)$ as desired.

Exercise 5.6.4.

Use the Fundamental Theorem of Calculus Part II to calculate the following integrals. In each case, draw a graph and check that your answer is reasonable.

- $\int_{x=0}^{x=\pi/6} \sin(x)\,\mathtt{d}x$
- $\int_{x=0}^{x=\pi/6} \cos(x)\,\mathtt{d}x$
- $\int_{x=0}^{x=1} \frac{4}{1+x^2}\,\mathtt{d}x$
- $\int_{x=0}^{x=1/2} \frac{1}{1+4x^2}\,\mathtt{d}x$

Exercise 5.6.5.

Find the following antiderivatives. You may find the following formula helpful:

$$\frac{\mathtt{d}}{\mathtt{d}x}\left(a^x\right) = \ln(a)a^x.$$

- $\int 2^x\,\mathtt{d}x$
- $\int 2^{3x-1}\,\mathtt{d}x$
- $\int x2^{\left(x^2+1\right)}\,\mathtt{d}x$

Exercise 5.6.6.

Find the following antiderivatives.

- $\int 2x+1\,\mathtt{d}x$
- $\int \frac{1}{2x+1}\,\mathtt{d}x$
- $\int e^x \sec\left(e^x\right)\tan\left(e^x\right)\mathtt{d}x$
- $\int \frac{1}{e^x}\,\mathtt{d}x$
- $\int \frac{\cos(\ln(2x))}{x}\,\mathtt{d}x$
- $\int \left(2^x\right)^2\mathtt{d}x$
- $\int \frac{1}{2+2x+x^2}\,\mathtt{d}x$
- $\int \frac{1}{1+2x+x^2}\,\mathtt{d}x$

Chapter 6

Applications of Integrals

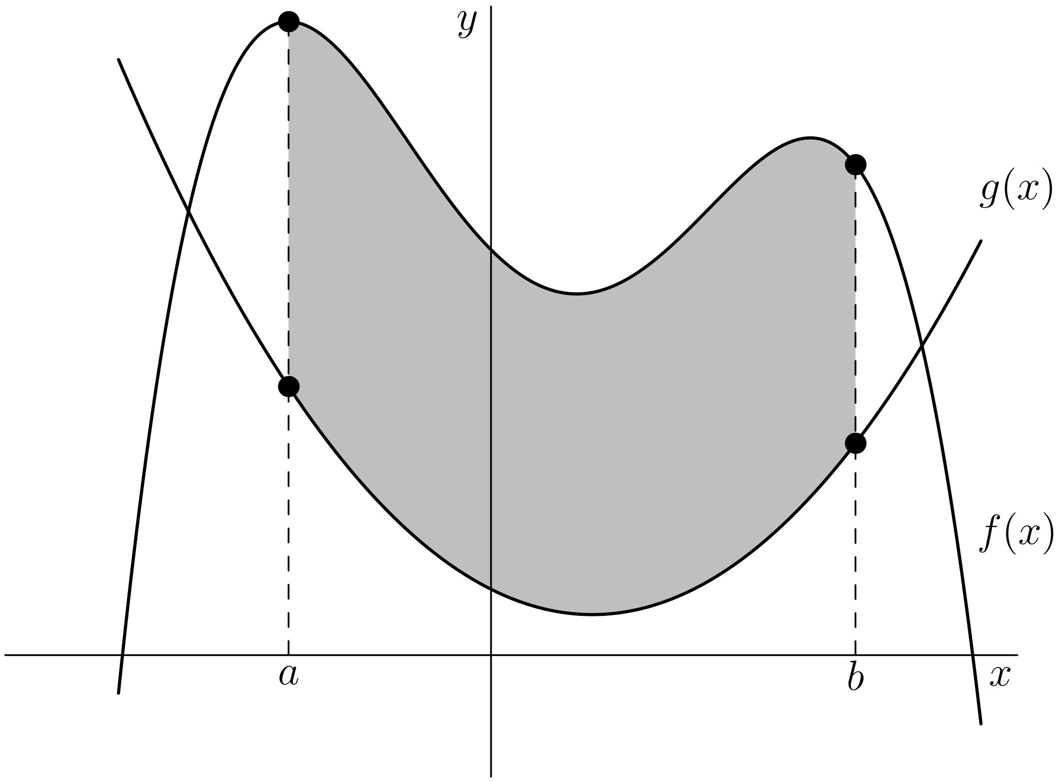
6.1 Area Between Curves

Recall that a definite integral calculates the signed area under a curve. Thus, we can find the signed area between two curves by taking their difference and integrating.

Formula 6.1.1. Area Between Curves

Let $g(x) \leq f(x)$ for all x in an interval $[a, b]$. Then the area A bounded by the graphs $x = a$, $x = b$, $y = f(x)$, and $y = g(x)$ is

$$A = \int_a^b (f(x) - g(x))\, \mathtt{d}x.$$

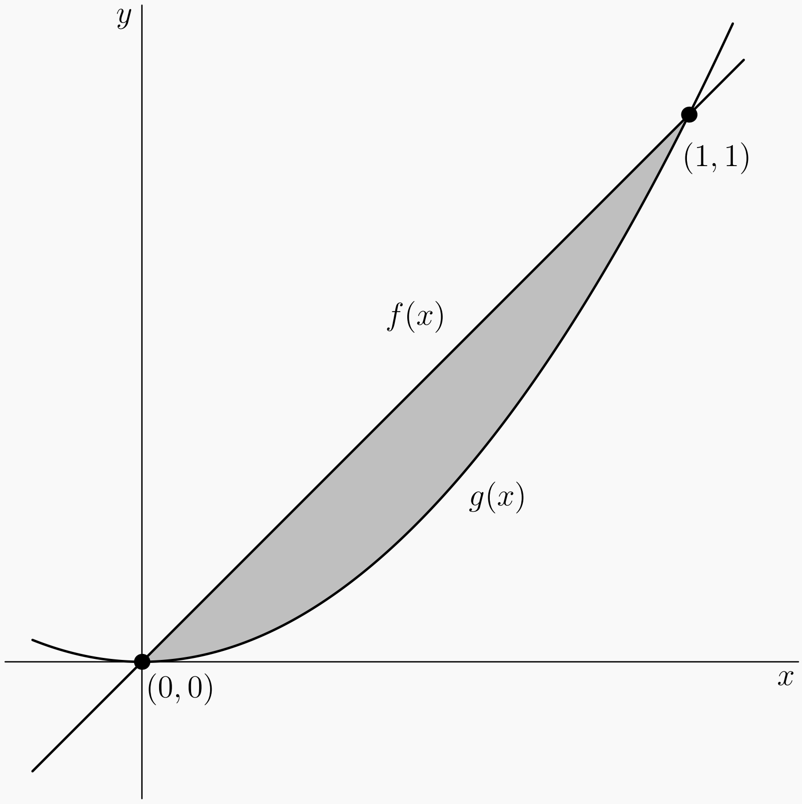


Example 6.1.2. Quadrature of a Parabola

Suppose we wish to find the area between curves $f(x) = x$ and $g(x) = x^2$. To accomplish this, we set the two formulas equal to each other to solve for the points of intersection. The line and parabola meet where

$$x^2 = x \implies x^2 - x = 0 \implies x(x-1) = 0 \implies x = 0 \text{ or } x = 1.$$

Thus the points of intersection are at $(0, 0)$ and $(1, 1)$.



Thus, the area between curves is

$$\begin{aligned}\int_{x=0}^{x=1} \left(x - x^2\right) \mathtt{d}x &= \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{x=1} \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{1}{6}.\end{aligned}$$

Exercise 6.1.3. Area Between x and x^n ☕☕

For each of the listed $n \in \mathbb{N}$, take the following steps:

- Graph $f(x) = x$ and $g(x) = x^n$ on the same axes and shade the region bounded by the curves in the first quadrant. Include labels of the intersection points of the curves.
- Use an integral to find the area between curves.
- Write the area as a decimal approximation.

Compile your results in the table below.

n	Graph of x and x^n in QI	Area Between Curves in QI	Decimal Approximation
3			
4			
5			
6			
7			
8			
9			
10			

- What does the area seem to be approaching as n keeps getting larger?

- What shape does the region between the curves seem to be approaching as n keeps getting larger? Using just basic geometry, what would the area of that shape be?

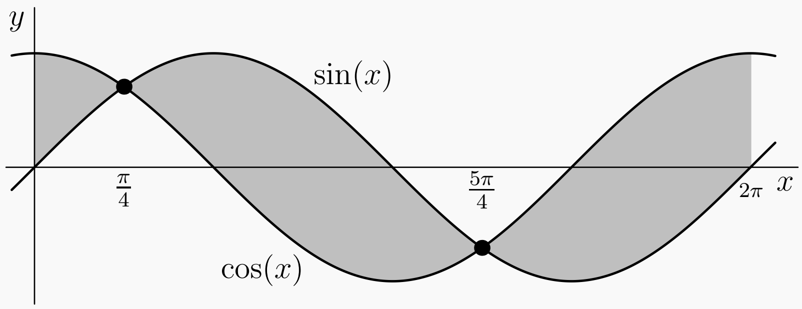
Note that if the curves intersect multiple times, you might have to split the integral onto the corresponding intervals.

Example 6.1.4. A Region with More Crossings

Find the area between the graphs of sine and cosine between $x = 0$ and $x = 2\pi$.
Again, to accomplish this, we set the two formulas equal to each other to solve for the points of intersection.

$$\sin(x) = \cos(x) \implies \tan(x) = 1 \implies x = \pi/4 \text{ or } x = 5\pi/4$$

Thus the points of intersection are at $\left(\pi/4, \sqrt{2}/2\right)$ and $\left(5\pi/4, -\sqrt{2}/2\right)$.



We now compute the area.

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos(x) - \sin(x))\, \mathtt{d}x + \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\, \mathtt{d}x + \int_{5\pi/4}^{2\pi} (\cos(x) - \sin(x))\, \mathtt{d}x \\ &= (\sin(x) + \cos(x))\,\big|_0^{\pi/4} + (-\cos(x) - \sin(x))\,\big|_{\pi/4}^{5\pi/4} + (\sin(x) + \cos(x))\,\big|_{5\pi/4}^{2\pi} \end{aligned}$$

Exercise 6.1.5. Complete the Example ☕☕

Finish the computation and verify the area is $4\sqrt{2}$.

Exercise 6.1.6. A Common Mistake ☕

Briefly write in words, why would simply evaluating

$$\int_{x=0}^{x=2\pi} \cos(x) - \sin(x)\, \mathtt{d}x$$

in the example above not give the area of the shaded region?

Some Other Regions for Practice

Find the area between the following curves. Graph the curves and shade the region!

Exercise 6.1.7. Other Regions ☕☕

- $y = |x|$ and $y = \frac{1}{2}x + 1$

- $y = \sqrt{x}$ and $y = \frac{1}{2}x^2$

- $f(x) = x^3 - x^2 - x + 1$ and $g(x) = x^3 + x^2 - x - 1$

6.2 Average Value of a Function

Discrete Averages

An *average* is a measure of the central tendency of a data set. One of the most common measures is the *arithmetic mean*: the sum of all entries in a data set divided by how many entries there are in the data set. This method works just fine as long as your data set is finite!

Example 6.2.1. Finding the Mean

Suppose we have a data set consisting of four numbers: $1, 3, 4, 3$. Then the mean is

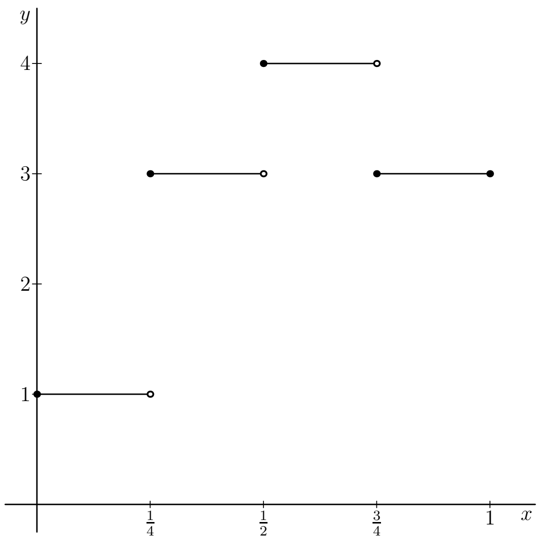
$$\frac{1+3+4+3}{4} = \frac{11}{4} = 2.75.$$

One interpretation of this is as follows: suppose a barista has four customers in a row tip her \$1, \$3, \$4, and \$3. Then her tips for that run of customers would have been exactly the same if each customer had tipped exactly \$2.75, since

$$1+3+4+3 = 4 \cdot 2.75.$$

Here is another way to frame that same question. Suppose we represent our data set as heights of a step function. For those tips, consider

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{4} \\ 3 & \text{if } \frac{1}{4} \le x < \frac{1}{2} \\ 4 & \text{if } \frac{1}{2} \le x < \frac{3}{4} \\ 3 & \text{if } \frac{3}{4} \le x \le 1. \end{cases}$$



It would be reasonable to define the *average value* of $f(x)$ on the interval as the height of the constant function that would have the same area, since the sum of areas of those rectangles is

$$\frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 3 + \frac{1}{4} \cdot 4 + \frac{1}{4} \cdot 3 = 2.75.$$

Notice that this is just a slight rewriting of the fraction used to compute the mean in Example 6.2.1. To state this area property in terms of integrals, we would say

$$\int_0^1 f(x)\,\mathrm{d}x = 1 \cdot 2.75.$$

That is, the area under that step function is the same as the area under a 1×2.75 rectangle.

A Continuous Average

It is much less clear what to do if there are infinitely many data points! For example, suppose the data set we are working with is a set of temperatures over an interval of time. Here we cannot just add all values and divide by how many there are, since you would just get $\frac{\infty}{\infty}$ which isn't among the most meaningful expressions we've seen.

Example 6.2.2. Average Temperature During a Day

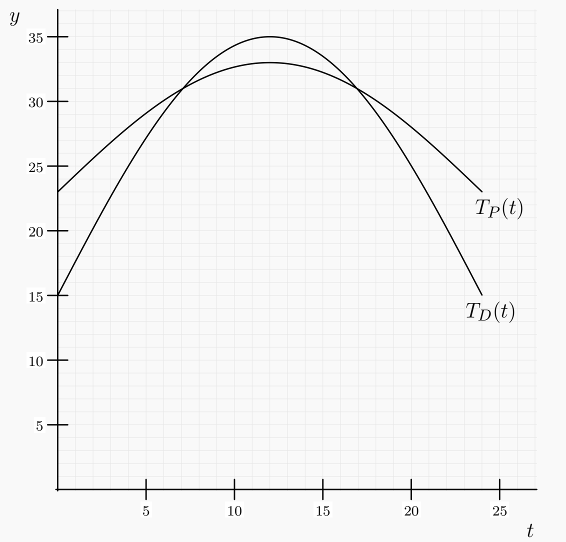
For a 24-hour period in Denver, the temperature at time t is given by

$$T_D(t) = 20\sin\left(\pi\frac{t}{24}\right) + 15,$$

where t is the time in hours elapsed since 5am. So for example, at 10am That same day in Philadelphia, we have the temperature given by

$$T_P(t) = 10\sin\left(\pi\frac{t}{24}\right) + 23.$$

Which city had the hotter day?



On one hand, it is easy to say that Denver had the higher maximum temperature that day. On the other hand, for most of the day it was hotter in Philadelphia! So, one sensible way to answer the question is to determine which city had the higher *average* temperature. The trouble is, we have infinitely many temperatures being measured at infinitely many times! So we cannot do the easy little "add up all values and divide by how many" trick. What we might do instead is approximate the functions with a simpler functions, namely step functions! Suppose we just sampled the temperature at the start of every hour and then assumed the temperature did not change for the course of that hour. This would give us a decent approximation of the average value, namely

$$\frac{1}{24}\sum_{n=1}^{24} T_D(n)$$

for Denver, and similarly for Philly. To make this approximation better, we could measure the temperature every half hour and get

$$\frac{1}{48}\sum_{n=1}^{48} T_D(n/2).$$

If we take more and more frequent samples, these numbers will converge on

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T_D(24n/N) = \frac{1}{24}\lim_{N\to\infty}\sum_{n=1}^{N} T_D(24n/N)\frac{24}{N},$$

which is just the Riemann sum definition of

$$\frac{1}{24}\int_0^{24} T_D(x)\,\mathtt{d}x.$$

We thus define the average value to be this value that the discrete averages are converging upon. So, to answer which day was hotter on average, we will compare the above value to

$$\frac{1}{24}\int_0^{24} T_P(x)\,\mathtt{d}x.$$

Exercise 6.2.3. Finishing the Example

- Calculate each of $\frac{1}{24}\int_0^{24} T_D(x)\,\mathtt{d}x$ and $\frac{1}{24}\int_0^{24} T_P(x)\,\mathtt{d}x$ to find the average temperature in each city for that day. Which day was hotter?

- In the above example, why did we change the fraction $\frac{1}{N}$ to be $\frac{24}{N}$? What part of the

Riemann sum does this quantity $\frac{24}{N}$ represent?

- How did we pay for magically inserting that 24 in the numerator of that fraction?

We generalize the construction of the above example to define *average value* as follows.

Definition 6.2.4. Average Value of a Continuous Function

Let $f(x)$ be a continuous function on $[a, b]$. Then the *average value* of $f(x)$ is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathtt{d}x.$$

Exercise 6.2.5. Sine ☕

- Just from looking at the graph, what would you expect the average value of $\sin(x)$ to be on the interval $[0, 2\pi]$.

- Use the definition above to compute the average value and verify your suspicion.

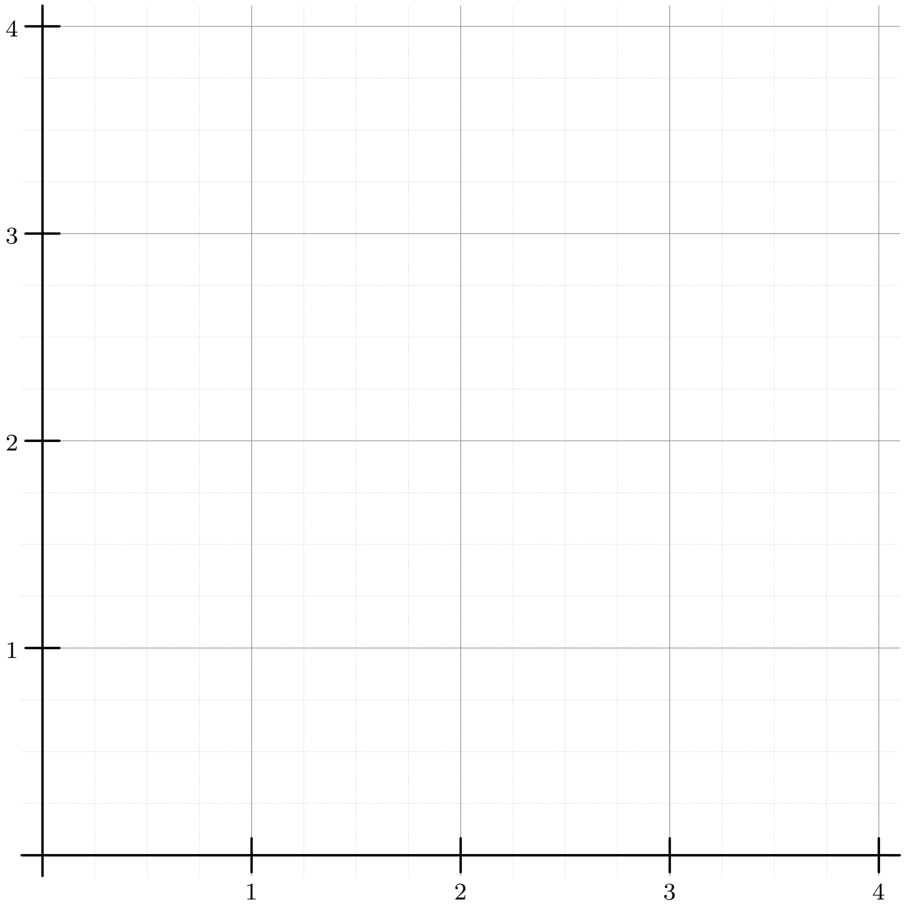
Exercise 6.2.6. A Race! ☕☕☕

Two very algorithmic frogs, Linearibbit and Toadratic, have a race. Each second, the referee announces a random number x between 0 and 15. Linearibbit will leap forward $10x$ centimeters. Toadratic will leap forward x^2 centimeters. After one minute, who would you expect to be in the lead, and by how far? Justify your answer using an average value integral or integrals.

Exercise 6.2.7. A Square Root ☕☕

- What is the average value of $f(x) = \sqrt{x}$ on $[0, 4]$?

- Draw a rectangle on that same interval whose height is the average value. How does the area of that rectangle compare to the area under the curve $f(x)$?

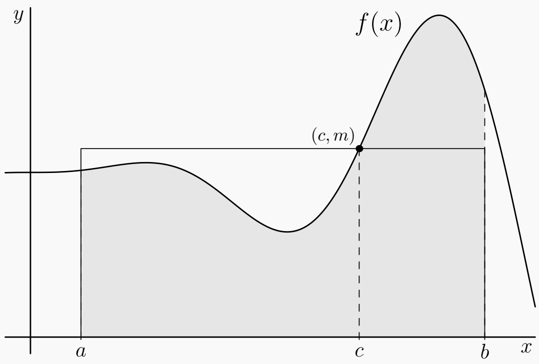


- Where does the top of the rectangle intersect the graph of $f(x)$? Label this point on your graph.

We generalize the above example to a theorem.

Theorem 6.2.8. Mean Value Theorem for Integrals

Let $f(x)$ be a continuous function on $[a, b]$ and let m be the average value of $f(x)$ on that interval. Then there exists a $c \in [a, b]$ such that $f(c) = m$.



Exercise 6.2.9. Proof of MVT for Integrals

Fill in the blanks in the proof below.

Proof. We proceed by contradiction. In particular, assume $f(x) \neq m$ for all $x \in [a, b]$. Consider the value $f(a)$. If $f(a) > m$, then $f(x) > m$ for all $x \in [a, b]$. This is because if even a single $f(x)$ value crossed below the line $y = m$, there would be a place where $f(x) = m$ by ______________. However, if all $f(x) > m$, then we have the average value of f is

$$m = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x > \frac{1}{b-a}\int_a^b m\,\mathrm{d}x = \frac{m}{b-a}\int_a^b 1\,\mathrm{d}x = \underline{\hspace{6cm}}.$$

But a number cannot be greater than itself, so we have reached a contradiction.
If $f(a) < m$, we reach a contradiction by a similar argument.
Thus, our original assumption must have been wrong, so there must exist a $c \in [a, b]$ for which $f(c) = m$. □

This is a rather nice analogy to the original MVT. That said that a derivative would at some particular point equal the average rate of change on an interval. This says that every function will at some point be equal to its average value on an interval.

Exercise 6.2.10. Whoa. ☕☕☕

What happens if you substitute $f'(x)$ for $f(x)$ in the Mean Value Theorem for Integrals?

Exercise 6.2.11. Nice. ☕☕☕

What happens if you substitute $F(x) = \int_0^x f(t)\,\mathrm{d}t$ for $f(x)$ in the original Mean Value Theorem?

6.3 Probability

A French noble and naturalist, Georges Louis Leclerc, Comte de Buffon, asked the following question in 1777:

If you drop a short needle onto ruled paper, what is the probability that the needle crosses one of the lines?

Before we go about answering such a question, we provide a tiny bit of background on probability. (This is by no means comprehensive; you will see a far more detailed treatment of this in an actual Probability Theory course!)

Discrete Probability

A *discrete event* is an event with finite or countably many possibilities. If one wishes to find the probability of a discrete event, often one can simply sum the probability of each outcome, weighted by how likely that outcome is to occur. Here is a really simple example.

Example 6.3.1. Dice

How likely is it to roll an odd number on a standard six-sided die?
There are six possible outcomes: roll 1,2,3,4,5, or 6. The probability of rolling a 1 is 1/6, and if you roll a 1, there is a 100% chance that you rolled an odd. The probability of rolling a 2 is also 1/6. If you roll a 2, there is a zero percent chance of rolling an odd, since you already rolled an even. And so on, through six. Thus the probability is

$$\frac{1}{6}\cdot 1+\frac{1}{6}\cdot 0+\frac{1}{6}\cdot 1+\frac{1}{6}\cdot 0+\frac{1}{6}\cdot 1+\frac{1}{6}\cdot 0=\frac{3}{6}=\frac{1}{2},$$

as we knew it should have been.

Notice how similar this calculation ends up being to computing a mean. Similarly, average value will be the right tool for computing a continuous probability!

Continuous Probability

In discrete cases, we sum over all possible outcomes. However a *continuous event* is one with uncountably many possible outcomes, so we cannot simply sum over the outcomes. Instead we use the "continuous" version of a sum: an integral!

In particular, if you want to know the probability of an event P given that it has probability $p(x)$ of happening for equally likely outcomes x in $[a,b]\subset\mathbb{R}$, the probability is given by:

$$\frac{1}{b-a}\int_a^b p(x)\,\mathrm{d}x.$$

Notice that this is just the average value of the probability function $p(x)$.

Exercise 6.3.2. Reframing Monte Carlo Integration

Consider the 2×2 *square with corners* $(0,0)$ *and* $(2,2)$*. If a point is selected at random, what is the probability it lies below the curve* $f(x)=\frac{1}{2}x^2$*?*

- Geometric intuition tells us it should be the area under the curve divided by the area of the square. Compute this quantity and sketch a diagram illustrating what you computed.

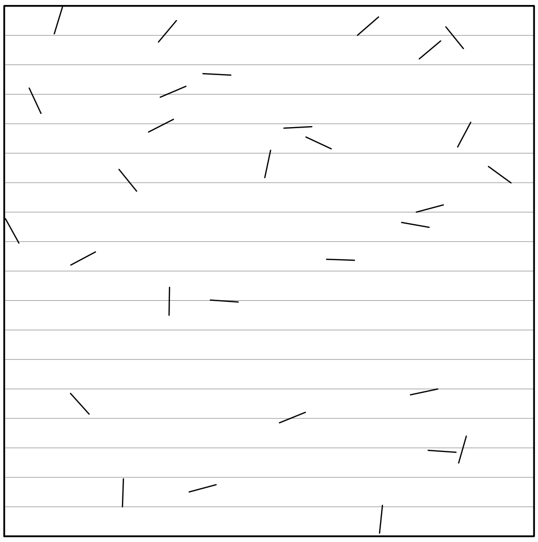
- Another way to look at it in light of the integration formula above: if we pick an x-coordinate inbetween 0 and 2, then $f(x)/2$ is the probability that a randomly selected y-coordinate will form a point (x, y) that is below the curve f (since $f(x)$ out of the 2 units worth of y-coordinates lie below the curve). Thus, our probability is

$$\frac{1}{2-0}\int_{x=0}^{x=2} \frac{f(x)}{2}\,\mathtt{d}x.$$

Evaluate the above integral and verify you get the same probability.

Buffon's Needle Problem

Back to the original question. To be a little more precise, let us say that our needle has length l and our rulings are distance d apart. Then in this case we can take "short" to mean that $l < d$. That is to say, we don't need to worry about the possibility of a needle crossing more than one ruling.



Exercise 6.3.3. Gathering Some Data

Try this out on your own. Find a short needle (say a toothpick for example) and drop it repeatedly onto a page with rulings at distance d apart (say a sheet of normal lined paper but only using every sixth line or so, enough to make the gaps between lines longer than a toothpick). Drop the needle at least 30 times. How many times did it cross? What proportion of needles actually cross a line?

We now compute the theoretical probability of this event using our integration trick from above. Let's walk through the computation of the answer to Buffon's needle question. Let x represent the angle measure in radians that the needle makes with the direction of the rulings.

Exercise 6.3.4. Computing the Probability ☕☕

- Draw and label a picture of a random needle and rulings, showing x, l, and d with your rulings going horizontally.

- Use your diagram to explain why the vertical height of the needle is $l\sin(x)$. (The vertical height is the length that is left if you project away any horizontal component. So for example, a needle at angle zero would have height zero, but a needle at angle $\frac{\pi}{2}$ would have height l.)

- Based on the above, the probability of a needle at angle x crossing a ruling is $l\texttt{sin}(x)/d$ since d is the total vertical distance between rulings. Compute the corresponding integral for probability, using the fact that the angle can only be between 0 and π.

- How does your observed proportion of needles crossing compare to the theoretical predicted value?

- How could this probability be used to set up a Monte Carlo experiment for numerically approximating π? Explain!

6.4 Position, Velocity, Acceleration

One common physics application of derivatives and integrals is to study one-dimensional motion. In this setting, the independent variable is time and the dependent variable represents the position of a moving object relative to some initial starting point on the axis of motion being studied. Some notation:

- Let $s(t)$ represent the **position** of the object at time t. (s for *spatium*, Latin for distance)
- Let $v(t) = s'(t)$ represent the **velocity** of the object at time t.
- Let $a(t) = v'(t)$ represent the **acceleration** of the object at time t.

A simple way to think of this is if you go on a road trip (without backtracking), then $s(t)$ represents the distance traveled so far at time t, while $v(t)$ represents whatever your speedometer reads at time t. The quantity $a(t)$ measures whether your speed is increasing or decreasing at time t.

Exercise 6.4.1. Second Derivative ☕

Given the above, why is $a(t) = s''(t)$?

Newton's Second Law and Projectiles

Newton's Second Law is most easily stated as

$$F = ma$$

where F is the force acting on an object, m is the mass of an object, and a is the acceleration. We begin with a simple application of this, namely an object in freefall. For distances not too far from the surface of the earth, we can say that the acceleration is constant, namely

$$a = F/m$$

since the mass of the object is certainly not changing, and unless you're launching into space, the force due to gravity is roughly the same regardless of your position. For example, you do not feel heavier at the bottom of a staircase than you did at the top of a staircase.

Exercise 6.4.2. A Physics Applecation ☕☕

Suppose an apple is in freefall, falling from a height of 20 feet out of a tree. Acceleration due to gravity is roughly 32ft/s^2. Assume that air resistance is negligible, along with any other force that might be acting on the apple. Let $s(t)$ be the apple's position relative to the ground t seconds after it fell.

- Explain in terms of this model why $v(0) = 0$ and $s(0) = 20$.

- The information above can be written as the equation
$$a(t) = -32\text{ft/s}^2.$$

Antidifferentiate both sides with respect to t to get an equation for $v(t)$.

- When you took the antiderivative above, you inevitably produced a constant of integration. Solve for the constant using the fact that $v(0) = 0$.

- Antidifferentiate both sides of your equation for $v(t)$ with respect to t once again to obtain an equation for $s(t)$.

- Solve for the constant using the fact that $s(0) = 20$.

- Use your equation for $s(t)$ to determine how long it takes the apple to hit the ground.

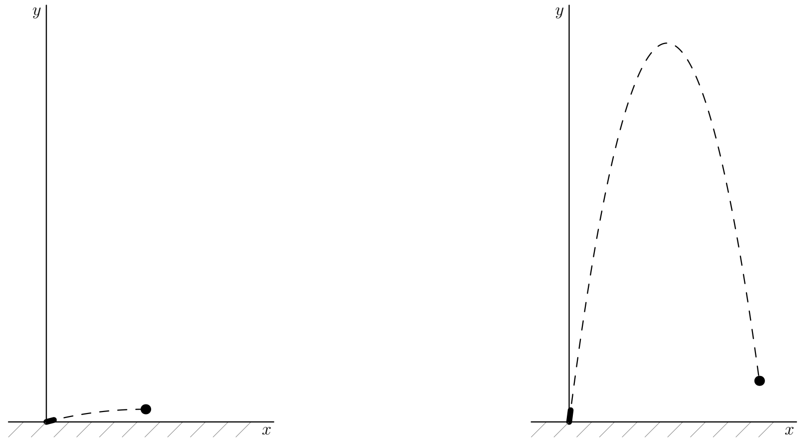
It is worth noting that the above fact, that projectiles in a vacuum follow paths given by quadratic equations, was one of the major accomplishments of Galileo during the early Renaissance (significantly before the development of calculus)! Next, we explore a classic optimization problem that will use the principles of freefall described above.

Exercise 6.4.3. The Best Angle to Launch an Object ☕☕☕

Consider the following optimization problem:

At what angle do you point a cannon to maximize the horizontal distance the projectile travels?

On the one hand, if you point the cannon too low, the object will not be in the air very long and just splat on the ground. On the other hand, if you point the cannon too high, the projectile will just move up and down a whole lot without really covering much horizontal distance.



Somewhere in between should be the sweet spot, that perfect place where the object has enough height to stay in the air for a long time but also is covering a lot of horizontal distance.

- Assume the total force expended by the cannon is constant. For sake of clean numbers, call it 1 unit of force. Draw a diagram below that shows a cannon firing across flat ground at an angle θ to the horizontal. (Assume the height of the cannon is negligible; one can consider the ground to be the x axis and the cannon to be located at the origin.)

- Use your diagram to explain why the horizontal component of the force will be $\cos(\theta)$ and the vertical component of the force will be $\sin(\theta)$.

- Call $s_x(t)$, $v_x(t)$, and $a_x(t)$, the position, velocity, and acceleration of the projectile in the x direction. Assume no significant forces are acting on the projectile in the x direction (for example, no air resistance). Thus, Newton's Second Law tells us that $a_x(t) = 0$ for all t. Furthermore, assume $v_x(0) = \cos(\theta)$ since it is proportional to the initial force acting on it. Use antiderivatives to solve this for an equation for $s_x(t)$.

- Call $s_y(t)$, $v_y(t)$, and $a_y(t)$, the position, velocity, and acceleration of the projectile in the y direction. Assume no significant forces are acting on the projectile in the y direction other than gravity. Thus, Newton's Second Law tells us that $a_y(t) = -32$. Furthermore, assume $v_y(0) = \sin(\theta)$ since it is proportional to the initial force acting on it. Use antiderivatives to solve this for an equation for $s_y(t)$.

- Find an expression in terms of θ for the roots of $s_y(t)$. Interpret the larger root in terms of this model.

- Use the information from the previous part along with your formula for $s_x(t)$ to obtain a formula in terms of θ for the total horizontal distance traveled by the projectile before hitting the ground. Call this formula $D(\theta)$.

- Find the θ value that maximizes D using Fermat's Theorem.

- Write your final answer to the question in a short sentence below!

6.5 Chapter Summary

There are a vast number of applications of integrals across mathematics and throughout the sciences. Here we demonstrated just a few of these applications.

- **Area between curves.** If you have a strangely shaped region that you need the area of, whose x-values go from $x = a$ to $x = b$, a good strategy can be to express the top boundary as a function $f(x)$, the bottom boundary as a function $g(x)$, and compute the area

  $$A = \int_{x=a}^{x=b} f(x) - g(x)\, \mathtt{d}x.$$

  Remember that if your functions $f(x)$ and $g(x)$ cross each other, you will have to split the area calculation into separate integrals.

- **Average value of a continuous function.** The average of a list of numbers $x_1, x_2, \ldots, x_N$ can be calculated as

  $$\frac{\sum_{n=1}^{N} x_n}{N}.$$

  If rather than a list of finitely many numbers, you have infinitely many values given by a continuous function $f(x)$ on an interval $[a, b]$, you can calculate the **average value** as

  $$\frac{\int_{x=a}^{x=b} f(x)\, \mathtt{d}x}{b - a}.$$

- **Probability.** Here we had a very similar formula to average value, but with a different interpretation. Specifically, if an event P has probability of $p(x)$ of happening over a domain of possibilities $x \in [a, b]$, the probability is given by

  $$\frac{\int_{x=a}^{x=b} p(x)\, \mathtt{d}x}{b - a}.$$

- **Position, velocity, acceleration.** If the position of an object at time t is given by a function $s(t)$, then the velocity is $v(t) = s'(t)$ and the acceleration is $a(t) = v'(t) = s''(t)$. Since we can differentiate to move from position to velocity to acceleration, we can antidifferentiate to move the other way. More formally, we can say the following:

  - $v(t) = \int a(t)\, \mathtt{d}t$
  - $s(t) = \int v(t)\, \mathtt{d}t$

  Notice these integrals will result in a "$+C$". One can solve for the C by plugging in an initial position or initial velocity as appropriate.

6.6 Mixed Practice

Exercise 6.6.1. ☕

- If we have a constant function $f(x) = h$ on an interval $[a, b]$, what is the average value of the function? (Compute it just from looking at the graph.)
- Use the formula for average value of a function to again calculate the average value of $f(x) = h$ on $[a, b]$. Verify your answers match!

Exercise 6.6.2. ☕☕

Decide if each of the statements below is always true or if it can be false.

- If m is the average value of $f(x)$ on an interval $[a, b]$, then $f(x) > m$ for exactly half of the interval, and $f(x) < m$ for exactly half of the interval.
- Suppose m_1 is the average value of $f(x)$ on an interval $[a, b]$ and m_2 is the average value of $g(x)$ on $[a, b]$. If $m_1 > m_2$, then $f(x) > g(x)$ for all $x \in [a, b]$.
- Suppose m is the average value of an increasing function $f(x)$ on an interval $[a, b]$. Then there must exist a point $x \in [a, b]$ at which $f(x) > m$, and there must exist a point $x \in [a, b]$ at which $f(x) < m$.

Exercise 6.6.3. ☕☕☕

Notice that the graphs of $f(x) = \cos\left(\frac{\pi}{2}x\right)$ and $g(x) = \sqrt{1 - x^2}$ look kind of similar on the interval $[-1, 1]$. Graph them! Compute the average value of each. How do they compare?

Exercise 6.6.4. ☕☕☕

An object is thrown upwards from a height of 10 feet with an initial velocity of 20 feet per second. How long until it hits the ground? Assume no forces are acting on the object except for acceleration due to gravity of -32 feet per second squared.

Exercise 6.6.5. ☕☕☕

Suppose a car linearly accelerates from 0mph to 50mph over the course of 1 hour. It then travels at 50mph for 1 hour. The car then decelerates linearly from 50mph back to 0mph.

- Plot $v(t)$ on the interval [0,3], where $v(t)$ represents the car's velocity measured in miles per hour at time t.
- Write a piecewise formula for $v(t)$.
- Write a piecewise formula for $a(t)$, the acceleration at time t, measured in miles per hour squared.
- Use geometry to find the area under the curve $v(t)$. Explain why this value represents the total distance traveled by the car.
- Use an antiderivative to again find the total distance traveled by the car.
- What was the average velocity across the entire trip?

Selected Answers and Hints

Exercise 0.3.1. The first is a function, the second is a quantity, and the third is a statement.

Exercise 0.3.6. The first is a quantity, the second is a function, and the third through fifth are garbage. (Note that for comparing sets, there is in fact an entirely different relation called *subset*, written $\subseteq$, which means that the elements of one set are contained in the other set. So what was written in the problem was garbage, but one could say $\mathbb{N} \subseteq \mathbb{Z}$.) The sixth is a statement (that happens to be false), and the last two are statements (that happen to be true). Note that the last one may be a bit counterintuitive, saying that 3 is a complex number. However, the set of complex numbers contains the natural numbers (where 3 certainly lives), so it is valid to say 3 is a complex number. It is sometimes helpful to think of it as $3 + 0i$ to see that it does fit the standard form of a complex number.

Exercise 0.3.7. •In words, it says "For all persons x, there exists a person y, such that x is the mother of y." This is accurate but maybe sounds a bit rigid in English, where we would likely say something more like "Every person had a mother." Thus, this sentence is true, since every human came from some mother. (At least at the time of writing this, monkeys have been cloned, but not humans!) •In words, it says "There exists a person x, such that for all persons y, x is the mother of y. A more natural way to say this is "All people have the same mother!" which is clearly false, since the sentence implies that the one special person x is the mother of every person who has ever existed! •In words, the sentence says "Every real number has another real number bigger than it." This is certainly true. For example, one could satisfy the statement above by choosing $y = x + 1$, which is greater than x no matter what x is. •This statement says that there exists one real number that is bigger than all real numbers! This is false. One might think of infinity as a symbol which plays this role, but infinity is not an element of the set of real numbers.

Exercise 0.4.1. All statements are true except the last one. You cannot say that the polynomial $p(x) = 0$ has any degree at all, because we require the leading coefficient a_n to be nonzero, and here no such coefficient exists! Thus, the degree of the polynomial $p(x) = 0$ is undefined. It is also incorrect to say it has no roots; every number r satisfies $p(r) = 0$, so it actually has infinitely many roots.

Exercise 0.4.6. The roots are $x = \frac{-2\pm\sqrt{2^2-4\cdot 2\cdot 2}}{2\cdot 2} = \frac{-2\pm\sqrt{-12}}{4} = \frac{-2\pm 2\sqrt{3}i}{4}$ which simplifies to the roots $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Thus, it factors as $p(x) = 2\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$.

Exercise 0.4.9. •Here nothing needs to be done: the polynomial already has no linear term. If one wanted to really force it, it would be simply $p(x) = 1\left(x-0\right)^2 - 1$. •$p(x) = \left(x - 1/2\right)^2 + 3/4$ •$p(x) = 3(x-1)^2 - 2$

Exercise 0.4.10. •The multiplications all work out just fine; remember that $i^2 = -1$. •The polynomial $p(x) = 4x^4 - 9$ can be viewed as a difference of two squares: $p(x) = \left(2x^2\right)^2 - 3^2$. Choosing $A = 2x^2$ and $B = 3$ produces a factorization using the difference of two squares formula: $p(x) = \left(2x^2 - 3\right)\left(2x^2 + 3\right)$. Notice the first factor is again a difference of two squares, as it can be written as $\left(\left(\sqrt{2}x\right)^2 - \left(\sqrt{3}\right)^2\right) = \left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x + \sqrt{3}\right)$. If we are factoring only using real numbers, we would then be done, and leave the factorization as $p(x) = \left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x + \sqrt{3}\right)\left(2x^2 + 3\right)$. However, if we allow complex numbers, we can factor further by recognizing the final factor as a sum of two squares (or by using the quadratic formula, but this is excessive). This produces the factorization $p(x) = \left(\sqrt{2}x - \sqrt{3}\right)\left(\sqrt{2}x + \sqrt{3}\right)\left(\sqrt{2}x - \sqrt{3}i\right)\left(\sqrt{2}x + \sqrt{3}i\right)$. •The polynomial factors over the real numbers as $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$ and over the complex numbers as $x^6 - 1 = (x-1)(x+1)\left(x - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)$.

Exercise 0.4.15. The polynomial factors as $(x+1)^2(2x-3)$.

Exercise 0.4.16. •Simply plug it in and calculate $p(25/6)$. It will be zero. •Long division produces $p(x) = (6x-25)(x^2+x+2)$.

Exercise 0.4.18. First factor the common $4x^2$ out of the degree three and degree two terms. Then think of the $x-9$ as $1\cdot(x-9)$. The two groupings will have the common $(x-9)$ which can then be factored out.

Exercise 0.4.19. Hint: The first one is actually the trickiest. Rewrite the polynomial $x^4-x^3+2x^2-x+1$ as $x^4 - x^3 + x^2 + x^2 - x + 1$, and proceed to factor x^2 out of the first three terms.

Exercise 0.4.21. •We notice the $1, 5, 10, 10, 5, 1$ in the coefficients. This leads us to think about the row $n = 5$ of Pascal's Triangle, and tells us that the polynomial will factor in the form $(A+B)^5$. It only remains to decide what the A and B are. A little strategic guess and check will show that $A = -x^2$ and $B = 1$ works. Thus, the polynomial factors as $-x^{10} + 5x^8 - 10x^6 + 10x^4 - 5x^2 + 1 = \left(-x^2+1\right)^5$. Notice this actually factors further by reversing $-x^2+1$ to be $1-x^2$ and then applying the Difference of Two Squares. Thus, we have $-x^{10} + 5x^8 - 10x^6 + 10x^4 - 5x^2 + 1 = ((1-x)(1+x))^5 = (1-x)^5(1+x)^5$. •The coefficients lead us to row $n = 2$, since we see the $1, 2, 1$. Thus, $x^6 + 2x^3 + 1 = (x^3+1)^2$. We then factor the inside further using Sum of Two Cubes: $x^6+2x^3+1 = (x+1)^2(x^2-x+1)^2$. If we are factoring over the real numbers, we are done, or over the complex numbers one could continue by applying the quadratic formula to the degree 2 factor.

Exercise 0.5.3. As a hint, reference Example 0.5.2! One needs only to swap the roles of input and output, but also to be mindful of the domain/range restrictions. Arccosine outputs values between 0 and π; thus $\arccos\left(-\frac{1}{2}\right) = 2\pi/3$ works just fine. However, we cannot say that $\arcsin\left(\frac{\sqrt{3}}{2}\right) = 2\pi/3$ as well, because it is not in the proper interval. Arcsine outputs values between $-\pi/2$ and $\pi/2$. So, we need an angle $\theta \in [-\pi/2, \pi/2]$ that also satisfies $\sin(\theta) = \frac{\sqrt{3}}{2}$. A visual inspection of the unit circle reveals $\theta = \pi/3$ will do just fine. Thus, $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \pi/3$.

Exercise 1.1.4. The right-hand limit is 3 as well.

Exercise 1.1.8. The limits are 2,2,2,1,2, and DNE.

Exercise 1.1.10. The limits are $-1, -1, -1, -\infty, \infty$, and DNE.

Exercise 1.1.12. The limits are $-\pi/2, \pi/2, 0, 0$, and 0. The first two respectively correspond to horizontal asymptotes at $y = -\pi/2$ and $y = \pi/2$.

Exercise 1.1.13. The limits are $1, 1, -\infty, -\infty$, and $-\infty$.

Exercise 1.1.14. The limits are DNE, -2, DNE, $1, 1, 1, 1, 0$, DNE, ∞, ∞, ∞.

Exercise 1.2.2. The δ values are 2/3 and 1/5. The δ value will always be twice our ϵ value. That is, $\delta = 2\epsilon$, which works no matter how tiny ϵ is chosen.

Exercise 1.2.6. In this case, $\delta = \frac{1}{3}\epsilon$ will work.

Exercise 1.2.7. Use $\delta = \epsilon/m$ and then follow the template for a $\delta - \epsilon$ proof.

Exercise 1.2.8. For a horizontal line, the slope is zero, so the above argument fails since we would be dividing by zero in our choice of δ. However, for a horizontal line there is no error in the y coordinate no matter how much the x coordinate changes. So, a simple choice of δ like $\delta = 1$ will work just fine!

Exercise 1.2.11. A value of $\delta = 0.007$ or smaller will work. That is to say, if the radius is measured to a precision of ± 0.007 cm, then the area can be guaranteed to be within 0.01 cm^2 of $A = 4\pi$.

Exercise 1.2.21. •The values seem to just be bouncing around randomly! •Plugging the value into $f(x)$ will result in the output $\sin\left(2\pi n + \frac{\pi}{2}\right)$. The copies of 2π are irrelevant, so all are equal to $\sin(\pi/2) = 1$. •Similar to previous. •By choosing n sufficiently large, we can find an x value that is less than any δ, no matter how tiny, both of the form $\frac{1}{2\pi n + \frac{\pi}{2}}$ and $\frac{1}{\pi n}$. •The limit does not exist.

Exercise 1.2.27. The limit is zero. There are many ways to build this out of linear functions. One way is to just use the linear function x, multiplied with itself three times, and then added to the linear function 1.

Exercise 1.3.4. The argument can be essentially repeated, but you must reverse which of x and which of $-x$ is the upper bound vs lower bound. Think about why!

Exercise 1.3.10. The function is increasing on the first interval: if the x-coordinate increases, so does the y-coordinate. It is not increasing on the second interval. For example, consider $x_1 = \pi/2$, which is less than $x_2 = 3\pi/4$, but $\sin(x_1) = 1$ is actually greater than $\sin(x_2) = \sqrt{2}/2$.

Exercise 1.3.13. Yes for the first and third, since it is increasing and then decreasing, respectively. No for the second, since it is neither increasing nor decreasing on that interval.

Exercise 1.3.17. In the first case, the function is not monotone. In the second, it is not bounded.

Exercise 1.5.7. •Since the special limit for sine evaluates to 1, the result is $\ln(1)$ which is zero. •The sign issues are nonissues; if x is negative but close to zero then $\sin(x)$ is negative as well. Thus the quotient of the two is positive.

Exercise 1.5.8. •0 •1/2

Exercise 1.5.10. The first two statements can be written as the claims $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to b} g(x) = g(b)$. Putting these together, we have $\lim_{x\to a} (g \circ f)(x) = \lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)) = g(f(a)) = (g \circ f)(a)$. Thus, the composition of the two functions is continuous at a.

Exercise 1.5.12. There is not a unique correct way to express that as a composition, but one way is to call the inner function $f(x) = \cos(\pi x)$ and the outer function $g(x) = \frac{x}{x+1}$. To be thorough, one could even apply the theorem a second time, with the inner function πx being composed with the outer function $\cos(x)$.

Exercise 1.5.13. •4 •Though $f(x)$ is continuous at 0, $g(x)$ is not continuous at $1 = f(0)$, so the theorem does not apply.

Exercise 1.6.3. •The limits are equal because the two functions are identical for $x \neq 3$, and limits are only computed using x values that approach a, not equal to. •The first two expressions are both equal to $r(x)$, since they both are undefined at $x = 3$. The second equality is only valid if $x \neq 3$, since the final expression $\frac{x+3}{x+2}$ is defined at $x = 3$. The final expression is equal to $g(x)$ rather than $r(x)$. •There is no value we could define the function to be at $x = -2$ to make it become continuous. The limit at that value does not exist.

Exercise 1.6.4. The function is continuous for all real numbers except $x = 0$, which is nonremovable. No matter how you redefine the y coordinate of the function at $x = 0$, the limit will still not exist, and thus cannot equal the y coordinate.

Exercise 1.6.5. The function is discontinuous at $x = -3, -1, 0, 3$, and 5. All are nonremovable except $x = 3$ which is removable. The function $g(x)$ defined as $f(x)$ for $x \neq 3$ but 2 for $x = 3$ is the corresponding function that is continuous at $x = 3$.

Exercise 1.6.6. •Continuous on $(-\infty, \infty)$. •Continuous on $(-\infty, \infty)$. •Continuous on $(-\infty, \infty)$. •Continuous on $(-\infty, 0) \cup (0, \infty)$. Removable discontinuity at $x = 0$. •Continuous on $(-\infty, 0) \cup (0, \infty)$. Nonremovable discontinuity at $x = 0$. •Continuous on $[0, \infty)$. •Continuous on $(-\infty, \infty)$. •Continuous on $(-\infty, 4) \cup (4, \infty)$. Removable discontinuity at $x = 4$. •Continuous on $[-1, 0) \cup (0, \infty)$. Removable discontinuity at $x = 0$.

Exercise 1.7.1. The function is undefined at $x = -2$ and undefined at $x = 3$. However, the limit exists and can be written as $\lim_{x\to 3} r(x) = 1$.

Exercise 1.7.3. $\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} = \frac{1+0}{\sqrt{1}+1} = \frac{1}{2}$.

Exercise 1.7.9. Since $t = 2x$, you can divide both sides by 2 to get $x = t/2$.

Exercise 1.7.10. The function can be split into the product $\frac{\sin(2x)}{x} = (2\cos(x)) \cdot \left(\frac{\sin(x)}{x}\right)$. The limit of the first factor becomes 2 and the second becomes 1.

Exercise 1.7.11. •0 •0 •-1 •1 •DNE •1 •-1

Exercise 1.7.12. $L = 0$ as before.

Exercise 1.7.14. The limit evaluates to 2.

Exercise 1.8.3. Consider the left-hand side to be a function, namely $f(x) = e^{x^2+1} + \ln(x)$. The function is continuous on $(0, \infty)$. If one chooses an x-value near 0, say $x = 1/10$, one will get a y-value less than 2 (in this case something like 0.44). If one chooses a larger x-value, one will get a y value larger than 2 (say for example $f(1) = e^2$). Thus, somewhere between 1/10 and 1 there must be a solution to the equation, as a result of applying IVT to the function $f(x)$ on the interval $[1/10, 1]$.

Exercise 1.8.5. The polynomial must have a root between -1 and 0 since $p(-1)$ is positive, while $p(0)$ is negative. Thus, the intermediate y value of zero must be attained by some x_0 between -1 and 0. The root happens to be $x_0 = -\frac{1}{3}$. The factorization of the polynomial is $(3x + 1)(x - 2)(x + 5)$.

Exercise 1.8.7. It does not guarantee such an x_0 will exist; the given y-value 3 does not lie between the y-values on the endpoints of the interval. In fact, no such x_0 exists.

Exercise 1.8.10. The value $f(0)$ is positive while $f(\pi)$ is negative because P was assumed to be warmer than Q. Since temperature is a continuous function of location, and the difference of continuous functions is again continuous, we have that f is continuous. Thus, we can apply IVT and conclude there must exist a zero inbetween, since the function is positive at one endpoint and negative at the other.

Exercise 1.8.11. Let $D_1(t)$ be her distance from the trailhead at time t on day 1. Let $D_2(t)$ be her distance from the trailhead at time t on day 2. Then let $f(t) = D_1(t) - D_2(t)$ and apply IVT.

Exercise 1.10.1. The first is a vertical asymptote since the y-coordinate is approaching infinity. The second is a horizontal asymptote since the x-coordinate is approaching infinity.

Exercise 1.10.2. The limits are DNE, DNE, $-\infty$, DNE, 0,0,0, and ∞.

Exercise 1.10.3. The small positive x-values seem to indicate a limit of $0.1666\ldots$. Small negative x-values would produce the same y-values since all instances of x in the formula are squared and the negatives would vanish. Applying the conjugate and simplifying calculates the limit as one-sixth, which as a decimal matches what we found above!

Exercise 1.10.4. Let $\epsilon > 0$, an arbitrarily chosen real number. Using this ϵ, we define the value $\delta = -\epsilon$. Note this choice is motivated by the fact that ϵ represents change in y while δ represents change in x. Thus the slope of the line, -1, must satisfy $\epsilon/\delta = -1$. Assume that x is an input chosen within distance of δ from 2, but not equal to 2. That is, we assume

$$|x - 2| < \delta.$$

Under these conditions, we wish to show that any corresponding output $f(x) = 2 - x$ will be within ϵ of $L = 0$. Proceeding, we have $|f(x) - L| = |2 - x - 0| = |(-1) \cdot (x - 2)| < |(-1) \cdot \delta| = |(-1) \cdot -\epsilon| = |\epsilon| = \epsilon$. Thus, we have verified that the values of the function can be forced to be arbitrarily close to 0 by restricting our inputs to a domain sufficiently close to 2. This proves that $\lim_{x\to 2} 2 - x = 0$ as desired.

Exercise 1.10.5. The limit is five-thirds. In words, this means that the value of $\frac{1}{3}x + 1$ can be made arbitrarily close to 5/3 by restricting to x-values sufficiently close to 2. For $\epsilon = 0.1$, we can use $\delta = 0.3$ (or smaller). The $\delta - \epsilon$ proof can be constructed by following the template with $\delta = 3\epsilon$.

Exercise 1.10.6. The above limit evaluates to 2. In words, this means that the value of $\sqrt{x}$ can be made arbitrarily close to 2 by restricting to x-values sufficiently close to 4. The intersection point of the line $y = 1.9$ with the graph at the point roughly $(3.61, 1.9)$ indicates that $\delta_1 = 0.39$ should work on the left. On the right however, we have an intersection point of the at roughly $(4.41, 2.1)$ indicates a value of $\delta_2 = 0.41$ on the right. Thus, we must restrict to the smaller $\delta = 0.39$ in order to guarantee the outputs are within $\epsilon = 0.1$ of the limit. This is because if we chose the larger, $\delta = 0.41$, that would allow inputs like $x = 3.60$ to be used, and $\sqrt{3.60}$ is not within 0.1 of 2.

Exercise 1.10.7. Tangent is continuous on $[0, 1]$ but not the other two domains. Arctangent is continuous on all three.

Exercise 1.10.8. There are certainly many valid ways to address this. However, one in particular would be to look at a function like $\sin(1/x)$ at $x = 0$. Can the graph be drawn without lifting the pen? Can your pen move infinitely fast or cover infinite distance? It is not a well-defined mathematical notion. However, our limit definition will always return a definitive "yes" or "no". (It returns "no" in the example given here.)

Exercise 1.10.9. There is just one removable discontinuity at $x = 10$. If $f(10)$ were redefined to be two-thirtieths, the function would be continuous.

Exercise 1.10.10. There are removable discontinuities at $x = 10$ and $x = -10$. Both values could be redefined to be 1 to produces a continuous function.

Exercise 1.10.11. There is a nonremovable discontinuity at $x = 100$. It is an infinite discontinuity and as such there is not a single value we could redefine to make the function continuous at that point.

Exercise 1.10.12. •IVT does not apply because the y-value 0 is not between $f(0)$ and $f(1)$. •IVT does apply since the y-value 0 is between $f(0)$ and $f(1)$ and $f(x)$ is continuous on that interval. Thus $f(x)$ must have a root in that interval (and in fact it does, at $x \approx 1.618$). •Although 0 is between the values of $f(0)$ and $f(2)$, IVT does not apply because $f(x)$ is not continuous on $[0, 2]$.

Exercise 2.1.4. If $a = b$, then the average rate of change formula has division by zero.

Exercise 2.1.8. The slope of the tangent line is one-twelfth.

Exercise 2.1.9. The tangent line has slope twelve.

Exercise 2.1.10. The cubed root is the inverse function of the cubic polynomial function. When we take an inverse function, we interchange the x- and y-axes. Thus, when computing slopes, rise becomes run and run becomes rise, so slopes get reciprocated. Sure enough, one-twelfth is the reciprocal of twelve and vice versa.

Exercise 2.1.11. The limit does not exist, therefore the slope of the tangent line is undefined. The function is not differentiable at 0.

Exercise 2.1.12. The cosine angle sum formula will be very helpful in evaluating the limit.

Exercise 2.1.15. It shifts the graph up three units. Translating the graph upwards did not affect the derivative at all, which makes sense since it is just moving the entire curve up uniformly but not modifying the slopes in any way.

Exercise 2.1.18. No matter what the starting point, the tangent line will always just be the original line. Thus, the slope is always just m. The limit definition will confirm this, that $f'(x) = m$, the constant function.

Exercise 2.1.20. There are! Play around and you'll find more than zero of them.

Exercise 2.2.11. •0 •0 •$2x - 3$ •$-2x^{-4/3}$ •$-\frac{1}{2}x^{-3/2}$ •$50x^{3/2}$

Exercise 2.2.16. •$2xe^x + x^2e^x$ •$\sqrt{x}\cos(x) + \frac{\sin(x)}{2\sqrt{x}}$ •$e^x \sin(x)$

Exercise 2.2.18. They all match and are all zero. Heh.

Exercise 2.2.23. All answers will match and equal $4(2x + 3)$.

Exercise 2.2.24. All answers match and are equal to $6x^5$.

Exercise 2.2.27. The derivative is $\sec^2(x)$.

Exercise 2.2.28. •$\sec(x)\tan(x)$ •$-\csc(x)\cot(x)$ •$-\csc^2(x)$

Exercise 2.2.31. It comes out to zero.

Exercise 2.2.33. Both are equal to $-e^{-x}$.

Exercise 2.2.34. $\bullet e^{-x}\left(\cos^2(x)-\sin^2(x)-\cos(x)\sin(x)\right)$ $\bullet\frac{1-2x-\sqrt{1-4x}}{2x^2\sqrt{1-4x}}$

Exercise 2.3.5. The function is self-inverse because it composes to the identity map. That is, $1/(1/x)=x$. The Inverse Function Theorem produces $1/(-1/(1/x)^2)=-1/x^2$.

Exercise 2.3.7. The derivative of the base-two exponential is $2^x\cdot\ln(2)$. The derivative of the base-two logarithm is $\frac{1}{\ln(2)x}$. Analogous formulas hold if the 2 is replaced with a positive real a.

Exercise 2.3.10. The derivatives are $-\frac{1}{\sqrt{1-x^2}}$, $\frac{1}{1+x^2}$, $\frac{1}{x\sqrt{x^2-1}}$, $-\frac{1}{x\sqrt{x^2-1}}$, and $-\frac{1}{1+x^2}$.

Exercise 2.3.19. $\bullet\operatorname{arcsinh}(x)=\ln\left(x+\sqrt{1+x^2}\right)$ $\bullet$The composition will be x as desired. $\bullet$The derivative can clean up to $\frac{1}{\sqrt{1+x^2}}$. $\bullet$It will also produce $\frac{1}{\sqrt{1+x^2}}$.

Exercise 2.3.20. $\bullet\operatorname{arctanh}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $\bullet$The composition will be x as desired. $\bullet$The derivative can clean up to the surprisingly nice $\frac{1}{1-x^2}$. $\bullet$It will also produce $\frac{1}{1-x^2}$.

Exercise 2.3.21. The formula becomes $\sin^2(x)+\cos^2(x)=1$ which is true since it is just the Pythagorean Identity.

Exercise 2.3.23. The sines cancel and you are left with $\cos(x)$, which makes perfect sense since that is in fact the derivative of sine.

Exercise 2.3.26. Implicit differentiation results in $x/2+2y\cdot y'=0$ which implies that $y'=-x/(4y)$. Thus the slope of the tangent line at P is -1/2 and the equation of the tangent line (using point-slope form) is $y-\sqrt{2}/2=-\frac{1}{2}(x-\sqrt{2})$. On the other hand, if we solve for the explicit formula, we have $y=\sqrt{1-x^2/4}$. The derivative is $y'=\frac{-x}{4\sqrt{1-x^2/4}}$, which if we substitute $x=\sqrt{2}$ becomes $\frac{-\sqrt{2}}{4\sqrt{(1/2)}}=-\frac{1}{2}$, which verifies the slope we obtained via implicit differentiation.

Exercise 2.4.3. There are many valid answers, but one of the most familiar is something like $f(x)=|x|$ at 0. The function is continuous at that point but not differentiable.

Exercise 2.4.4. The first and last bullet create functions that are neither differentiable nor continuous. The second is continuous but not differentiable. The third is both continuous and differentiable. The results are in agreement with the theorem, since none of our choices for a and b created an example that was differentiable but not continuous, which the theorem shows is impossible to do.

Exercise 2.5.3. All polynomials are continuous, so this function is. The max occurs at both endpoints and the min occurs at the origin.

Exercise 2.5.4. Though removable, the function has a discontinuity at $x=0$. The absolute min would have been at that point, but it is a hole instead. It does happen to attain its absolute max at the endpoints, -2 and 2.

Exercise 2.5.5. It does have a min at the right hand endpoint. It has no max since it goes to infinity as it approaches zero from the right. It did not satisfy the EVT preconditions since it was continuous only on the interval $(0,2]$ and not $[0,2]$.

Exercise 2.5.9. There are many correct answers, but any interval containing the point $x=-\frac{\sqrt{(3)}}{3}$ that keeps away from y-values higher than $y = 2\sqrt{3}/9$ is fine. For example, $[-1, 0]$ works just fine, but so does $[-300, 0.2]$. Similarly for the other point.

Exercise 2.5.13. The derivative is $-\sin(x)$, which is zero at each local max and min

Exercise 2.6.2. Not needed since we stated the function was differentiable, and differentiable implies continuous.

Exercise 2.6.4. The slope of the secant line is 3. The slope of the tangent line is equal to that at $c = 3/2$ in that interval.

Exercise 2.6.6. MVT requires the function be differentiable on the interval in question; the function $f(x) = |x|$ is not differentiable at 0.

Exercise 2.6.7. The slope of the secant line is zero. The slope of the tangent line is equal to that at $c = 2$ in that interval.

Exercise 2.8.1. The slope of the line $f(x) = x$ is 1, since in slope-intercept form the equation would be $f(x) = 1 \cdot x + 0$. The power rule also tells us the slope is 1, since $(x)' = \left(x^1\right)' = 1x^0 = 1$.

Exercise 2.8.2. •Polynomials are continuous everywhere, so the EVT applies. The minimum value is at the vertex of the parabola, $(1/2, -5/4)$. The maximum value is -1 and occurs at the endpoints. •Rational functions can only be discontinuous at zeros of the denominator. This rational function has zeros at $x = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $x = \frac{1-\sqrt{5}}{2} \approx -0.618$. Since neither of these is in the interval $[0.1]$, we know $f(x)$ is continuous on $[0, 1]$ and thus EVT applies. The max and min values are simply the reciprocals of the values in the previous part. •EVT does not apply here since the function is discontinuous.

Exercise 2.8.3. •The Extreme Value Theorem requires the function be continuous on an interval, whereas Fermat's Theorem has the stronger requirement that the function also be differentiable. The Extreme Value Theorem promises the existence of a max and min for a continuous function on a closed interval; if that continuous function happens to also be differentiable then the derivative will be zero at that max and mix if it occurs on the interior of the interval (as opposed to at an endpoint). •Again, there are many valid examples, but an easy one is to consider the function $f(x) = -|x|$. It has an absolute maximum at the origin but $f'(0)$ is undefined.

Exercise 2.8.4. We first apply the definition of derivative: $f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{x+h}+1-\left(\sqrt[3]{x}+1\right)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$. We then apply the difference of two cubes formula $A^3 - B^3 = (A-B)\cdot\left(A^2 + AB + B^2\right)$ to remove the discontinuity as follows: $f'(x) = \lim_{h\to 0} \frac{\left(\sqrt[3]{x+h}-\sqrt[3]{x}\right)\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\,\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\,\sqrt[3]{x}+\sqrt[3]{x^2}\right)}$. Carrying out the multiplication produces $f'(x) = \lim_{h\to 0} \frac{\left(\sqrt[3]{x+h}\right)^3-\left(\sqrt[3]{x}\right)^3}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\,\sqrt[3]{x}+\sqrt[3]{x^2}\right)} = \lim_{h\to 0} \frac{x+h-x}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\,\sqrt[3]{x}+\sqrt[3]{x^2}\right)}$. The x and $-x$ in the numerator cancel, which then lets us cancel the h and then substitute $h = 0$ as follows: $f'(x) = \lim_{h\to 0} \frac{h}{h\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\,\sqrt[3]{x}+\sqrt[3]{x^2}\right)} = \lim_{h\to 0} \frac{1}{\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\,\sqrt[3]{x}+\sqrt[3]{x^2}} = \frac{1}{\left(\sqrt[3]{x+0}\right)^2+\sqrt[3]{x+0}\,\sqrt[3]{x}+\sqrt[3]{x^2}}$. Algebra cleanup produces $f'(x) = \frac{1}{\left(\sqrt[3]{x}\right)^2+\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x^2}} = \frac{1}{3\left(\sqrt[3]{x}\right)^2} = \frac{1}{3}x^{-2/3}$. On the other hand, we could use the power rule and produce $f'(x) = \left(x^{1/3}\right)' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$, which matches! •To match the notation of the IFT, we rename our function $f^{-1}(x) = \sqrt[3]{x} + 1$. We use the "swap x and y and solve" trick to find the inverse of this function. Specifically, if $x = \sqrt[3]{y} + 1$, then $x - 1 = \sqrt[3]{y}$ and $y = (x-1)^3$. We now apply IFT to the functions $f(x) = (x-1)^3$, $f'(x) = 3(x-1)^2$, and $f^{-1}(x) = \sqrt[3]{x} + 1$. This computes $\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3(f^{-1}(x)-1)^2} = \frac{1}{3\left(\sqrt[3]{x}+1-1\right)^2} = \frac{1}{3\left(\sqrt[3]{x}\right)^2 = \frac{1}{3}x^{-2/3}}$. IT WORKS YET AGAAAAAIN!

Exercise 2.8.5. •The chain rule produces $f'(x) = 2\cos(2x)$. •Applying the product rule to the double-angle formula produces $f'(x) = 2\cos(x)\cos(x) + 2\sin(x)(-\sin(x))$. This simplifies to $f'(x) = 2\cos^2(x) - 2\sin^2(x)$. •The cosine double-angle identity $\cos(2x) = \cos^2(x) - \sin^2(x)$ shows the answers are in fact compatible.

Exercise 2.8.6. •The graph is a secant graph with a horizontal compression by a factor of π. The vertical asymptotes are at the integer and half-integer x-values. •It does not. Since the function is discontinuous at $x = 1/2$, it fails to be differentiable at that point as well. •It does! The function is differentiable on that interval. Since $f(x) = \sec(\pi x)$ is an even function, we know $f(1/4) = f(-1/4)$, so the average rate of change on that interval is zero. From looking at the graph, we can see the instantaneous rate of change is zero at the point $(0, 1)$. Thus $c = 0$. Or, rather than appealing to the graph, one can calculate $f'(c) = \pi\sec(\pi c)\tan(\pi c)$ and set it equal to zero (the average rate of change). Solving this equation, we have $\pi\sec(\pi c)\tan(\pi c) = 0 \implies \sec(\pi c) = 0$ or $\tan(\pi c) = 0$. However, secant can never be zero. Thus the solution must come from tangent, which is zero when the input is zero (on this interval). Thus, $\pi c = 0 \implies c = 0$.

Exercise 3.1.1. Using this principle, Friday would be predicted to be 22 degrees, as Thursday's temperature is taken to be the best guess at Friday's temperature.

Exercise 3.1.3. The weather seemed to be gaining one degree every two days, so it looks like Friday's high should be 23 degrees.

Exercise 3.1.7. It should square to exactly 4.1 if it were perfect. The square of our approximation is 4.10065, which is very close to 4.1!

Exercise 3.1.8. The linearization is $L(x) = 2 + \frac{1}{12}(x - 8)$. This produces the approximation $\sqrt[3]{8.1} \approx 2.008333\ldots$, which cubes to roughly 8.10041 (depending on how many digits you carry it out to). Not bad!

Exercise 3.1.9. The linearization is $L(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \pi/3)$. This produces an approximation of $\cos(1) \approx 0.54$.

Exercise 3.1.10. The linearization is $L(x) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$. This produces the approximation $\arctan(4/5) \approx 0.685$.

Exercise 3.2.3. The end behavior is up/up. The roots are at 2 and -1 and both have multiplicity two. There is a local max at $(1/2, 81/16)$.

Exercise 3.2.4. The function must attain an absolute max and absolute min on that closed interval by EVT since all polynomials are continuous everywhere. The absolute max is $y = -1$ which occurs at both the endpoint $x = 0$ and the critical point $x = -1$. The absolute minimum of $y = -31/27$ occurs at the critical point $x = -1/3$.

Exercise 3.2.5. The derivative is $f'(x) = 2ax + b$, which has $-\frac{b}{2a}$ as its only zero.

Exercise 3.3.9. Notice that the circumference of the cone base is equal to the circle's original circumference minus the amount that was removed in trimming the sector. Also notice that the radius of the original disc is equal to the distance from the vertex of the cone to any point on the edge of the circular base of the cone. These observations will lead to the formula $V(\theta) = \frac{1}{3}\sqrt{1 - \left(\frac{2\pi-\theta}{2\pi}\right)^2} \cdot \pi\left(\frac{2\pi-\theta}{2\pi}\right)^2$. The optimal value is the not exactly intuitive $\theta = \frac{2\pi}{3}\left(3 - \sqrt{6}\right)$.

Exercise 3.4.2. It is convex on the entire real number line. It is decreasing on $(\infty, 0)$ and increasing on $(0, \infty)$.

Exercise 3.5.3. The lifter should gain 20 lbs per month on leg press.

Exercise 3.5.4. It is always 500mph.

Exercise 3.5.5. After 1 second, $dh/dt = 1.845$ inches per second. After 2 seconds, it is 1.162 inches per second. It makes sense that the height changes more slowly as time goes on, since the cone keeps getting wider, so it takes more liquid to fill each vertical inch than it took to fill the previous.

Exercise 3.5.6. $\frac{\mathrm{d}P}{\mathrm{d}t}V + P\frac{\mathrm{d}V}{\mathrm{d}t} = nR\frac{\mathrm{d}T}{\mathrm{d}t}$

Exercise 3.5.7. Use the tangent function to relate the height of the room, the horizontal distance of the cat from center of the room, and the angle of the pointer from vertical. The dot will be traveling at a brutal speed of roughly 42.84mph when the dot hits the corner. Ouch! Slow that motor down!

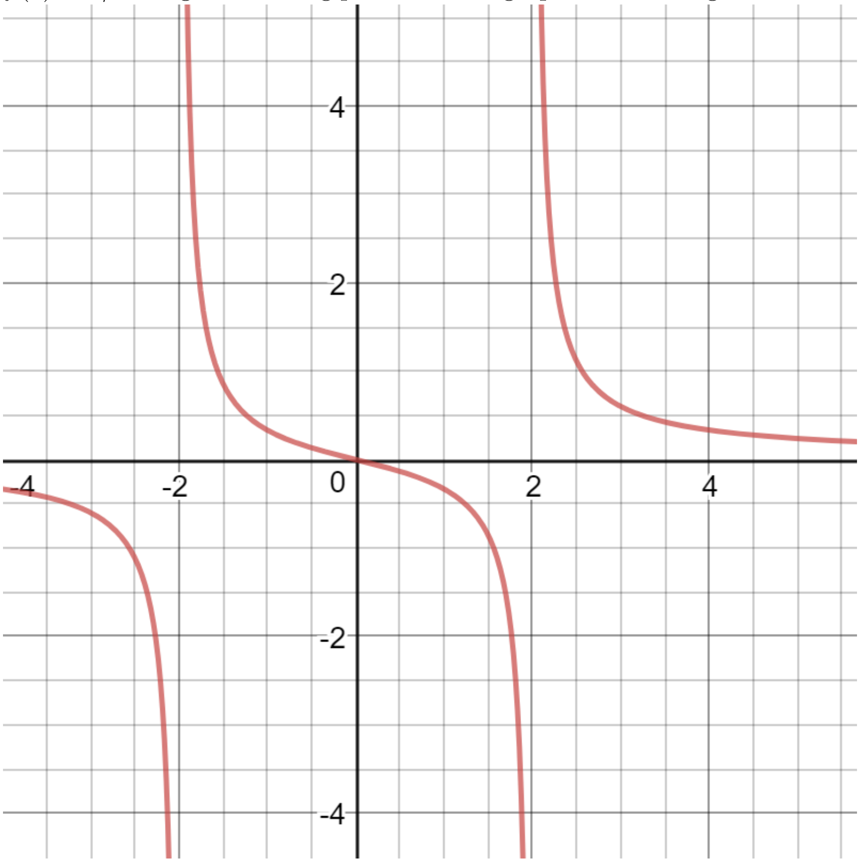
Exercise 3.7.1. The linearization $L(x) = 1 + 3(x - 1)$ calculates $L(1.1) = 1.3$, which is very close to the true value of $f(1.1) = 1.1^3 = 1.331$.

Exercise 3.7.2. All three statements are false.

Exercise 3.7.3. •The value of $\tan(0.8)$ should be approximated more accurately by the linearization since we are linearizing at $\pi/4$ which is very close to 0.8. The further you get from the point at which you linearize, the more error there will typically be in the approximation. •The linearization is $L(x) = 1 + 2\,(x - \pi/4)$. This approximates $\tan(0.8) \approx L(0.8) = 1.02920\ldots$ and $\tan(1) \approx L(1) = 1.4292\ldots$. The true values are $\tan(0.8) = 1.0296\ldots$ and $\tan(1) = 1.5574\ldots$. Thus the linearization was much more accurate at 0.8 as expected.

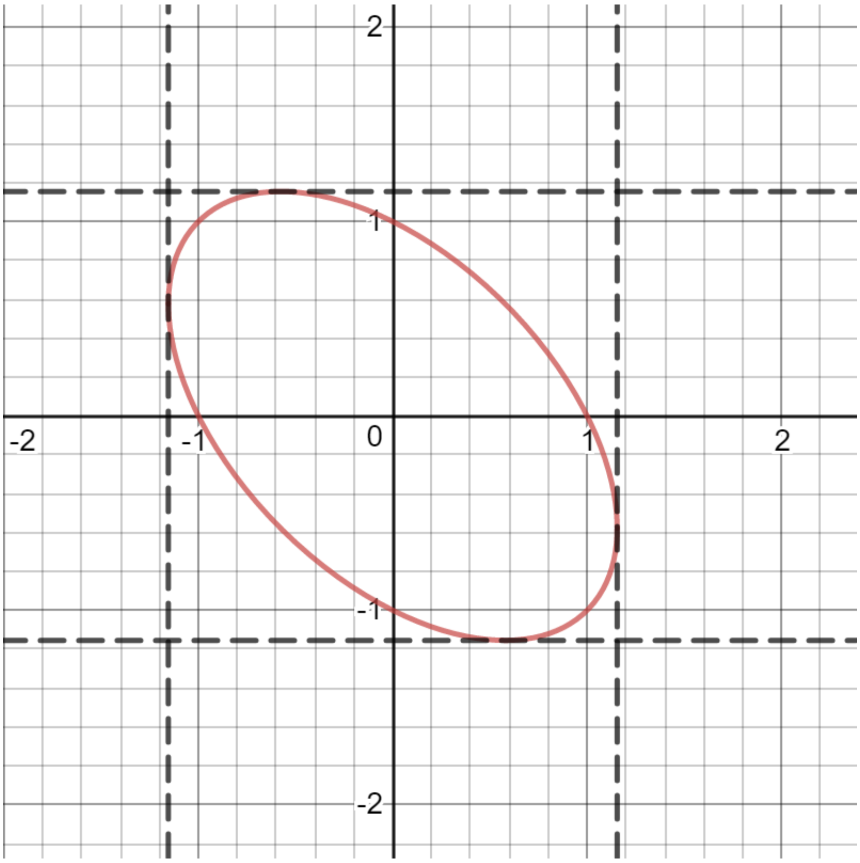
Exercise 3.7.4. •Since the graph already is a line, the linearization should just be the original line. That is, $L(x) = f(x)$. •The derivative is the constant function $f'(x) = m$. Thus we have $L(x) = f(a) + f'(a)(x - a) = ma + b + m(x - a) = ma + b + mx - ma = mx + b$. Thus $L(x) = f(x)$ as the graph indicated.

Exercise 3.7.5. •Division by zero is the only potential problem here, and $x^2 - 4 = 0$ if and only if $x = \pm 2$, so the domain is $(\infty, -2) \cup (-2, 2) \cup (2, \infty)$. This also tells us that there are vertical asymptotes at the lines $x = -2$ and $x = 2$. •The first derivative can be computed using the quotient rule as follows: $\frac{d}{dx}\left(\frac{x}{x^2-4}\right) = \frac{\left(x^2-4\right)(x)' - \left(x^2-4\right)'(x)}{(x^2-4)^2} = \frac{\left(x^2-4\right)1 - (2x)(x)}{(x^2-4)^2} = \frac{x^2-4-2x^2}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2}$, which is always negative. Thus, the graph is decreasing on the entire domain and has no local max or mins. •To calculate the second derivative, we take the first derivative of the first derivative, which we have already obtained above. Proceeding, we have $\frac{d}{dx}\left(\frac{-x^2-4}{(x^2-4)^2}\right) = \frac{\left(x^2-4\right)^2\left(-x^2-4\right)' - \left(\left(x^2-4\right)^2\right)'\left(-x^2-4\right)}{(x^2-4)^4} = \frac{\left(x^2-4\right)^2(-2x) - \left(2\left(x^2-4\right)2x\right)\left(-x^2-4\right)}{(x^2-4)^4} = \frac{\left(x^2-4\right)(-2x) - (4x)\left(-x^2-4\right)}{(x^2-4)^3} = \frac{-2x^3+8x+4x^3+16x}{(x^2-4)^3} = \frac{2x^3+24x}{(x^2-4)^3} = \frac{2x\left(x^2+12\right)}{(x^2-4)^3}$. Since the numerator is zero only at $x = 0$ and the denominator is zero only at 2 and -2, we need to determine the sign on the following intervals: **Interval** $(-\infty, -2)$**.** Trying the test point $x = -3$ produces the value $\frac{2(-3)\left((-3)^2+12\right)}{((-3)^2-4)^3} < 0$. **Interval** $(-2, 0)$**.** Trying the test point $x = -1$ produces the value $\frac{2(-1)\left((-1)^2+12\right)}{((-1)^2-4)^3} > 0$. **Interval** $(0, 2)$**.** Trying the test point $x = 1$ produces the value $\frac{2(1)\left((1)^2+12\right)}{((1)^2-4)^3} < 0$. **Interval** $(0, 2)$**.** Trying the test point $x = 3$ produces the value $\frac{2(3)\left((3)^2+12\right)}{((3)^2-4)^3} > 0$. Thus, the graph is convex on $(-2,0)\cup(2,\infty)$ and concave on $(-\infty,-2)\cup(0,2)$. with a point of inflection at the origin. •Lastly, we compute a few points just to get started. The values $f(-3) = -3/5, f(0) = 0$, and $f(3) = 3/5$ are good starting points for the graph. Assembling all of this information, we draw the graph.



Exercise 3.7.6. $\bullet Q(x) = 750 - 50x$ $\bullet R(x) = xQ(x) = 750x - 50x^2$ •The first derivative $R'(x) = 750 - 100x$ has a zero at $x = 7.5$ and the second derivative $R''(x) = -100$ verifies the original revenue function R has a max at that point. It is in fact an absolute max for the function since the graph is a downward-facing parabola. Thus the revenue is maximized when the martinis are sold for \$7.50 each.

Exercise 3.7.7. •If the variables x and y are swapped, the equation is unchanged. Thus, it must be symmetric across the line $y = x$. •If $x = 0$, $y^2 = 1$ which implies $y = \pm 1$. Similarly if $y = 0$ then $x = \pm 1$. Thus the intercepts are at $(0,1),(0,-1),(1,0)$, and $(-1,0)$. •Implicit differentiation produces $2x + x\frac{\mathrm{d}y}{\mathrm{d}x} + 1\cdot y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ which when solved for $\frac{\mathrm{d}y}{\mathrm{d}x}$ becomes $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+y}{x+2y}$. •If the numerator is zero, we have a horizontal tangent. If the denominator is zero, we have a vertical tangent. In particular, a horizontal tangent occurs wherever $y = -2x$, which when plugged into our original equation becomes $x^2 + x(-2x) + (-2x)^2 = 1$. Thus $x = \pm\sqrt{3}/3$. So, the horizontal tangents occur at $\left(\sqrt{3}/3, -2\sqrt{3}/3\right)$ and $\left(-\sqrt{3}/3, 2\sqrt{3}/3\right)$. Similarly, the vertical tangents occur at $\left(2\sqrt{3}/3, -\sqrt{3}/3\right)$ and $\left(-2\sqrt{3}/3, \sqrt{3}/3\right)$. •Assembling the above info, we graph the curve, showing the tangents as dashed lines.



Exercise 3.7.8. •If the radius is small, it doesn't take much ice cream to grow the cone. However, if the radius is large, it takes a large amount of ice cream to increase the radius and height of the cone. Since it pours are a constant volume per second, the radius should be increasing more quickly when the cone is small. •We see that V, h, and r are all functions of t. Fortunately, we are given that $h = r$. We substitute this into the volume formula to get $V = \frac{\pi}{3}r^3$. Differentiating both sides with respect to t produces $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{3}3r^2\frac{\mathrm{d}r}{\mathrm{d}t} = \pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$. Since the rate of change of volume is always just $1\frac{\mathrm{cm}^3}{s}$, we can solve for the radius 5cm rate of change by solving $1 = \pi 25\frac{\mathrm{d}r}{\mathrm{d}t}$. Thus, the rate of change at 5cm is $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{25\pi}\mathrm{cm}^3/s$. •By a similar argument, at radius 10 cm the rate of change is $\frac{1}{100\pi}\mathrm{cm}^3/s$. •It did match the prediction; as the cone became larger, the rate of change of the radius decreased.

Exercise 4.1.4. The totals are 12, -6, and 15.

Exercise 4.1.5. It is easiest to just expand the sums on both sides and see what the terms look like. For example, in the first case the left-hand side is $(ca_j + ca_{j+1} + \cdots + ca_k)$, whereas the right-hand side is $c\,(a_j + a_{j+1} + \cdots + a_k)$. These two expressions are equal, because we can factor the c out of the left-hand side to produce the right-hand side. For the last two summations, think about our discussion of fencepost problems above!

Exercise 4.2.5. The totals are 500500, 1501500, 214214, and 245.

Exercise 4.2.7. In the first case, both sides equal 55. In the second case, both sides equal 225.

Exercise 4.2.8. The degrees are 2,3,4, and 5.

Exercise 4.2.11. The common ratio $r = 10$. The first term is 1. The number of terms is 6. Putting this all together in the Geometric Series Formula produces $1 \cdot \frac{1-10^6}{1-10} = \frac{-99999}{-9} = 11111$.

Exercise 4.2.12. A finite sum of consecutive powers of two, starting at one, is equal to one less than the next power of two.

Exercise 4.2.15. Sure! If you further factor $A^2 - B^2$ via difference of two squares and further factor $A^3 + A^2B + AB^2 + B^3$ via grouping, you will end up with the same factorizations.

Exercise 4.3.10. The true area is 1/4.

Exercise 4.3.11. The first region has area $\frac{23+16\sqrt{2}}{12} \approx 3.8$. The second region has signed area $\frac{16\sqrt{2}-23}{12} \approx -0.03$.

Exercise 4.3.12. You will need the geometric series formula to evaluate the summation! The area of the region is $e - 1 \approx 1.8$.

Exercise 4.3.13. The area is $\frac{1}{2}e^2 - 1$.

Exercise 4.4.1. The second property is false but the rest are true.

Exercise 4.5.2. $\bullet 5/8 = .625$ $\bullet 14/27 \approx .518\ldots$ $\bullet 15/32 \approx .468\ldots$ $\bullet 11/25 = 0.44$

Exercise 4.7.1. One way is that area above the x-axis is counted as positive, and it could cancel out with area under the x-axis, which is counted as negative. Another way is that maybe the function is just $f(x) = 0$. Yet another way is that maybe the upper and lower bounds of the integral are equal to each other.

Exercise 4.7.2. The first and third are true while the second is false.

Exercise 4.7.3. The function is $f(x) = \sqrt{1-x^2}$, and the area under the curve is half of π because the whole interior of $x^2 + y^2 = 1$ is the unit circle. The four-rectangle Riemann sum approximation gives $\pi = 2\int_{x=-1}^{x=1} f(x)\,\mathtt{d}x \approx 2.73$. This is quite a bit smaller than π, which makes sense because we are only using four rectangles. The eight-rectangle Riemann sum approximation gives $\pi = 2\int_{x=-1}^{x=1} f(x)\,\mathtt{d}x \approx 2.99$ which is looking quite a bit better. The twenty-rectangle approximation produces $\pi = 2\int_{x=-1}^{x=1} f(x)\,\mathtt{d}x \approx 3.1044$.

Exercise 4.7.4. The graph is just an arc of an upwards sloping parabola connecting the point $(1,3)$ to the point $(2,12)$. For convenience, call the area under the curve A. For the Riemann sum, we begin with the definition and the formula for $f(x)$: $A = \int_1^2 f(x)\,\mathtt{d}x = \lim_{N\to\infty}\sum_{n=1}^{N} f(x_n)\,\Delta x = \lim_{N\to\infty}\sum_{n=1}^{N} 3x_n^2\Delta x$. We next pull the constant out of the summation and plug in the formulas for x_n and Δx: $A = 3\lim_{N\to\infty}\sum_{n=1}^{N}(1+n\Delta x)^2\,\Delta x = 3\lim_{N\to\infty}\sum_{n=1}^{N}\left(1+\frac{n}{N}\right)^2\frac{1}{N}$. Next, we break up the expression with algebra and summation properties: $A = 3\lim_{N\to\infty}\sum_{n=1}^{N}\left(1+\frac{2n}{N}+\frac{n^2}{N^2}\right)\frac{1}{N} = 3\lim_{N\to\infty}\left(\sum_{n=1}^{N}1+\frac{2}{N}\sum_{n=1}^{N}n+\frac{1}{N^2}\sum_{n=1}^{N}n^2\right)\frac{1}{N}$. At last, we are ready to apply summation formulas: $A = 3\lim_{N\to\infty}\left(N+\frac{2}{N}\frac{N(N+1)}{2}+\frac{1}{N^2}\frac{N(N+1)(2N+1)}{6}\right)\frac{1}{N} = 3\lim_{N\to\infty}\left(\frac{N}{N}+\frac{2N(N+1)}{2N^2}+\frac{N(N+1)(2N+1)}{6N^3}\right)$. I lied, that apparently was second to last. Now really at last, we use the fact that the three terms we are taking the limit of all represent rational function with the same degree in the numerator as denominator. The ratio of the leading coefficients will give us the limit: $A = 3\left(\frac{1}{1}+\frac{2}{2}+\frac{2}{6}\right) = 7$. Thus, the area under the curve is 7.

Exercise 4.7.5. The result is negative because integrals calculate signed area, and the graph lies entirely below the x-axis in that region. The integral evaluates to $-1/6$.

Exercise 4.7.6. The region is a trapezoid with altitude 1 and bases 7 and 6. Thus, the area is $1 \cdot \frac{7+6}{2} = 6.5$. The Riemann sum produces the same result.

Exercise 4.7.7. 750

Exercise 5.1.5. FTC I tells us that $F'(2) = 8$, the height of the cubic at $x = 2$. The values $F(2) = 4$ and $F(2.1) = 4.86\ldots$ (computed via Riemann sums) verify this, showing an average rate of change of 8.6.

Exercise 5.2.3. Both methods produce the answer of $m(b-a)$.

Exercise 5.2.4. Both methods produce an area of $a_2b^2 - a_2a^2 + a_1b - a_1a$.

Exercise 5.2.5. To verify the antiderivative, simply calculate the derivative of $x\ln(x) - x$ and verify it produces $\ln(x)$. The exact area is 1. The triangle approximates it as $\frac{1}{2}(e-1)$.

Exercise 5.3.5. It is always $\frac{1}{n+1}x^{n+1}$ unless $n = -1$, in which case that formula would give you division by zero. Fortunately, in that case the antiderivative is just $\ln(x)$.

Exercise 5.3.8. The antiderivative is $2x + \arctan(x) + C$.

Exercise 5.4.1. $\int f'(g(x)) \cdot g'(x)\,\mathtt{d}x = \int (f(g(x)))'\,\mathtt{d}x = f(g(x)) + C$

Exercise 5.4.6. Use the substitutions $u = x^2 + x + 8, \ln(x)$, and $-x^2$. In the last case, the $\mathtt{d}u$ term has nothing to cancel the x with!

Exercise 5.4.14. The antiderivative is $\operatorname{arctanh}(3x-2) + C$.

Exercise 5.4.15. The antiderivative is $\operatorname{arcsec}(x+1) + C$.

Exercise 5.6.1. Differentiate it! If you get the original function back, you win. The derivative of $x\ln(x) - x + C$ (which will need product rule) will produce just $\ln(x)$ when simplified.

Exercise 5.6.2. If a function $f(x)$ has antiderivative $F(x)$, that means that $F'(x) = f(x)$. However, if that equation is true, then so is $(F(x) + C)' = f(x)$ for any real number C. Thus an antiderivative can always be adjusted up or down by a constant.

Exercise 5.6.3. •By Fundamental Theorem of Calculus Part I, we have $f'(x) = \frac{\mathtt{d}}{\mathtt{d}x}\left(\int_{t=1}^{t=x} \frac{1}{t}\,\mathtt{d}t\right) = \frac{1}{x}$. •The composition can be expressed as $f \circ g(x) = \int_{t=1}^{t=x^2} \frac{1}{t}\,\mathtt{d}t$. •The inner function is $g(x) = x^2$, while the outer is $f(x) = \ln(x)$. Thus, the derivative is $\frac{1}{x^2} \cdot 2x = \frac{2}{x}$. •This derivative is also $\frac{2}{x}$ by simply pulling out the constant and differentiating the logarithm. •Their derivatives are equal. Thus, they can only differ by a constant. This fact is Corollary 2.6.10. •If we set $x = 1$, we get the equation $0 = 0 + C$, so $C = 0$ and the functions are equal.

Exercise 5.6.4. •$1 - \frac{\sqrt{3}}{2}$ •$\frac{1}{2}$ •π •$\pi/8$ (**Hint:** Read the Chapter Summary for help with the antiderivative.)

Exercise 5.6.5. •Setting $a = 2$ and dividing both sides of the formula given by $\ln(2)$ produces $\int 2^x \, \mathtt{d}x = \frac{1}{\ln(2)} 2^x + C$. •We use the result of the previous problem along with the u-substitution $u = 3x - 1$ and $\frac{\mathtt{d}u}{\mathtt{d}x} = 3$ to produce $\int 2^{3x-1} \, \mathtt{d}x = \frac{1}{3\ln(2)} 2^{3x-1} + C$. •Applying the u-sub $u = x^2 + 1$ with $\frac{\mathtt{d}u}{\mathtt{d}x} = 2x$ produces $\int x 2^{\left(x^2+1\right)} \, \mathtt{d}x = \frac{1}{2\ln(2)} 2^{x^2+1} + C$.

Exercise 5.6.6. •$\int 2x + 1 \, \mathtt{d}x = x^2 + x + C$ Here no u-sub is required. •We apply the substitution $u = 2x+1$, which implies $\mathtt{d}u / \mathtt{d}x = 2$ and $\mathtt{d}x = \mathtt{d}u/2$. Thus, $\int \frac{1}{2x+1} \, \mathtt{d}x = \int \frac{1}{u} \frac{\mathtt{d}u}{2} = \frac{1}{2} \int \frac{1}{u} \, \mathtt{d}u = \frac{1}{2} \ln(u) + C = \frac{1}{2} \ln(2x+1) + C$. •The occurrences of e^x inside parentheses motivate the choice $u = e^x$. Thus, $\mathtt{d}u / \mathtt{d}x = e^x$ as well and $\mathtt{d}x = \mathtt{d}u / e^x$. The integral becomes $\int e^x \sec\left(e^x\right) \tan\left(e^x\right) \mathtt{d}x = \int e^x \sec(u) \tan(u) \frac{\mathtt{d}u}{e^x} = \int \sec(u) \tan(u) \, \mathtt{d}u = \sec(u) + C = \sec\left(e^x\right) + C$. •Though the initial form might motivate the choice of $u = e^x$, upon trying this one finds the integral becomes worse rather than better. Instead, rewrite as a negative exponent and use $u = -x$ so $\mathtt{d}x = -\mathtt{d}u$. Thus, $\int \frac{1}{e^x} \, \mathtt{d}x = \int e^{-x} \, \mathtt{d}x = \int e^u (-1) \, \mathtt{d}u = -\int e^u \, \mathtt{d}u = -e^u + C = -e^{-x} + C$. •One can try the inner function $u = 2x$, but that ends up not cleaning things up enough. A more productive choice is $u = \ln(2x)$. That choice produces $\frac{\mathtt{d}u}{\mathtt{d}x} = \frac{1}{2x} \cdot 2 = \frac{1}{x}$. Solving for $\mathtt{d}x$ produces $\mathtt{d}x = x \cdot \mathtt{d}u$. We now substitute into the integral, creating $\int \frac{\cos(\ln(2x))}{x} \, \mathtt{d}x = \int \frac{\cos(u)}{x} x \cdot \mathtt{d}u = \int \cos(u) \cdot \mathtt{d}u = \sin(u) + C = \sin(\ln(2x)) + C$. •One can try $u = 2^x$, but it gets tangled up. Instead, try simplifying the integrand using properties of exponents to become 2^{2x}, at which point $u = 2x$ becomes a natural substitution. Taking the derivative produces $\mathtt{d}u / \mathtt{d}x = 2$ so $\mathtt{d}x = \mathtt{d}u/2$. Thus, $\int \left(2^x\right)^2 \mathtt{d}x = \int 2^{2x} \, \mathtt{d}x = \int 2^u \, \mathtt{d}u/2 = \frac{1}{2} \int 2^u \, \mathtt{d}u = \frac{1}{2} \frac{1}{\ln(2)} 2^u + C = \frac{1}{2 \cdot \ln(2)} 2^{2x} + C$. •This seems close enough to arctangent if you just glance at it! Let's complete the square on the denominator to get it there. In particular, $2 + 2x + x^2 = 1 + 1 + 2x + x^2 = 1 + (1+x)^2$, which motivates the u-sub $u = 1 + x$ with change of differential $\mathtt{d}u / \mathtt{d}x = 1$ which implies $\mathtt{d}u = \mathtt{d}x$. Therefore, $\int \frac{1}{2+2x+x^2} \, \mathtt{d}x = \int \frac{1}{1+(1+x)^2} \, \mathtt{d}x = \int \frac{1}{1+u^2} \, \mathtt{d}u = \arctan(u) + C = \arctan(1+x) + C$. •This appears so similar to the previous, yet it works out quite differently! The denominator is a perfect square: $(1+x)^2$. Thus, we simply apply u-sub with $u = 1 + x$, which has the squeaky clean consequence that $\mathtt{d}u / \mathtt{d}x = 1$ so $\mathtt{d}u = \mathtt{d}x$. This produces the integral $\int \frac{1}{1+2x+x^2} \, \mathtt{d}x = \int \frac{1}{(1+x)^2} \, \mathtt{d}x = \int \frac{1}{u^2} \, \mathtt{d}u = \int u^{-2} \, \mathtt{d}u = u^{-1}/(-1) + C = -\frac{1}{1+x} + C$.

Exercise 6.1.3. As the region converges upon an isosceles right triangle with legs of length one, the area converges upon one-half.

Exercise 6.1.7. •The curves intersect at $x = -2/3$ and $x = 2$. The area between them is $4/3$. •The curves intersect at $x = 0$ and $x = 2^{2/3}$. The area between them is $2/3$. •The curves $y = x^3 + x^2 - x - 1$ and $y = x^3 - x^2 - x + 1$ intersect on the x axis at -1 and 1 and have area $8/3$ between them.

Exercise 6.2.3. The average temperature was Denver is 27.7 degrees, while in Philly it was hotter on average, at 29.4 degrees. The fraction $24/N$ is Δt in the Riemann sum. We paid for that 24 in the numerator of the fraction by also dividing by 24, essentially multiplying the expression by 1. Harmless!

Exercise 6.2.5. The positive and negative y-coordinates perfectly balance each other, so 0 would be a reasonable guess for the average value. The integral confirms this.

Exercise 6.2.6. The average value of $x^2 - 10x$ on the interval $[0, 15]$ should give us the average gain of Toadratic over Linearibbit on a single jump. Multiplying this by sixty will produce a good estimate of how far ahead he should be after one minute.

Exercise 6.3.4. The integral that calculates the probability is $\frac{1}{\pi - 0} \int_0^\pi l \sin(x)/d \, \mathtt{d}x$. If a needle of length l is dropped upon a ruled plane with rulings d apart and $l < d$, the probability the needle crosses a ruling is $\frac{2l}{\pi d}$.

Exercise 6.4.2. It hits the ground in roughly 1.6 seconds.

Exercise 6.6.1. The average value is h.

Exercise 6.6.2. The first two can be false. The third is always true.

Exercise 6.6.3. The average value of $f(x)$ is $2/\pi \approx 0.63662$ and the average value of $g(x)$ is $\pi/4 \approx 0.7854$, which is quite a bit larger as a result of cosine dive-bombing towards the x-axis much more rapidly, while the circle takes longer to decay away from the point $(0, 1)$.

Exercise 6.6.4. Since $a(t) = -16$, the integral is $v(t) = -16t + C$. We can solve for C using the given initial velocity of $v(0) = 20$, so $v(t) = 20 - 16t$. We integrate again to find the position function of $s(t) = 20t - 8t^2 + C$. Again, we solve for C using the initial height of $s(0) = 10$, so $s(t) = 10 + 20t - 8t^2$. To find when it hits the ground, we use the quadratic formula to find the zeros of $s(t)$. Specifically, they are $t = \frac{-20 \pm \sqrt{400-4\cdot(-8)\cdot(10)}}{2(-8)} \approx -0.42$ or 2.92. The negative root does not make sense in the context of the problem however, so we conclude the object hits the time after $t \approx 2.92$ seconds.

Exercise 6.6.5. The region will form a trapezoid whose area can be found by multiplying the height by the average of the bases. The average velocity is $33.333\ldots$ miles per hour.